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THE DIMENSION OF THE SPACE OF STABLE MAPS TO THE RELATIVE LAGRANGIAN GRASSMANNIAN OVER A CURVE

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Abstract. Let C be a smooth projective curve and W a symplectic bundle over C of degree w. Let $\pi : \mathbb{LG}(W) \to C$ be the relative Lagrangian Grassmannian over C and $\mathcal{S}_d(W)$ be the space of Lagrangian subbundles of degree $w-d$. Then Kontsevich's space $\overline{\mathcal{M}}_g(\mathbb{LG}(W), \beta_d)$ of stable maps to $\mathbb{LG}(W)$ is a compactification of $\mathcal{S}_d(W)$. In this article, we give an upper bound on the dimension of $\overline{\mathcal{M}}_g(\mathbb{LG}(W), \beta_d)$, which is an analogue of a result in [8] for the relative Lagrangian Grassmannian.

1. Backgrounds

1.1. Relative Lagrangian Grassmannian

Let C be a smooth projective algebraic curve of genus $g \geq 0$ over $\mathbb{C}$. For a line bundle L over C, an L-valued symplectic form is a nondegenerate skew-symmetric bilinear form $\omega : W \otimes W \to L$. A vector bundle W equipped with an L-valued symplectic form ω over C is called an *L-valued sympelctic* bundle, or simply a *symplectic* bundle. A subbundle E of W is called *isotropic* if $\omega|_{E\otimes E} = 0$. By linear algebra, a symplectic bundle has even rank $2n$ for some n and any isotropic subbundle has rank at most n. An isotropic subbundle of rank n is called a *Lagrangian subbundle*. Let w be the degree of W. Then we have $w = n\ell$, where ℓ denotes the degree of L. For details on symplectic vector bundles, see [1] and [2].

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For a symplecitc vector bundle W, let $\pi : \mathbb{LG}(W) \to C$ be the relative Lagrangian Grassmannian of W. Lagrangian subbundles E of W are in one to one correspondence with sections $s_E : C \to \mathbb{LG}(W)$ and give rise to quotient bundles $F_E := W/E$ of W, each of which fits into the short exact sequence

$$0 \to E \to W \to F_E \to 0. \tag{1.1}$$

1.2. Second (co)homology classes of $\mathbb{LG}(W)$

Let T_V be the vertical tangent bundle of $\mathbb{LG}(W)$ relative to the map π. For a section $s_E : C \to \mathbb{LG}(W)$, the pull-back of T_V along s_E is the bundle over C

$$s_E^*(T_V) = L \otimes \mathrm{Sym}^2 E^*.$$

Note that the determinant of the bundle $\left(\mathrm{Sym}^k E^*\right)$ is $(\det E^*)^{\otimes t}$, where $t = \binom{n+k-1}{n}$. Thus, if E has a degree e, then we have

$$\deg\left(s_E^*(T_V)\right) = \ -(n+1)e + \frac{n(n+1)}{2}\ell.$$

For convenience, we express $\deg\left(s_E^*(T_V)\right)$ in terms of the degree of the quotient bundle F_E. Let d be the degree of the quotient bundle F_E in (1.1), so that $d = n\ell - e$. Then we can write

$$\deg\left(s_E^*(T_V)\right) = \ (n+1)d - \frac{n(n+1)}{2}\ell.$$

The second cohomology group $H^2(\mathbb{LG}(W), \mathbb{Q})$ is generated by the first Chern class $c_1(T_V)$ and the class, say $\mathbb{F}$, of a fibre of π. Thus any homology class $\beta \in H_2(\mathbb{LG}(W))$ is determined by the pairings with these two cohomology classes

$$\int_\beta \mathbb{F} \quad \text{and} \quad \int_\beta c_1(T_V).$$

Assume that $\beta \in H_2(\mathbb{LG}(W); \mathbb{Q})$ is a class of a section, i.e., $\beta = (s_E)_*[C]$ for some section s_E. Then we have

$$\int_\beta \mathbb{F} = 1, \tag{1.2}$$

$$\int_\beta c_1(T_V) = \deg(s_E^* T_V) = (n+1)d - \frac{n(n+1)}{2}\ell. \tag{1.3}$$

We see that once β satisfies (1.2), a class of section β and the the degree of a quotient bundle F_E are interrelated with each other via (1.3), and so one gives rise to the other.

