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A NOTE ON WEAK EXCLUDED MIDDLE LAW

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Abstract. Intuitionistic Zermelo-Fraenkel (IZF) set theory is a set theory without the axiom of choice and the law of excluded middle (LEM). The weak excluded middle law (WEM) states that $\neg\varphi \vee \neg\neg\varphi$ for any formula φ. In IZF we show that LEM is equivalent to WEM plus the condition that any set not equal to the empty set has an element.

1. Introduction

Intuitionistic set theory is a set theory based on intuitionistic logic which is just classical logic minus LEM. Its axiomatic formulation known as IZF system consists of the following axioms :

- Extensionality: $\forall x \forall y[\forall z(z \in x \leftrightarrow z \in y) \to x = y]$
- Empty set: $\exists x \forall y \neg(y \in x)$
- Pairing: $\forall x \forall y \exists z \forall w(w \in z \leftrightarrow w = x \vee w = y)$
- Union: $\forall x \exists y \forall z[z \in y \leftrightarrow \exists w(w \in x \wedge z \in w)]$
- Power set: $\forall x \exists y \forall z[z \in y \leftrightarrow \forall w(w \in z \to w \in x)]$
- Infinity: $\exists x[\emptyset \in x \wedge \forall y(y \in x \to y \cup \{y\} \in x)]$
- Separation schema: $\forall a \exists x \forall y(y \in x \leftrightarrow y \in a \wedge \phi(y))$ where x is not free in $\phi(y)$
- Collection schema: $\forall a[(\forall x \in a\ \exists y\ \phi(x, y)) \to (\exists b \forall x \in a\ \exists y \in b\ \phi(x, y))]$ for all formulas $\phi(x, y)$ where b is not free in $\phi(x, y)$
- $\in$ induction schema: $\forall a[(\forall y \in a\ \phi(y)) \to \phi(a)] \to \forall a \phi(a)$ for every formula $\phi(a)$

By extensionality the empty set is unique and denoted by $\emptyset$. These 9 axioms are chosen so that LEM is not deduced from them, and the combination of IZF and LEM gives Zermelo-Fraenkel (ZF) set theory. The above first 7 axioms constitute intuitionistic Zermelo (IZ) set theory.

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A weaker form of LEM is WEM and it is equivalent to De Morgan's law

$$\neg(\varphi \wedge \phi) \rightarrow (\neg\varphi \vee \neg\phi)$$

where φ and ϕ are any formulas.([1])

There are many conditions which are sufficient to derive LEM in IZ. For example, it is well-known from Diaconescu's theorem that IZ combined with the axiom of choice implies LEM.[1] One can also get LEM by adding to IZ the axiom of foundation (or regularity) stating that $\forall x[\exists y \in x \rightarrow \exists y(y \in x \wedge \forall z(z \in y \rightarrow \neg(z \in x)))]$, i.e. every inhabited set has an $\in$-minimal element. As usual "inhabited" means "having at least one element".

In this article, we add another theorem of such type by asking what condition is needed to get LEM from WEM. Indeed, we show

THEOREM 1.1. *In IZ, LEM is equivalent to WEM plus the condition that every set not equal to $\emptyset$ is inhabited. In IZF, the same statement is true.*

2. Proof of Theorem 1.1

It's enough to show the case of IZ.

One direction of the proof is almost trivial. Suppose LEM holds. Then first WEM holds obviously. Let X be any set which is not $\emptyset$. Assuming to the contrary that there does not exist element in X. By extensionality, $X = \emptyset$. By employing LEM, this contradiction confirms that X is inhabited.

To prove the other direction, let ψ be any formula. We have to show $\psi \vee \neg\psi$. By separation schema, we set

$$A := \{x \in \{0\} | (x = 0) \wedge \psi\}.$$

We claim that $A \neq \emptyset \leftrightarrow \psi$. If ψ holds, then $A = \{0\}$ and hence $A \neq \emptyset$. If $A \neq \emptyset$, then there exists an element $x \in A$ by the assumption. By the definition of A, x must be 0 and ψ holds.

Now by invoking WEM,

$$\neg(A = \emptyset) \vee \neg\neg(A = \emptyset) = \psi \vee \neg\psi$$

holds. This completes the proof.

[1] In fact it is provable in IZ that LEM holds iff every doubleton has a choice function.

3. Discussion

In [2] Brouwer, the founder of the doctrine of intuitionistic mathematics, called WEM the simple principle of testability, saying that every assignment φ of a property to a mathematical entity can be tested, i.e. proved to be either non-contradictory $\neg\neg\varphi$ or absurd $\neg\varphi$. Jankov [3] proved that WEM axiomatizes the strongest super-intuitionistic logic having the same positive fragment as the intuitionistic propositional calculus. By our result, WEM plus an acceptable existential claim that every set not equal to $\emptyset$ is inhabited, can revive LEM and hence many important existence theorems of traditional mathematics such as the Bolzano?Weierstrass theorem, the extreme value theorem, the intermediate value theorem, etc.

Intuitionistic logic proves triple negation elimination

$$\neg\neg\neg\varphi \equiv \neg\varphi$$

so that one cannot further weaken WEM simply by adding more negation signs such as $\neg\neg\varphi \vee \neg\neg\neg\varphi$. For other characterizations and generalizations of WEM, the interested reader may refer to [4, 5] and the references therein.

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