



GRAPH CONVERGENCE AND GENERALIZED CAYLEY OPERATOR WITH AN APPLICATION TO A SYSTEM OF CAYLEY INCLUSIONS IN SEMI-INNER PRODUCT SPACES

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Abstract. In this paper, we introduce and study a generalized Cayley operator associated to $H(\cdot, \cdot)$ -monotone operator in semi-inner product spaces. Using the notion of graph convergence, we give the equivalence result between graph convergence and convergence of generalized Cayley operator for the $H(\cdot, \cdot)$ -monotone operator without using the convergence of the associated resolvent operator. To support our claim, we construct a numerical example. As an application, we consider a system of generalized Cayley inclusions involving $H(\cdot, \cdot)$ -monotone operators and give the existence and uniqueness of the solution for this system. Finally, we propose a perturbed iterative algorithm for finding the approximate solution and discuss the convergence of iterative sequences generated by the perturbed iterative algorithm.

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1. INTRODUCTION

Variational inclusions, as the generalization of variational inequalities, have been widely studied in recent years because of their wide applications in optimization and control, economic and transportation equilibrium, and engineering sciences. It is well known that monotonicity and accretivity of the underlying mappings plays an important role in the theory and algorithms of variational inclusions. Many researchers investigated several classes of variational inclusions through generalized monotone and accretive operators and their corresponding resolvent operators, see for example [6, 7, 8, 10, 11, 12, 14, 15, 23, 24, 25, 28].

In 2011, Li and Huang [17] studied the graph convergence for $H(\cdot, \cdot)$ -accretive mappings and showed the equivalence between graph convergence and proximal-point mapping convergence for the $H(\cdot, \cdot)$ -accretive mappings in a Banach space. Since then many authors studied graph convergence for maximal monotone mappings and established the equivalence between the graph convergence and the proximal-point mapping convergence in different spaces, see for example [1, 9, 16, 21, 22, 26]. They extended the concept of graph convergence introduced and considered by Attouch [5]. Very recently, Ahmad *et al.* [3] and Akram *et al.* [4] generalized the notion of graph convergence to Yosida approximation operators to solve a new type of variational inclusions known as Yosida inclusions.

It is well known that most of the results and techniques in Hilbert spaces are developed in terms of the inner product space structure. On the other hand, this type of notion finds no real parallel in the general Banach space setting. With the aim of carrying over Hilbert space type arguments to the theory of Banach spaces, Lumer [18] defined a new type of inner product known as semi-inner product with a more general axiom system than that of Hilbert space. It is pertinent to mention that a semi-inner product provides one with sufficient structure to obtain certain nontrivial general results.

Motivated and inspired by the above research work and knowing the importance of Cayley operator to approximate the solution of variational inclusion problems, in this paper, we introduce and study a generalized Cayley operator associated to $H(\cdot, \cdot)$ -monotone operator in semi-inner product spaces. Using the notion of graph convergence, we establish an equivalence result between graph convergence and convergence of generalized Cayley operator for the $H(\cdot, \cdot)$ -monotone operators without using the convergence of the associated resolvent operator. In support of our claim, we construct a numerical example. As an application, we consider a system of generalized Cayley inclusions involving the $H(\cdot, \cdot)$ -monotone operators and give the existence and uniqueness of the of solution for this system. Finally, we propose a perturbed iterative

algorithm for finding the approximate solution and discuss the convergence of iterative sequences generated by the perturbed iterative algorithm. Our results refine, unify and generalize some known results in literature.

First we afford a view of semi-inner product and its important features that we use in our work.

Definition 1.1. ([18]) Let X be a vector space over the field \mathbb{F} of real or complex numbers. A functional $[\cdot, \cdot] : X \times X \rightarrow \mathbb{F}$ is called a semi-inner product if it satisfies the following conditions:

- (i) $[x + y, z] = [x, z] + [y, z]$, $\forall x, y, z \in X$;
- (ii) $[\lambda x, y] = \lambda[x, y]$, $\forall \lambda \in \mathbb{F}$ and $x, y \in X$;
- (iii) $[x, x] > 0$, for $x \neq 0$;
- (iv) $|[x, y]|^2 \leq [x, x][y, y]$.

The pair $(X, [\cdot, \cdot])$ is said to be a semi-inner product space.

Every semi-inner product space X is a normed linear space with the norm $\|x\| = [x, x]^{\frac{1}{2}}$. On the other hand, in a normed linear space, we can define semi-inner product in infinitely many ways. In 1967, Giles [13] had proved that if the underlying space X is a uniformly convex smooth Banach space, then it is possible to define a semi-inner product uniquely.

The sequence space $\ell^p, p > 1$ and the function space $L^p, p > 1$ are uniformly convex smooth Banach spaces. So we can define a semi-inner product on these spaces uniquely.

Example 1.2. ([20]) The real sequence space ℓ^p for $1 < p < \infty$ is a semi-inner product space with the semi-inner product defined by

$$[x, y] = \frac{1}{\|y\|_p^{p-2}} \sum_i x_i y_i |y_i|^{p-2}, \quad x, y \in \ell^p.$$

Example 1.3. ([13]) The real Banach space $L^p(\mathbb{R})$ for $1 < p < \infty$ is a semi-inner product space with the semi-inner product defined by

$$[f, g] = \frac{1}{\|g\|_p^{p-2}} \int_{\mathbb{R}} f(t) |g(t)|^{p-1} \operatorname{sgn}(g(t)) dt, \quad f, g \in L^p(\mathbb{R}).$$

Definition 1.4. ([27]) Let X be a real Banach space. The *modulus of smoothness* of X is the function $\rho_X : [0, +\infty) \rightarrow [0, +\infty)$ defined by

$$\rho_X(t) = \sup \left\{ \frac{\|x + y\| + \|x - y\|}{2} - 1 : x, y \in X, \|x\| = 1, \|y\| = t, t > 0 \right\}.$$

- (i) X is said to be *uniformly smooth*, if $\lim_{t \rightarrow 0} \frac{\rho_X(t)}{t} = 0$.

- (ii) X is said to be q -uniformly smooth, if there exists a positive real constant c such that

$$\rho_X(t) \leq ct^q, \quad q > 1.$$

- (iii) X is said to be 2-uniformly smooth, if there exists a positive real constant c such that

$$\rho_X(t) \leq ct^2.$$

Lemma 1.5. ([27]) *Let X be a smooth Banach space. Then the following statements are equivalent:*

- (i) X is 2-uniformly smooth;
(ii) There is a constant $c > 0$ such that

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, f_x \rangle + c\|y\|^2, \quad \forall x, y \in X, \quad (1.1)$$

where $f_x \in J(x)$ and $J(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 \text{ and } \|x^*\| = \|x\|\}$, is the normalized duality mapping.

Remark 1.6. Every normed linear space X is a semi-inner product space [18]. In fact, by Hahn-Banach theorem, for each $x \in X$, there exists at least one functional $f_x \in X^*$ such that $\langle x, f_x \rangle = \|x\|^2$. Given any such mapping f from X into X^* , it has been show that $[y, x] = \langle y, f_x \rangle$ defines a semi-inner product. Hence inequality (1.1) can be written as

$$\|x + y\|^2 \leq \|x\|^2 + 2[y, x] + c\|y\|^2, \quad \forall x, y \in X. \quad (1.2)$$

The constant c is called a constant of smoothness of X and is chosen with best possible minimum value.

Example 1.7. The function space L^p is 2-uniformly smooth for $p \geq 2$ and it is p -uniformly smooth for $1 < p < 2$. If $2 \leq p < \infty$, then we have for all $x, y \in L^p$,

$$\|x + y\|^2 \leq \|x\|^2 + 2[y, x] + (p - 1)\|y\|^2,$$

where $(p - 1)$ is the constant of smoothness of L^p .

2. PRELIMINARIES

Throughout the paper, unless otherwise stated, we assume that X is a real 2-uniformly smooth space equipped with norm $\|\cdot\|$ and semi-inner product $[\cdot, \cdot]$, 2^X is the power set of X .