1.3. Compactification of the space of sections of π

For an integer d, let $\mathcal{S}_d(W)$ be the space of Lagrangian subbundles E of W of degree $n\ell - d$ or quotient bundles F_E of degree d. It is known that $\mathcal{S}_d(W)$ is a quasi-projective variety. From the point of view of sections, we can compactify $\mathcal{S}_d(W)$. There are two popular compactifications of $\mathcal{S}_d(W)$. One is the Lagrangian Quot scheme of Lagrangian subsheaves of W ([6] and [3]), and the other is Kontsevich's space of stable maps to $\mathbb{LG}(W)$ ([4] and [5]). In general, in both cases, boundary points, i.e., newly added points are unwieldy. For instances, some components may have larger dimension than expected, or may consist only of boundary points. In this paper, we give an upper bound on the dimension of the Kontsevich's stable map compactification, following a result of Popa and Roth for the relative (ordinary) Grassmannian [8]. For a topology of Lagrangian Quot scheme, we refer the reader to [3].

1.4. Stable maps to $\mathbb{LG}(W)$

Let X be a smooth projective variety. For $\beta \in H_2(X, \mathbb{Q})$ and a non-negative integer m, the Kontsevich's space $\overline{\mathcal{M}}_{g,m}(X, \beta)$ consists of isomorphism classes of maps $f : C' \to X$, subject to

1. C' is a curve of arithmetic genus g with m markings,
2. $f_*[C'] = \beta$,
3. f satisfies the following stability condition:
   (a) If a rational component of C' collapses to a point by f, then it must contain at least three special points, i.e., markings or (and) nodes.
   (b) If a component of C' with the arithmetic genus one collapses to a point, then it must contain at least one special point.

Here an isomorphism between two maps is obviously defined. We shall write $[f]$ for the isomorphism class containing $f : C' \to X$, and also refer to $f : C' \to X$ as a *reducible* (resp. *irreducible*) map if the domain C' is reducible (resp. irreducible).

Let $\mathcal{M}_{g,m}(X, \beta)$ be the sublocus of $\overline{\mathcal{M}}_{g,m}(X, \beta)$ consisting of isomorphism classes of irreducible maps f. Originally, $\overline{\mathcal{M}}_{g,m}(X, \beta)$ have been constructed as a compactification of $\mathcal{M}_{g,m}(X, \beta)$. When $m = 0$, we shall write $\overline{\mathcal{M}}_g(X, \beta)$ and $\mathcal{M}_g(X, \beta)$ for $\overline{\mathcal{M}}_{g,0}(X, \beta)$ and $\mathcal{M}_{g,0}(X, \beta)$, respectively. More details on $\overline{\mathcal{M}}_{g,m}(X, \beta)$ can be found in [4] and [5].

1.5. Description of elements of $\overline{\mathcal{M}}_g(\mathbb{LG}(W), \beta_d)$

Now we describe elements of $\overline{\mathcal{M}}_g(\mathbb{LG}(W), \beta_d)$. The fact that β_d is a class of section and the genus condition impose a strong condition on C' and f. By the equality (1.2) with $\beta = [f_*C']$, $f(C')$ intersects a general fiber of π exactly at one point, and so there must be exactly one special component C_0 of C' which is isomorphically sent to C by $\pi \circ f$. This implies that the image curve $f(C_0)$ forms a section of π, and hence the component C_0 is a smooth projective curve of genus g. Other components of C' can have genus zero by the genus condition, and so form trees of rational curves, hanging off of C_0. Write $T_1, \dots, T_k$ for these trees, so that $C' = C_0 \cup \left(\bigcup_{i=1}^k T_i\right)$. By the condition (1.2), each T_i is sent to $\mathbb{LG}(W_{p_i})$ for some $p_i \in C$. Furthermore, in each tree T_i, there is a rational curve with exactly one node, and hence, by the stability condition, T_i cannot be completely collapsed to a point (in $\mathbb{LG}(W_{p_i})$). Thus $f_*[T_i] \neq 0$ in $H_2(\mathbb{LG}(W))$. Write $\alpha_i := f_*[T_i]$ for $i = 1, \dots, k$. Then α_i is characterized by

$$\int_{\alpha_i} \mathbb{F} \text{ and } \int_{\alpha_i} c_1(T_V).$$

Let us compute these parings. The first one has value 0 since T_i is sent to the fiber $\mathbb{LG}(W_{p_i})$. For the second pairing, note that the restriction of the vertical tangent bundle T_V to $\mathbb{LG}(W_{p_i})$ is the tangent bundle $\mathscr{T}_i := T\mathbb{LG}(W_{p_i})$ of the Lagrangian Grassmannian $\mathbb{LG}(W_{p_i})$. Thus we have