First, we recall some known definitions and results which are important to achieve the goal of this paper.

Definition 2.1. Let $A, B, T : X \rightarrow X$ and $H : X \times X \rightarrow X$ be single-valued mappings. Then

(i) T is said to be *monotone*, if

$$[Tx - Ty, x - y] \geq 0, \forall x, y \in X;$$

- (ii) T is said to be *strictly monotone*, if it is monotone and equality holds if and only if $x = y$;
 (iii) T is said to be *r -strongly monotone*, if there exists a constant $r > 0$ such that

$$[Tx - Ty, x - y] \geq r\|x - y\|^2, \forall x, y \in X;$$

(iv) T is said to be *m -relaxed monotone*, if there exists a constant $m > 0$ such that

$$[Tx - Ty, x - y] \geq (-m)\|x - y\|^2, \forall x, y \in X;$$

(v) T is said to be *s -Lipschitz continuous*, if there exists a constant $s > 0$ such that

$$\|Tx - Ty\| \leq s\|x - y\|, \forall x, y \in X;$$

(vi) $H(A, \cdot)$ is said to be *α -strongly monotone*, if there exists a constant $\alpha > 0$ such that

$$[H(Ax, u) - H(Ay, u), x - y] \geq \alpha\|x - y\|^2, \forall x, y, u \in X;$$

(vii) $H(\cdot, B)$ is said to be *β -relaxed monotone*, if there exists a constant $\beta > 0$ such that

$$[H(u, Bx) - H(u, By), x - y] \geq -\beta\|x - y\|^2, \forall x, y, u \in X;$$

(viii) $H(A, B)$ is said to be *$\alpha\beta$ -symmetric monotone*, if $H(A, \cdot)$ is α -strongly monotone and $H(\cdot, B)$ is β -relaxed monotone with $\alpha \geq \beta$ and $\alpha = \beta$ if and only if $x = y$, for all $x, y \in X$;

(ix) $H(A, \cdot)$ is said to be *τ_1 -Lipschitz continuous*, if there exists a constant $\tau_1 > 0$ such that

$$\|H(Ax, u) - H(Ay, u)\| \leq \tau_1\|x - y\|, \forall x, y, u \in X;$$

(x) $H(\cdot, B)$ is said to be *τ_2 -Lipschitz continuous*, if there exists a constant $\tau_2 > 0$ such that

$$\|H(u, Bx) - H(u, By)\| \leq \tau_2\|x - y\|, \forall x, y, u \in X.$$

Definition 2.2. A set-valued mapping $M : X \rightarrow 2^X$ is said to be

(i) *monotone*, if

$$[u - v, x - y] \geq 0, \forall x, y \in X, u \in M(x), v \in M(y);$$

(ii) *r -strongly monotone*, if there exists a constant $r > 0$ such that

$$[u - v, x - y] \geq r\|x - y\|^2, \forall x, y \in X, u \in M(x), v \in M(y);$$

(iii) *m-relaxed monotone*, if there exists a constant $m > 0$ such that

$$[u - v, x - y] \geq (-m)\|x - y\|^2, \quad \forall x, y \in X, u \in M(x), v \in M(y).$$

Definition 2.3. Let $A, B : X \rightarrow X, H : X \times X \rightarrow X$ be single-valued mappings. A set-valued mapping $M : X \rightarrow 2^X$ is said to be $H(\cdot, \cdot)$ -monotone with respect to A and B (or simply $H(\cdot, \cdot)$ -monotone in the sequel), if M is monotone and $(H(A, B) + \lambda M)(X) = X$, for all $\lambda > 0$.

Remark 2.4. If X is a Banach space, then the definition of $H(\cdot, \cdot)$ -monotonicity reduces to the definition of $H(\cdot, \cdot)$ -accretivity considered in [3, 17].

Lemma 2.5. Let $A, B : X \rightarrow X, H : X \times X \rightarrow X$ be single-valued mappings such that $H(A, B)$ is $\alpha\beta$ -symmetric monotone and $M : X \rightarrow 2^X$ be an $H(\cdot, \cdot)$ -monotone operator. Then the operator $(H(A, B) + \lambda M)^{-1}$ is single-valued.

Based on Lemma 2.5, we define the generalized resolvent operator for $H(\cdot, \cdot)$ -monotone operator M as follows:

Definition 2.6. Let $A, B : X \rightarrow X, H : X \times X \rightarrow X$ be single-valued mappings such that $H(A, B)$ is $\alpha\beta$ -symmetric monotone and $M : X \rightarrow 2^X$ be an $H(\cdot, \cdot)$ -monotone operator. Then the generalized resolvent operator $R_{M,\lambda}^{H(\cdot, \cdot)} : X \rightarrow X$ is defined by

$$R_{M,\lambda}^{H(\cdot, \cdot)}(x) = [(H(A, B) + \lambda M)^{-1}(x)], \quad \forall x \in X. \quad (2.1)$$

Definition 2.7. The generalized Cayley operator $C_{M,\lambda}^{H(\cdot, \cdot)} : X \rightarrow X$ associated with $H(\cdot, \cdot)$ -monotone operator M is defined as:

$$C_{M,\lambda}^{H(\cdot, \cdot)}(x) = [2R_{M,\lambda}^{H(\cdot, \cdot)} - I](x), \quad \forall x \in X \text{ and } \lambda > 0, \quad (2.2)$$

where $R_{M,\lambda}^{H(\cdot, \cdot)}$ is the generalized resolvent operator defined by (2.1).

Remark 2.8. The resolvent and the Cayley operators are connected by means of the following relation

$$C_{M,\lambda}^{H(\cdot, \cdot)}(x) \in [2I - (H(A, B) + \lambda M)] R_{M,\lambda}^{H(\cdot, \cdot)}(x).$$

The following lemmas give the Lipschitz continuity of generalized resolvent and generalized Cayley operators. For the sake of brevity, we omit the proofs.

Lemma 2.9. Let $H(A, B)$ be $\alpha\beta$ -symmetric monotone. Then, the generalized resolvent operator $R_{M,\lambda}^{H(\cdot, \cdot)} : X \rightarrow X$ is $\frac{1}{\alpha-\beta}$ -Lipschitz continuous, that is,

$$\left\| R_{M,\lambda}^{H(\cdot, \cdot)}(x) - R_{M,\lambda}^{H(\cdot, \cdot)}(y) \right\| \leq \frac{1}{\alpha - \beta} \|x - y\|, \quad \forall x, y \in X.$$

Lemma 2.10. *Let $H(A, B)$ be $\alpha\beta$ -symmetric monotone. Then, the generalized Cayley operator $C_{M,\lambda}^{H(\cdot,\cdot)} : X \rightarrow X$ is θ -Lipschitz continuous, that is,*

$$\left\| C_{M,\lambda}^{H(\cdot,\cdot)}(x) - C_{M,\lambda}^{H(\cdot,\cdot)}(y) \right\| \leq \theta \|x - y\|, \quad \forall x, y \in X,$$

where $\theta = \frac{2+(\alpha-\beta)}{\alpha-\beta}$.

Lemma 2.11. *Let $\{c_n\}$ and $\{t_n\}$ be two non-negative real sequences satisfying*

$$c_{n+1} \leq dc_n + t_n,$$

with $0 < d < 1$ and $t_n \rightarrow 0$. Then $\lim_{n \rightarrow \infty} c_n = 0$.

3. GRAPH CONVERGENCE FOR $H(\cdot, \cdot)$ -MONOTONE OPERATORS

Let $M : X \rightarrow 2^X$ be a set-valued mapping. The graph of the mapping M is defined by

$$\text{graph}(M) = \{(x, y) \in X \times X : y \in M(x)\}.$$

Definition 3.1. ([17]) Let $A, B : X \rightarrow X$ and $H : X \times X \rightarrow X$ be single-valued mappings. Let $M_n, M : X \rightarrow 2^X$ be $H(\cdot, \cdot)$ -monotone set-valued operators for $n = 0, 1, 2, \dots$. The sequence $\{M_n\}$ is said to be graph convergent to M , denoted by $M_n \xrightarrow{G} M$, if for every $(x, y) \in \text{graph}(M)$, there exist sequences $\{x_n\}, \{y_n\} \subset X$ with $(x_n, y_n) \in \text{graph}(M_n)$ such that

$$x_n \rightarrow x, \quad y_n \rightarrow y \quad \text{as } n \rightarrow \infty.$$

Theorem 3.2. ([17]) *Let $M_n, M : X \rightarrow 2^X$ be $H(\cdot, \cdot)$ -monotone operators for $n = 0, 1, 2, \dots$. Assume that $H : X \times X \rightarrow X$ is a single-valued mapping such that*

- (i) $H(A, B)$ is $\alpha\beta$ -symmetric monotone with $\alpha > \beta$;
- (ii) $H(A, \cdot)$ is τ_1 -Lipschitz continuous and $H(\cdot, B)$ is τ_2 -Lipschitz continuous.

Then $M_n \xrightarrow{G} M$ if and only if $R_{M_n,\lambda}^{H(\cdot,\cdot)}(x) \rightarrow R_{M,\lambda}^{H(\cdot,\cdot)}(x)$, where $R_{M_n,\lambda}^{H(\cdot,\cdot)}(x) = [(H(A, B) + \lambda M_n)]^{-1}(x)$.

Next, we prove an equivalence result between graph convergence and convergence of generalized Cayley operator for the $H(\cdot, \cdot)$ -monotone operator without using convergence of the associated resolvent operator.

Theorem 3.3. *Let $M_n, M : X \rightarrow 2^X$ be $H(\cdot, \cdot)$ -monotone operators for $n = 0, 1, 2, \dots$ and $H : X \times X \rightarrow X$ be a single-valued mapping such that assumptions (i) and (ii) of Theorem 3.2 hold. Then $M_n \xrightarrow{G} M$ if and only if $C_{M_n,\lambda}^{H(\cdot,\cdot)}(x) \rightarrow C_{M,\lambda}^{H(\cdot,\cdot)}(x)$, where $C_{M_n,\lambda}^{H(\cdot,\cdot)}(x) = [2R_{M_n,\lambda}^{H(\cdot,\cdot)} - I](x)$.*

Proof. Suppose $M_n \xrightarrow{G} M$. For any given $x \in X$, let

$$z_n = C_{M_n, \lambda}^{H(\cdot, \cdot)}(x) \quad \text{and} \quad z = C_{M, \lambda}^{H(\cdot, \cdot)}(x).$$

Now, $z = C_{M, \lambda}^{H(\cdot, \cdot)}(x) = [2R_{M, \lambda}^{H(\cdot, \cdot)} - I](x)$. It follows that

$$\mathbf{u} = \frac{x + z}{2} = R_{M, \lambda}^{H(\cdot, \cdot)}(x) = [H(A, B) + \lambda M]^{-1}(x),$$

which implies that

$$\frac{1}{\lambda} [x - H(A\mathbf{u}, B\mathbf{u})] \in M(\mathbf{u}).$$

That is

$$\left(\mathbf{u}, \frac{1}{\lambda} [x - H(A\mathbf{u}, B\mathbf{u})] \right) \in \text{graph}(M).$$

By definition of graph convergence there exist $(w_n, y_n) \in \text{graph}(M_n)$, such that

$$w_n \rightarrow \mathbf{u}, \quad y_n \rightarrow \frac{1}{\lambda} [x - H(A\mathbf{u}, B\mathbf{u})]. \quad (3.1)$$

Since $y_n \in M_n(w_n)$, we can write

$$H(Aw_n, Bw_n) + \lambda y_n \in [H(A, B) + \lambda M_n](w_n).$$

It follows that

$$\begin{aligned} w_n &= R_{M_n, \lambda}^{H(\cdot, \cdot)} [H(Aw_n, Bw_n) + \lambda y_n] \\ &= \frac{1}{2} \left(I + C_{M_n, \lambda}^{H(\cdot, \cdot)} \right) [H(Aw_n, Bw_n) + \lambda y_n], \end{aligned}$$

which implies that

$$2w_n = H(Aw_n, Bw_n) + \lambda y_n + C_{M_n, \lambda}^{H(\cdot, \cdot)} [H(Aw_n, Bw_n) + \lambda y_n]. \quad (3.2)$$

Note that

$$\begin{aligned} \|z_n - z\| &= \|C_{M_n, \lambda}^{H(\cdot, \cdot)}(x) - 2w_n + 2w_n - z\| \\ &= \|C_{M_n, \lambda}^{H(\cdot, \cdot)}(x) - H(Aw_n, Bw_n) - \lambda y_n \\ &\quad - C_{M_n, \lambda}^{H(\cdot, \cdot)} [H(Aw_n, Bw_n) + \lambda y_n] + 2w_n - z\| \\ &\leq \|C_{M_n, \lambda}^{H(\cdot, \cdot)}(x) - C_{M_n, \lambda}^{H(\cdot, \cdot)} [H(Aw_n, Bw_n) + \lambda y_n]\| \\ &\quad + \|x - H(Aw_n, Bw_n) - \lambda y_n + 2w_n - x - z\|. \end{aligned}$$

Using the Lipschitz continuity of the Cayley operator, we have

$$\begin{aligned}
\|z_n - z\| &\leq \theta \|x - H(Aw_n, Bw_n) - \lambda y_n\| + \|x - H(Aw_n, Bw_n) - \lambda y_n\| \\
&\quad + \|2w_n - x - z\| \\
&= (\theta + 1) \|x - H(Aw_n, Bw_n) - \lambda y_n\| + \|2w_n - x - z\| \\
&= (\theta + 1) \|x - H(Aw_n, Bw_n) + H(\mathbf{A}\mathbf{u}, B\mathbf{u}) - H(\mathbf{A}\mathbf{u}, B\mathbf{u}) - \lambda y_n\| \\
&\quad + \|2w_n - x - z\| \\
&\leq (\theta + 1) \|x - H(\mathbf{A}\mathbf{u}, B\mathbf{u}) - \lambda y_n\| \\
&\quad + (\theta + 1) \|H(\mathbf{A}\mathbf{u}, B\mathbf{u}) - H(Aw_n, Bw_n)\| + \|2w_n - x - z\|. \quad (3.3)
\end{aligned}$$

Since H is τ_1 -Lipschitz continuous with respect to A and τ_2 -Lipschitz continuous with respect to B , we have

$$\begin{aligned}
\|H(\mathbf{A}\mathbf{u}, B\mathbf{u}) - H(Aw_n, Bw_n)\| &\leq \|H(\mathbf{A}\mathbf{u}, B\mathbf{u}) - H(Aw_n, B\mathbf{u})\| \\
&\quad + \|H(Aw_n, B\mathbf{u}) - H(Aw_n, Bw_n)\| \\
&\leq (\tau_1 + \tau_2) \|\mathbf{u} - w_n\|. \quad (3.4)
\end{aligned}$$

Thus it follows from (3.3) and (3.4) that

$$\begin{aligned}
\|z_n - z\| &\leq (\theta + 1) \|x - H(\mathbf{A}\mathbf{u}, B\mathbf{u}) - \lambda y_n\| \\
&\quad + (\theta + 1)(\tau_1 + \tau_2) \|\mathbf{u} - w_n\| + \|2w_n - x - z\| \\
&\leq (\theta + 1) \|x - H(\mathbf{A}\mathbf{u}, B\mathbf{u}) - \lambda y_n\| \\
&\quad + ((\theta + 1)(\tau_1 + \tau_2) + 2) \|\mathbf{u} - w_n\|. \quad (3.5)
\end{aligned}$$