$$\int_{\alpha_i} c_1(T_V) = \int_{\alpha_i} c_1(\mathscr{T}_i).$$

On the other hand, since the determinant of the bundle $\mathscr{T}_i$ is ample and $f_*[T_i]$ is an effective class, the pairing $\int_{\alpha_i} c_1(\mathscr{T}_i)$ is a positive number. More precisely, applying (1.3) to the new trivial symplectic bundle $W = W_{p_i} \times C$, we have

$$\int_{\alpha_i} c_1(T_V) = \int_{\alpha_i} c_1(\mathscr{T}_i) = (n+1)d_i$$

for some positive $d_i < d$. Recall from (1.3) that $\int_{f_*[C']} c_1(T_V) = (n+1)d - \frac{n(n+1)}{2}\ell$. Since $f_*[C'] = f_*[C_0] + \sum_i^k f_*[T_i]$, letting $d_0 := d - \sum_{i=1}^k d_i$, we have

$$\int_{f_*[C_0]} c_1(T_V) = (n+1)d_0 - \frac{n(n+1)}{2}\ell.$$

Thus $f_*[C_0]$ is the class of section β_{d_0}. This implies that if $[f]$ is a point of $\overline{\mathcal{M}}_g(\mathbb{LG}(W), \beta_d)$ with C' reducible, then the quotient bundle corresponding to the section $f(C_0)$ of π has degree d_0 with $0 < d_0 < d$.

Recall that the tangent bundle of the Lagrangian Grassmannian (in fact, a homogeneous manifold) is generated by global sections([7]). Thus, as in the Grassmannian ([8]), nodes connecting two rational curves can be smoothed. This implies that the general point of any component of $\overline{\mathcal{M}}_g(\mathbb{LG}(W), \beta_d)$ corresponds to a map with a domain $C' = C_0 \cup \left(\bigcup_{i=1}^k C_i\right)$ for some $k \geq 0$, where C_0 is a special component of C^prime and each C_i a rational curve hanging off of C_0 such that $f_*[C_i] \neq 0$ in $H_2(\mathbb{LG}(W))$. Other points in the component correspond to maps with a domain $C' = C_0 \cup \left(\bigcup_{i=1}^k T_i\right)$, where T_i is a degeneration of C_i.

As explained above, all points in $\mathcal{M}_g(\mathbb{LG}(W), \beta_d)$ correspond to a map f with $C' = C_0$. Since C_0 is isomorphic to C, we can identify

$$\mathcal{S}_d(W) = \mathcal{M}_g(\mathbb{LG}(W), \beta_d).$$

Thus, the space $\overline{\mathcal{M}}_g(\mathbb{LG}(W), \beta_\beta)$ is treated as a compactification of $\mathcal{S}_d(W)$.

2. Main theorem and its proof

Let

$$d_n := \min\{\deg\, F \mid F = W/E \text{ for a Lagrangian bundle } E\}.$$

We obtain an upper bound on the dimesion of $\overline{\mathcal{M}}_g(\mathbb{LG}(W), \beta_d)$, as an analogue of Theorem 3.1 in [8].

THEOREM 2.1. *For an integer $d \geq d_n$, we have*

$$\dim \overline{\mathcal{M}}_g(\mathbb{LG}(W), \beta_d) \leq \frac{n(n+1)}{2} + (d - d_n)(n+1). \tag{2.1}$$

To prove Theorem 2.1, we need the following lemma whose proof can be copied from that of Lemma 3.1 in [8].

LEMMA 2.2. *If M is an irreducible component of $\overline{\mathcal{M}}_g(\mathbb{LG}(W), \beta_d)$ whose generic point corresponds to an irreducible map, then one of the following is true.*

- $\dim M \leq \frac{n(n+1)}{2}$,
- *there is a (nonempty) divisor $D \subset M$ which consists of maps with a reducible domain.*

Proof of Theorem 2.1. The proof is an adaptation of Theorem 3.1 of [8]. We use induction on d. First, as a base case of the induction, we consider the case $d = d_n$. For this case, we claim that each irreducible component M of $\overline{\mathcal{M}}_g(\mathbb{LG}(W), \beta_{d_n})$ consists only of (isomorphism classes of) irreducible maps. Indeed, if there is an irreducible component M that contains a reducible map f, then f gives rise to a section $f(C_0)$ of π. By the description in Subsection 1.5, the degree d_0 corresponding to the section $f(C_0)$ is strictly less than d_n But this is impossible by minimality of d_n. Thus each component of $\overline{\mathcal{M}}_g(\mathbb{LG}(W), \beta_{d_n})$ consists only of irreducible maps.