In view of (3.1), it follows that

$$\|w_n - \mathbf{u}\| \rightarrow 0 \quad \text{and} \quad \|x - H(\mathbf{A}\mathbf{u}, B\mathbf{u}) - \lambda y_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Thus, it follows from (3.5) that $\|z_n - z\| \rightarrow 0$ as $n \rightarrow \infty$, which implies that

$$C_{M_n, \lambda}^{H(\cdot, \cdot)}(x) \longrightarrow C_{M, \lambda}^{H(\cdot, \cdot)}(x).$$

Conversely, suppose that

$$C_{M_n, \lambda}^{H(\cdot, \cdot)}(x) \longrightarrow C_{M, \lambda}^{H(\cdot, \cdot)}(x), \quad \forall x \in X, \quad \lambda > 0.$$

For any $(x, y) \in \text{graph}(M)$, where $y \in M(x)$ and hence

$$H(Ax, Bx) + \lambda y \in [H(A, B) + \lambda M](x).$$

Therefore,

$$x = R_{M, \lambda}^{H(\cdot, \cdot)} [H(Ax, Bx) + \lambda y] = \frac{1}{2} (C_{M, \lambda}^{H(\cdot, \cdot)} + I) [H(Ax, Bx) + \lambda y].$$

Let $x_n = \frac{1}{2}(C_{M_n, \lambda}^{H(\cdot, \cdot)} + I) [H(Ax, Bx) + \lambda y]$. This implies that

$$\frac{1}{\lambda} [H(Ax, Bx) - H(Ax_n, Bx_n) + \lambda y] \in M_n(x_n).$$

Let $y_n = \frac{1}{\lambda} [H(Ax, Bx) - H(Ax_n, Bx_n) + \lambda y]$ and using the same arguments as for (3.4), we have

$$\begin{aligned} \|y_n - y\| &= \left\| \frac{1}{\lambda} [H(Ax, Bx) - H(Ax_n, Bx_n) + \lambda y] - y \right\| \\ &= \frac{1}{\lambda} \|H(Ax, Bx) - H(Ax_n, Bx_n)\| \\ &\leq \frac{1}{\lambda} \left\{ \|H(Ax, Bx) - H(Ax_n, Bx)\| \right. \\ &\quad \left. + \|H(Ax_n, Bx) - H(Ax_n, Bx_n)\| \right\} \\ &\leq \left(\frac{\tau_1 + \tau_2}{\lambda} \right) \|x_n - x\|. \end{aligned} \quad (3.6)$$

Using above arguments, we have

$$\begin{aligned} \|x_n - x\| &= \frac{1}{2} \left\| (C_{M_n, \lambda}^{H(\cdot, \cdot)} + I) [H(Ax, Bx) + \lambda y] \right. \\ &\quad \left. - (C_{M, \lambda}^{H(\cdot, \cdot)} + I) [H(Ax, Bx) + \lambda y] \right\| \\ &= \frac{1}{2} \left\| (C_{M_n, \lambda}^{H(\cdot, \cdot)} - C_{M, \lambda}^{H(\cdot, \cdot)}) [H(Ax, Bx) + \lambda y] \right\|. \end{aligned} \quad (3.7)$$

Since $C_{M_n, \lambda}^{H(\cdot, \cdot)}(x) \rightarrow C_{M, \lambda}^{H(\cdot, \cdot)}(x)$, we have $\|x_n - x\| \rightarrow 0$ as $n \rightarrow \infty$. Thus from (3.6), it follows that $\|y_n - y\| \rightarrow 0$ as $n \rightarrow \infty$. That is, $M_n \xrightarrow{G} M$. This completes the proof. \square

Remark 3.4. One can easily verify that the convergence of the resolvent operator $R_{M_n, \lambda}^{H(\cdot, \cdot)}(x) \rightarrow R_{M, \lambda}^{H(\cdot, \cdot)}(x)$ and the convergence of generalized Cayley operator $C_{M_n, \lambda}^{H(\cdot, \cdot)}(x) \rightarrow C_{M, \lambda}^{H(\cdot, \cdot)}(x)$ are equivalent if and only if $M_n \xrightarrow{G} M$.

We now construct the following example which shows that the mapping M is $H(\cdot, \cdot)$ -monotone with respect to A and B , $M_n \xrightarrow{G} M$ and $C_{M_n, \lambda}^{H(\cdot, \cdot)}(x) \rightarrow C_{M, \lambda}^{H(\cdot, \cdot)}(x)$. Through MATLAB programming, we show some graphics for the convergence of $H(\cdot, \cdot)$ -monotone and generalized Cayley operators.

Example 3.5. Let $X = \mathbb{R}$ and $A, B : \mathbb{R} \rightarrow \mathbb{R}$ be single-valued mappings defined by

$$A(x) = \frac{x^3}{4}, \quad B(x) = \frac{2x}{5}, \quad \forall x \in \mathbb{R}.$$

Let $H : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a mapping defined by

$$H(A(x), B(x)) = \frac{1}{2} [A(x) - B(x)], \quad \forall x \in \mathbb{R}$$

with the condition that $x^2 + y^2 + xy \geq 1$. Then for any $u \in \mathbb{R}$, we have

$$\begin{aligned} [H(A(x), u) - H(A(y), u), x - y] &= \frac{1}{2} [A(x) - A(y), x - y] \\ &= \frac{1}{8} [x^3 - y^3, x - y] \\ &= \frac{1}{8} [(x^2 + y^2 + xy)(x - y), x - y] \\ &\geq \frac{1}{8} \|x - y\|^2. \end{aligned}$$

Hence, $H(A, \cdot)$ is $\frac{1}{8}$ -strongly monotone and

$$\begin{aligned} [H(u, B(x)) - H(u, B(y)), x - y] &= -\frac{1}{2} [B(x) - B(y), x - y] \\ &= -\frac{1}{5} \|x - y\|^2 \\ &\geq -\frac{1}{4} \|x - y\|^2. \end{aligned}$$

Hence, $H(\cdot, B)$ is $\frac{1}{4}$ -relaxed monotone. Thus, $H(A, B)$ is $\alpha\beta$ -symmetric monotone with $\alpha = \frac{1}{8}$ and $\beta = \frac{1}{4}$.

Let $M : \mathbb{R} \rightarrow \mathbb{R}$ be a mapping defined by $M(x) = \left\{ \frac{x}{5} \right\}$, for all $x \in \mathbb{R}$. It can be easily verified that M is monotone. Also, for any $x \in \mathbb{R}$ and $\lambda > 0$, we have

$$(H(A, B) + \lambda M)(x) = H(A(x), B(x)) + \lambda M(x) = \frac{x^3}{8} + (\lambda - 1) \frac{x}{5}.$$

Clearly the right hand side of above equation generates the whole space \mathbb{R} , that is,

$$(H(A, B) + \lambda M)(\mathbb{R}) = \mathbb{R}, \quad \forall \lambda > 0.$$

Hence, M is $H(\cdot, \cdot)$ -monotone with respect to A and B . Further, let $M_n : \mathbb{R} \rightarrow \mathbb{R}$ be a sequence of mappings defined by

$$M_n(x) = \frac{x}{5} + \frac{3}{n^2}, \quad \forall n \in \mathbb{N}.$$

By using the same arguments as above one can easily show that $\{M_n\}$ is a sequence of $H(\cdot, \cdot)$ -monotone operators.

Now, we show that $M_n \xrightarrow{G} M$. For any $(x, y) \in \text{graph}(M)$, there exist $(x_n, y_n) \in \text{graph}(M_n)$, where let

$$x_n = \left(1 + \frac{3}{n}\right)x \quad \text{and} \quad y_n = M_n(x_n) = \frac{x_n}{5} + \frac{3}{n^2}, \quad \forall n \in \mathbb{N}.$$

Since

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \left[\left(1 + \frac{3}{n}\right)x \right] = x,$$

we have $x_n \rightarrow x$ as $n \rightarrow \infty$. Also

$$\lim_{n \rightarrow \infty} y_n = \lim_{n \rightarrow \infty} \left(\frac{x_n}{5} + \frac{3}{n^2} \right) = \frac{x}{5} = M(x) = y,$$

which shows that $y_n \rightarrow y$ as $n \rightarrow \infty$ and consequently $M_n \xrightarrow{G} M$.

Finally, we show that $C_{M_n, \lambda}^{H(\cdot, \cdot)} \rightarrow C_{M, \lambda}^{H(\cdot, \cdot)}$ as $M_n \xrightarrow{G} M$. Now for $\lambda = 1$, the resolvent operators are given by

$$R_{M_n, \lambda}^{H(\cdot, \cdot)}(x) = (H(A, B) + \lambda M_n)^{-1}(x) = 2\sqrt[3]{\left(x - \frac{3}{n^2}\right)}$$

and

$$R_{M, \lambda}^{H(\cdot, \cdot)}(x) = (H(A, B) + \lambda M)^{-1}(x) = 2\sqrt[3]{x}.$$

Also, for $\lambda = 1$, the Cayley operators are given by

$$C_{M_n, \lambda}^{H(\cdot, \cdot)}(x) = \left(2R_{M_n, \lambda}^{H(\cdot, \cdot)} - I\right)(x) = 4\sqrt[3]{\left(x - \frac{3}{n^2}\right)} - x$$

and

$$C_{M, \lambda}^{H(\cdot, \cdot)}(x) = \left(2R_{M, \lambda}^{H(\cdot, \cdot)} - I\right)(x) = 4\sqrt[3]{x} - x.$$

From above, it is clear that

$$\left\| C_{M_n, \lambda}^{H(\cdot, \cdot)}(x) - C_{M, \lambda}^{H(\cdot, \cdot)}(x) \right\| \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Thus $C_{M_n, \lambda}^{H(\cdot, \cdot)} \rightarrow C_{M, \lambda}^{H(\cdot, \cdot)}$ as $M_n \xrightarrow{G} M$.

Using the above example, the convergence of $H(\cdot, \cdot)$ -monotone operators $M_n(x)$ to $M(x)$ and convergence of generalized Cayley operators $C_{M_n, \lambda}^{H(\cdot, \cdot)}(x)$ to $C_{M, \lambda}^{H(\cdot, \cdot)}(x)$ is illustrated in the following figure for $n = 2, 3, 5, 10$.

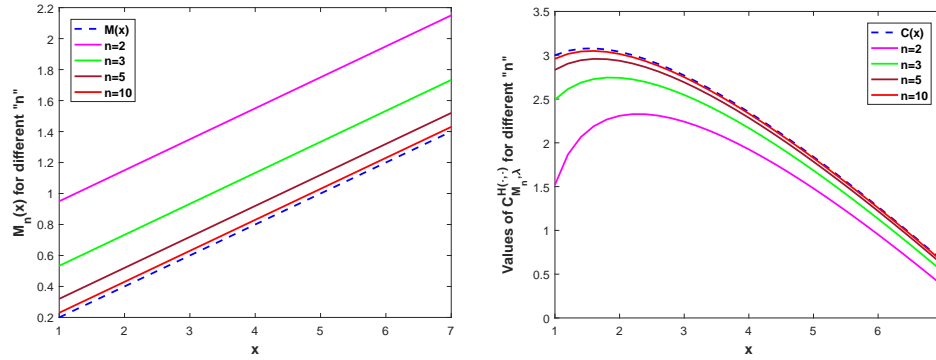


FIGURE 1. Convergence of $M_n(x)$ and $C_{M_n, \lambda}^{H(\cdot, \cdot)}(x)$.

4. SYSTEM OF GENERALIZED CAYLEY INCLUSIONS, EXISTENCE OF SOLUTION AND PERTURBED ITERATIVE ALGORITHM

In what follows, we assume that X_1 and X_2 are 2-uniformly smooth Banach spaces. Let $A_1, B_1 : X_1 \rightarrow X_1, A_2, B_2 : X_2 \rightarrow X_2, F : X_1 \times X_2 \rightarrow X_1, G : X_1 \times X_2 \rightarrow X_2, H_1 : X_1 \times X_1 \rightarrow X_1, H_2 : X_2 \times X_2 \rightarrow X_2$ be single-valued mappings. Let $M : X_1 \rightarrow 2^{X_1}$ be an $H_1(A_1, B_1)$ -monotone operator and $N : X_2 \rightarrow 2^{X_2}$ be an $H_2(A_2, B_2)$ -monotone operator. We consider the following system of generalized Cayley inclusions (SGCI): Find $(x, y) \in X_1 \times X_2$ such that

$$\begin{cases} 0 \in C_{M, \lambda}^{H_1(\cdot, \cdot)}(x) + F(x, y) + M(x), \\ 0 \in C_{N, \rho}^{H_2(\cdot, \cdot)}(x) + G(x, y) + N(x), \end{cases} \tag{4.1}$$

where $C_{M, \lambda}^{H_1(\cdot, \cdot)}$ and $C_{N, \rho}^{H_2(\cdot, \cdot)}$ are the generalized Cayley operators.

By taking $F \equiv G \equiv 0, N \equiv M$ and A -monotonicity of the set-valued mapping M instead of $H(\cdot, \cdot)$ -monotonicity, SGCI (4.1) reduces to the problem of finding $x \in X$ such that

$$0 \in C_{M, \lambda}^A(x) + M(x).$$

This problem was considered and studied by Ahmad *et al.* [2] in the setting of q -uniformly smooth Banach spaces.

Here we remark that for suitable choices of the mappings involved in the formulation of SGCI (4.1), one can obtain different classes of variational inclusions present in the literature, see for example [3, 6, 12, 16, 17, 19, 20, 22, 26], which can be efficiently solved using the techniques presented in this paper.

Lemma 4.1. *The system of generalized Cayley inclusions (4.1) admits a solution $(x, y) \in X_1 \times X_2$ if and only if it satisfies the following equations:*

$$\begin{aligned} x &= R_{M,\lambda}^{H_1(\cdot,\cdot)} \left[H_1(A_1, B_1)x - \lambda(F(x, y) + C_{M,\lambda}^{H_1(\cdot,\cdot)}(x)) \right], \\ y &= R_{N,\rho}^{H_2(\cdot,\cdot)} \left[H_2(A_2, B_2)y - \rho(G(x, y) + C_{N,\rho}^{H_2(\cdot,\cdot)}(y)) \right], \end{aligned}$$

where $R_{M,\lambda}^{H_1(\cdot,\cdot)}(x) = [H_1(A_1, B_1) + \lambda M]^{-1}(x)$ and $R_{N,\rho}^{H_2(\cdot,\cdot)}(x) = [H_2(A_2, B_2) + \rho N]^{-1}(x)$.

Proof. The proof immediately follows from the definition of resolvent operator. So, we omit the proof here. \square

Theorem 4.2. *Let $A_1, B_1 : X_1 \rightarrow X_1, A_2, B_2 : X_2 \rightarrow X_2, H_1 : X_1 \times X_1 \rightarrow X_1, H_2 : X_2 \times X_2 \rightarrow X_2, F : X_1 \times X_2 \rightarrow X_1, G : X_1 \times X_2 \rightarrow X_2$ be single-valued mappings such that $H_1(A_1, B_1)$ is $\alpha_1\beta_1$ -symmetric monotone and $H_2(A_2, B_2)$ is $\alpha_2\beta_2$ -symmetric monotone, $H_1(A_1, B_1)$ is τ_1, τ_2 -Lipschitz continuous with respect to A_1, B_1 and $H_2(A_2, B_2)$ is γ_1, γ_2 -Lipschitz continuous with respect to A_2, B_2 , respectively, $F(\cdot, \cdot)$ is k_1 -Lipschitz continuous and σ_1 -strongly monotone with respect to first argument and k_2 -Lipschitz continuous with respect to second argument, $G(\cdot, \cdot)$ is ξ_1 -Lipschitz continuous and σ_2 -strongly monotone with respect to first argument and ξ_2 -Lipschitz continuous with respect to second argument. Let $M : X_1 \rightarrow 2^{X_1}$ be an $H_1(A_1, B_1)$ -monotone set-valued operator and $N : X_2 \rightarrow 2^{X_2}$ be an $H_2(A_2, B_2)$ -monotone set-valued operator. In addition, if the constants satisfy the following inequalities:*