Now let d be an integer with $d > d_n$. Assume that the theorem holds for all d' with $d' < d$. Let M be any irreducible component of $\overline{\mathcal{M}}_g(\mathbb{LG}(W), \beta_d)$. It is enough to show

$$\dim M \le \frac{n(n+1)}{2} + (d - d_n)(n+1).$$

We treat two cases separately.

Case 1: Generic point of M corresponds to an irreducible map.
In this case, if $\dim M \le \frac{n(n+1)}{2}$, then there is nothing to prove. Thus we assume $\dim M > \frac{n(n+1)}{2}$. Then by Lemma 2.2, there is a (nonempty) divisor $Z \subset M$ which consists of irreducible maps. Let Z' be an irreducible component of Z (so that $\dim Z' = \dim Z = \dim M - 1$). Then if $[f]$ is a point of Z', then f has a domain $C' = C_0 \cup \left(\bigcup_{i=1}^k T_i\right)$ for trees T_i of rational curves for $i = 1, \ldots, k$. Note that the number k and all the homology classes $\alpha_i = f_*[T_i]$ for $i = 1, \ldots, k$ do not depend on points $[f]$ in Z'. Recall that $\int_{\alpha_i} c_1(T_V) = (n+1)d_i$ for some integer $0 < d_i < d$. For induction, we consider two spaces with evaluation maps ev at markings.

$$ev : \overline{\mathcal{M}}_{g,1}(\mathbb{LG}(W), \beta_{d-d_k}) \to \mathbb{LG}(W)$$

and

$$ev : \overline{\mathcal{M}}_{0,1}(\mathbb{LG}(W), \alpha_k) \to \mathbb{LG}(W).$$

We have a natural morphism Φ from the fibre product of these spaces taken along the evaluations ev

$$\Phi : \overline{\mathcal{M}}_{g,1}(\mathbb{LG}(W), \beta_{d-d_k}) \times_{\mathbb{LG}(W)} \overline{\mathcal{M}}_{0,1}(\mathbb{LG}(W), \alpha_k) \to \overline{\mathcal{M}}_g(\mathbb{LG}(W), \beta_d)$$

Note that Φ functions to gluing together two maps whose markings are sent to the same point of $\mathbb{LG}(W)$. We can easily see that Z' is in the image of Φ and hence the dimension of Z' is bounded above by the dimension of the fibre product. Recall from [5] that we have

$$\dim \overline{\mathcal{M}}_{0,1}(\mathbb{LG}(W), \alpha_k) = (n+1)d_k + \frac{n(n+1)}{2} - 1$$

and

$$\dim \overline{\mathcal{M}}_{g,1}(\mathbb{LG}(W), \beta_{d-d_k}) = \dim \overline{\mathcal{M}}_g(\mathbb{LG}(W), \beta_{d-d_k}) + 1.$$

Then, the dimension of the fibre product is

$$\big(\dim \overline{\mathcal{M}}_g(\mathbb{LG}(W), \beta_{d-d_k}) + 1\big) + \left((n+1)d_k + \frac{n(n+1)}{2} - 1\right) - \left(\frac{n(n+1)}{2} + 1\right).$$

Thus we have

$$\dim Z' \leq \dim \overline{\mathcal{M}}_g(\mathbb{LG}(W), \beta_{d-d_k}) + (n+1)d_k - 1. \tag{2.2}$$

By induction hypothesis, we obtain

$$\dim Z' \leq \frac{n(n+1)}{2} + (d - d_n)(n+1) - 1,$$

and hence

$$\dim M = \dim Z' + 1 \leq \frac{n(n+1)}{2} + (d - d_n)(n+1).$$

Case 2: A generic point of M corresponds to a reducible map. Assume that the general point of M corresponds to a map f with reducible domain C' such that $C' = C_0 \cup \left(\bigcup_{i=1}^k C_i\right)$. Writing $\alpha_i := f_*[C_i]$, we may think of M as Z' (for Z' in Case 1). Thus, by (2.2) of Case 1, we have

$$\dim M \leq \dim \overline{\mathcal{M}}_g(\mathbb{LG}(W), \beta_{d-d_k}) + (n+1)d_k - 1.$$

Repeating the same argument as in Case 1 for each of the other $(k-1)$ trees $T_{k-1}, \dots, T_1$, we obtain

$$\dim M \leq \frac{n(n+1)}{2} + (d - d_n)(n+1) - k.$$

This completes the proof of theorem.

$\square$

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