$$\begin{cases} \vartheta_1 &= m_1 + \eta_2\rho\xi_2 < 1, \\ \vartheta_2 &= m_2 + \eta_1\lambda k_2 < 1, \end{cases} \quad (4.2)$$

where,

$$\begin{aligned} m_1 &= \eta_1 \left\{ \sqrt{1 + c(\tau_1 + \tau_2)^2 - 2(\alpha_1 - \beta_1)} + \sqrt{1 + c\lambda^2 k_1^2 - 2\lambda\sigma_1 + \lambda\theta_1} \right\}, \\ m_2 &= \eta_2 \left\{ \sqrt{1 + c(\gamma_1 + \gamma_2)^2 - 2(\alpha_2 - \beta_2)} + \sqrt{1 + c\rho^2 \xi_1^2 - 2\rho\sigma_2 + \rho\theta_2} \right\}, \\ \eta_1 &= \frac{1}{\alpha_1 - \beta_1} \quad \text{and} \quad \eta_2 = \frac{1}{\alpha_2 - \beta_2}. \end{aligned}$$

Then SGCI (4.1) admits a unique solution.

Proof. For any given $\lambda, \rho > 0$, define $T_\lambda : X_1 \times X_2 \rightarrow X_1$ and $S_\rho : X_1 \times X_2 \rightarrow X_2$ by

$$T_\lambda(a, b) = R_{M,\lambda}^{H_1(\cdot, \cdot)} \left[H_1(A_1(a), B_1(a)) - \lambda \left(F(a, b) + C_{M,\lambda}^{H_1(\cdot, \cdot)}(a) \right) \right] \quad (4.3)$$

and

$$S_\rho(a, b) = R_{N,\rho}^{H_2(\cdot, \cdot)} \left[H_2(A_2(b), B_2(b)) - \rho \left(G(a, b) + C_{N,\rho}^{H_2(\cdot, \cdot)}(b) \right) \right]. \quad (4.4)$$

For the sake of brevity, let

$$J_1(u, v) = H_1(A_1(u), B_1(v)) \quad \text{and} \quad J_2(u, v) = H_2(A_2(u), B_2(v)).$$

In view of (4.3) and Lemma 2.9, we have for any $(a_1, b_1), (a_2, b_2) \in X_1 \times X_2$

$$\begin{aligned} & \|T_\lambda(a_1, b_1) - T_\lambda(a_2, b_2)\| \\ &= \left\| R_{M,\lambda}^{H_1(\cdot, \cdot)} \left[J_1(a_1, a_1) - \lambda \left(F(a_1, b_1) + C_{M,\lambda}^{H_1(\cdot, \cdot)}(a_1) \right) \right] \right. \\ &\quad \left. - R_{M,\lambda}^{H_1(\cdot, \cdot)} \left[J_1(a_2, a_2) - \lambda \left(F(a_2, b_2) + C_{M,\lambda}^{H_1(\cdot, \cdot)}(a_2) \right) \right] \right\| \\ &\leq \eta_1 \left\{ \left\| \left[J_1(a_1, a_1) - \lambda \left(F(a_1, b_1) + C_{M,\lambda}^{H_1(\cdot, \cdot)}(a_1) \right) \right] \right. \right. \\ &\quad \left. \left. - \left[J_1(a_2, a_2) - \lambda \left(F(a_2, b_2) + C_{M,\lambda}^{H_1(\cdot, \cdot)}(a_2) \right) \right] \right\| \right\} \\ &\leq \eta_1 \left\{ \left\| J_1(a_1, a_1) - J_1(a_2, a_2) - (a_1 - a_2) \right\| \right. \\ &\quad + \left\| (a_1 - a_2) - \lambda \left(F(a_1, b_1) - F(a_2, b_1) \right) \right\| \\ &\quad + \lambda \left\| F(a_2, b_1) - F(a_2, b_2) \right\| \\ &\quad \left. + \lambda \left\| C_{M,\lambda}^{H_1(\cdot, \cdot)}(a_1) - C_{M,\lambda}^{H_1(\cdot, \cdot)}(a_2) \right\| \right\}. \quad (4.5) \end{aligned}$$

By given assumptions and (1.2), we have

$$\begin{aligned} & \left\| J_1(a_1, a_1) - J_1(a_2, a_2) - (a_1 - a_2) \right\|^2 \\ & \leq \|a_1 - a_2\|^2 - 2[J_1(a_1, a_1) - J_1(a_2, a_2), (a_1 - a_2)] \\ & \quad + c \|J_1(a_1, a_1) - J_1(a_2, a_1)\|^2. \quad (4.6) \end{aligned}$$

According to hypothesis $H_1(A_1, B_1)$ is τ_1 -Lipschitz continuous with respect to A_1 and τ_2 -Lipschitz continuous with respect to B_1 , therefore we have

$$\begin{aligned} \left\| J_1(a_1, a_1) - J_1(a_2, a_2) \right\| &\leq \left\| J_1(a_1, a_1) - J_1(a_2, a_1) \right\| \\ &\quad + \left\| J_1(a_2, a_1) - J_1(a_2, a_2) \right\| \\ &\leq \tau_1 \|a_1 - a_2\| + \tau_2 \|a_1 - a_2\| \\ &= (\tau_1 + \tau_2) \|a_1 - a_2\|. \quad (4.7) \end{aligned}$$

In light of the fact that $H_1(A_1, B_1)$ is $\alpha_1\beta_1$ -symmetric monotone, we have

$$\begin{aligned} [J_1(a_1, a_1) - J_1(a_2, a_2), (a_1 - a_2)] &= [J_1(a_1, a_1) - J_1(a_2, a_1), (a_1 - a_2)] \\ &\quad + [J_1(a_2, a_1) - J_1(a_2, a_2), (a_1 - a_2)] \\ &\geq (\alpha_1 - \beta_1)\|a_1 - a_2\|^2. \end{aligned} \quad (4.8)$$

By (4.6)-(4.8), we obtain

$$\begin{aligned} &\|J_1(a_1, a_1) - J_1(a_2, a_2) - (a_1 - a_2)\|^2 \\ &\leq [1 + c(\tau_1 + \tau_2)^2 - 2(\alpha_1 - \beta_1)] \|a_1 - a_2\|^2. \end{aligned} \quad (4.9)$$

From the Lipschitz continuity and the monotonicity of F , we have

$$\begin{aligned} &\|(a_1 - a_2) - \lambda(F(a_1, b_1) - F(a_2, b_1))\|^2 \\ &\leq \|a_1 - a_2\|^2 - 2\lambda[F(a_1, b_1) - F(a_2, b_1), a_1 - a_2] \\ &\quad + c\lambda^2\|F(a_1, b_1) - F(a_2, b_1)\|^2 \\ &\leq (1 + c\lambda^2k_1^2 - 2\lambda\sigma_1)\|a_1 - a_2\|^2 \end{aligned} \quad (4.10)$$

and

$$\|F(a_2, b_1) - F(a_2, b_2)\| \leq k_2\|b_1 - b_2\|. \quad (4.11)$$

Also,

$$\|C_{M,\lambda}^{H_1(\cdot, \cdot)}(a_1) - C_{M,\lambda}^{H_1(\cdot, \cdot)}(a_2)\| \leq \theta_1\|a_1 - a_2\|, \quad (4.12)$$

where $\theta_1 = \frac{2+(\alpha_1-\beta_1)}{\alpha_1-\beta_1}$.

Using (4.9)-(4.12) in (4.5), we obtain

$$\begin{aligned} \|T_\lambda(a_1, b_1) - T_\lambda(a_2, b_2)\| &\leq \eta_1 \left\{ \sqrt{1 + c(\tau_1 + \tau_2)^2 - 2(\alpha_1 - \beta_1)} \right. \\ &\quad \left. + \sqrt{1 + c\lambda^2k_1^2 - 2\lambda\sigma_1 + \lambda\theta_1} \right\} \|a_1 - a_2\| \\ &\quad + \eta_1\lambda k_2\|b_1 - b_2\|. \end{aligned} \quad (4.13)$$

As for $S_\rho(a, b)$, similar to the deduction of inequality (4.13), we can conclude that

$$\begin{aligned} \|S_\rho(a_1, b_1) - S_\rho(a_2, b_2)\| &\leq \eta_2 \left\{ \sqrt{1 + c(\gamma_1 + \gamma_2)^2 - 2(\alpha_2 - \beta_2)} \right. \\ &\quad \left. + \sqrt{1 + c\rho^2\xi_1^2 - 2\rho\sigma_2 + \rho\theta_2} \right\} \|b_1 - b_2\| \\ &\quad + \eta_2\rho\xi_2\|a_1 - a_2\|, \end{aligned} \quad (4.14)$$

where $\theta_2 = \frac{2+(\alpha_2-\beta_2)}{\alpha_2-\beta_2}$.

From (4.13) and (4.14), we have

$$\begin{aligned} &\|T_\lambda(a_1, b_1) - T_\lambda(a_2, b_2)\| + \|S_\rho(a_1, b_1) - S_\rho(a_2, b_2)\| \\ &\leq \vartheta_1\|a_1 - a_2\| + \vartheta_2\|b_1 - b_2\|, \end{aligned} \quad (4.15)$$

where

$$\begin{aligned}\vartheta_1 &= \eta_1 \left\{ \sqrt{1 + c(\tau_1 + \tau_2)^2 - 2(\alpha_1 - \beta_1)} + \sqrt{1 + c\lambda^2 k_1^2 - 2\lambda\sigma_1 + \lambda\theta_1} \right\} + \eta_2 \rho \xi_2, \\ \vartheta_2 &= \eta_2 \left\{ \sqrt{1 + c(\gamma_1 + \gamma_2)^2 - 2(\alpha_2 - \beta_2)} + \sqrt{1 + c\rho^2 \xi_1^2 - 2\rho\sigma_2 + \rho\theta_2} \right\} + \eta_1 \lambda k_2.\end{aligned}$$

Setting

$$\vartheta_{\lambda,\rho} = \max \{ \vartheta_1, \vartheta_2 \}, \quad (4.16)$$

then (4.15) becomes

$$\|T_\lambda(a_1, b_1) - T_\lambda(a_2, b_2)\| + \|S_\rho(a_1, b_1) - S_\rho(a_2, b_2)\| \leq \vartheta_{\lambda,\rho} (\|a_1 - a_2\| + \|b_1 - b_2\|). \quad (4.17)$$

Define $\|(\cdot, \cdot)\|_1$ on $X_1 \times X_2$ by

$$\|(a, b)\|_1 = \|a\| + \|b\|, \quad \forall (a, b) \in X_1 \times X_2.$$

Then clearly $(X_1 \times X_2, \|\cdot\|_1)$ is a Banach space.

For any given $\lambda, \rho > 0$, define $D_{\lambda,\rho} : X_1 \times X_2 \rightarrow X_1 \times X_2$ by

$$D_{\lambda,\rho}(a, b) = (T_\lambda(a, b), S_\rho(a, b)), \quad \forall (a, b) \in X_1 \times X_2. \quad (4.18)$$

From (4.17) and (4.18), we have

$$\|D_{\lambda,\rho}(a_1, b_1) - D_{\lambda,\rho}(a_2, b_2)\|_1 \leq \vartheta_{\lambda,\rho} \|(a_1, b_1) - (a_2, b_2)\|_1. \quad (4.19)$$

From (4.2) and (4.16), we know that $0 \leq \vartheta_{\lambda,\rho} < 1$. Therefore, it follows from (4.19) that $D_{\lambda,\rho}$ is a contraction mapping. Hence, there exists a unique point $(x, y) \in X_1 \times X_2$ such that

$$D_{\lambda,\rho}(x, y) = (x, y), \quad (4.20)$$

which implies that

$$x = T_\lambda(x, y) = R_{M,\lambda}^{H_1(\cdot, \cdot)} [H_1(A_1(x), B_1(x)) - \lambda(F(x, y) + C_{M,\lambda}^{H_1(\cdot, \cdot)}(x))]$$

and

$$y = S_\rho(x, y) = R_{N,\rho}^{H_2(\cdot, \cdot)} [H_2(A_2(y), B_2(y)) - \rho(G(x, y) + C_{N,\rho}^{H_2(\cdot, \cdot)}(y))].$$

Thus we conclude from Lemma 4.1 that (x, y) is the unique solution of SGCI (4.1). This completes the proof. \square

Based on Lemma 4.1, we suggest and analyze the following perturbed iterative algorithm for finding an approximate solution for SGCI (4.1).

Algorithm 4.3.

- (i) Let $(x_0, y_0) \in X_1 \times X_2$ be an initial point.

(ii) Given $(x_n, y_n) \in X_1 \times X_2$, compute $(x_{n+1}, y_{n+1}) \in X_1 \times X_2$ by the iterative schemes:

$$\begin{aligned} x_{n+1} = & \alpha_n x_n + (1 - \alpha_n) R_{M_n, \lambda}^{H_1(\cdot, \cdot)} [H_1(A_1(x_n), B_1(x_n)) \\ & - \lambda(F(x_n, y_n) + C_{M_n, \lambda}^{H_1(\cdot, \cdot)}(x_n))], \end{aligned} \quad (4.21)$$

and

$$\begin{aligned} y_{n+1} = & \alpha_n y_n + (1 - \alpha_n) R_{N_n, \rho}^{H_2(\cdot, \cdot)} [H_2(A_2(y_n), B_2(y_n)) \\ & - \rho(G(x_n, y_n) + C_{N_n, \rho}^{H_2(\cdot, \cdot)}(y_n))], \end{aligned} \quad (4.22)$$

for $n = 0, 1, 2, \dots$, where $0 \leq \alpha_n < 1$ with $\lim_{n \rightarrow \infty} \sup \alpha_n < 1$,

$$R_{M_n, \lambda}^{H(\cdot, \cdot)}(x) = [(H(A, B) + \lambda M_n)^{-1}(x),$$

$$R_{N_n, \rho}^{H(\cdot, \cdot)}(x) = [(H(A, B) + \rho N_n)^{-1}(x),$$

$$C_{M_n, \lambda}^{H(\cdot, \cdot)}(x) = [2R_{M_n, \lambda}^{H(\cdot, \cdot)} - I](x)$$

and

$$C_{N_n, \rho}^{H(\cdot, \cdot)}(x) = [2R_{N_n, \rho}^{H(\cdot, \cdot)} - I](x).$$

Theorem 4.4. *Let $X_1, X_2, A_1, B_1, A_2, B_2, H_1, H_2, M, N, F, G$ be same as in Theorem 4.2. Assume that the constants satisfy inequality (4.2). Furthermore, let $M_n : X_1 \rightarrow 2^{X_1}$ be a sequence of $H_1(\cdot, \cdot)$ -monotone set-valued mappings such that $M_n \xrightarrow{G} M$ and $N_n : X_2 \rightarrow 2^{X_2}$ be a sequence of $H_2(\cdot, \cdot)$ -monotone set-valued mappings such that $N_n \xrightarrow{G} N$. Then the sequences $\{x_n\}, \{y_n\}$ generated by the Algorithm 4.3 converges strongly to the unique solution x, y of SGCI (4.1).*

Proof. By Theorem 4.2, SGCI (4.1) admits a unique solution (x, y) . It follows from Lemma 4.1 that

$$x = \alpha_n x + (1 - \alpha_n) R_{M, \lambda}^{H_1(\cdot, \cdot)} [H_1(A_1(x), B_1(x)) - \lambda(F(x, y) + C_{M, \lambda}^{H_1(\cdot, \cdot)}(x))] \quad (4.23)$$

and

$$y = \alpha_n y + (1 - \alpha_n) R_{N, \rho}^{H_2(\cdot, \cdot)} [H_2(A_2(y), B_2(y)) - \rho(G(x, y) + C_{N, \rho}^{H_2(\cdot, \cdot)}(y))]. \quad (4.24)$$

For the sake of brevity, denote

$$J_1(u, v) = H_1(A_1(u), B_1(v)) \text{ and } J_2(u, v) = H_2(A_2(u), B_2(v)).$$

Then, by (4.21) and (4.23), we have

$$\begin{aligned}
& \|x_{n+1} - x\| \\
&= \left\| \left\{ \alpha_n x_n + (1 - \alpha_n) R_{M_n, \lambda}^{H_1(\cdot, \cdot)} [J_1(x_n, x_n) - \lambda(F(x_n, y_n) + C_{M_n, \lambda}^{H_1(\cdot, \cdot)}(x_n))] \right\} \right. \\
&\quad \left. - \left\{ \alpha_n x + (1 - \alpha_n) R_{M, \lambda}^{H_1(\cdot, \cdot)} [J_1(x, x) - \lambda(F(x, y) + C_{M, \lambda}^{H_1(\cdot, \cdot)}(x))] \right\} \right\| \\
&\leq \alpha_n \|x_n - x\| + (1 - \alpha_n) \left\| R_{M_n, \lambda}^{H_1(\cdot, \cdot)} [J_1(x_n, x_n) - \lambda(F(x_n, y_n) + C_{M_n, \lambda}^{H_1(\cdot, \cdot)}(x_n))] \right. \\
&\quad \left. - R_{M, \lambda}^{H_1(\cdot, \cdot)} [J_1(x, x) - \lambda(F(x, y) + C_{M, \lambda}^{H_1(\cdot, \cdot)}(x))] \right\| \\
&\leq \alpha_n \|x_n - x\| + (1 - \alpha_n) \left\{ \left\| R_{M_n, \lambda}^{H_1(\cdot, \cdot)} [J_1(x_n, x_n) - \lambda(F(x_n, y_n) + C_{M_n, \lambda}^{H_1(\cdot, \cdot)}(x_n))] \right. \right. \\
&\quad \left. \left. - R_{M_n, \lambda}^{H_1(\cdot, \cdot)} [J_1(x, x) - \lambda(F(x, y) + C_{M_n, \lambda}^{H_1(\cdot, \cdot)}(x))] \right\| \right. \\
&\quad \left. + \left\| R_{M_n, \lambda}^{H_1(\cdot, \cdot)} [J_1(x, x) - \lambda(F(x, y) + C_{M_n, \lambda}^{H_1(\cdot, \cdot)}(x))] \right. \right. \\
&\quad \left. \left. - R_{M, \lambda}^{H_1(\cdot, \cdot)} [J_1(x, x) - \lambda(F(x, y) + C_{M, \lambda}^{H_1(\cdot, \cdot)}(x))] \right\| \right\}. \tag{4.25}
\end{aligned}$$

Following very similar argument from (4.5) to (4.13), we have

$$\begin{aligned}
& \left\| R_{M_n, \lambda}^{H_1(\cdot, \cdot)} [J_1(x_n, x_n) - \lambda(F(x_n, y_n) + C_{M_n, \lambda}^{H_1(\cdot, \cdot)}(x_n))] \right. \\
&\quad \left. - R_{M_n, \lambda}^{H_1(\cdot, \cdot)} [J_1(x, x) - \lambda(F(x, y) + C_{M_n, \lambda}^{H_1(\cdot, \cdot)}(x))] \right\| \\
&\leq \eta_1 \left\{ \sqrt{1 + c(\tau_1 + \tau_2)^2 - 2(\alpha_1 - \beta_1)} + \sqrt{1 + c\lambda^2 k_1^2 - 2\lambda\sigma_1 + \lambda\theta_1} \right\} \|x_n - x\| \\
&\quad + \eta_1 \lambda k_2 \|y_n - y\|. \tag{4.26}
\end{aligned}$$

Therefore it follows that

$$\begin{aligned}
\|x_{n+1} - x\| &\leq \alpha_n \|x_n - x\| + (1 - \alpha_n) m_1 \|x_n - x\| \\
&\quad + (1 - \alpha_n) \eta_1 \lambda k_2 \|y_n - y\| + (1 - \alpha_n) p_n, \tag{4.27}
\end{aligned}$$

where

$$\begin{aligned}
p_n &= \left\| R_{M_n, \lambda}^{H_1(\cdot, \cdot)} [J_1(x, x) - \lambda(F(x, y) + C_{M_n, \lambda}^{H_1(\cdot, \cdot)}(x))] \right. \\
&\quad \left. - R_{M, \lambda}^{H_1(\cdot, \cdot)} [J_1(x, x) - \lambda(F(x, y) + C_{M, \lambda}^{H_1(\cdot, \cdot)}(x))] \right\|.
\end{aligned}$$

Proceeding likewise, we can obtain

$$\begin{aligned}
\|y_{n+1} - y\| &\leq \alpha_n \|y_n - y\| + (1 - \alpha_n) m_2 \|y_n - y\| \\
&\quad + (1 - \alpha_n) \eta_2 \rho \xi_2 \|x_n - x\| + (1 - \alpha_n) q_n, \tag{4.28}
\end{aligned}$$

where

$$q_n = \left\| R_{N,\rho}^{H_1(\cdot,\cdot)} [J_2(y, y) - \rho(G(x, y) + C_{N,\rho}^{H_1(\cdot,\cdot)}(y))] \right. \\ \left. - R_{N,\rho}^{H_1(\cdot,\cdot)} [J_2(y, y) - \rho(G(x, y) + C_{N,\rho}^{H_1(\cdot,\cdot)}(y))] \right\|.$$

It follows from (4.16), (4.27) and (4.28) that

$$\begin{aligned} & \|x_{n+1} - x\| + \|y_{n+1} - y\| \\ & \leq \alpha_n (\|x_n - x\| + \|y_n - y\|) + (1 - \alpha_n) \vartheta_{\lambda,\rho} (\|x_n - x\| + \|y_n - y\|) \\ & \quad + (1 - \alpha_n) (p_n + q_n) \\ & = [\alpha_n + (1 - \alpha_n) \vartheta_{\lambda,\rho}] (\|x_n - x\| + \|y_n - y\|) + (1 - \alpha_n) (p_n + q_n). \end{aligned} \quad (4.29)$$

Let

$$\varepsilon_n = \alpha_n + (1 - \alpha_n) \vartheta_{\lambda,\rho}.$$

Then by choosing α_n and $\vartheta_{\lambda,\rho} < 1$ in such a way that $d = \limsup_{n \rightarrow \infty} \varepsilon_n < 1$.

Thus, (4.29) becomes

$$\|x_{n+1} - x\| + \|y_{n+1} - y\| \leq d (\|x_n - x\| + \|y_n - y\|) + (1 - \alpha_n) (p_n + q_n). \quad (4.30)$$

In view of Theorems 3.2 and 3.3, it follows that $p_n, q_n \rightarrow 0$. Let $c_n = \|x_n - x\| + \|y_n - y\|$ and $t_n = (1 - \alpha_n) (p_n + q_n)$, then (4.30) can be written as

$$c_{n+1} \leq dc_n + t_n.$$

Therefore by Lemma 2.11, $c_n \rightarrow 0$ as $n \rightarrow \infty$. Thus $x_n \rightarrow x$ and $y_n \rightarrow y$ as $n \rightarrow \infty$. Hence the sequences $\{x_n\}, \{y_n\}$ generated by the Algorithm 4.3 converges strongly to the unique solution x, y of SGCI (4.1). This completes the proof. \square

5. CONCLUSION

The introduction of graph convergence of operators was a great contribution by Attouch [5] to the classical resolvent methods and beyond because it empowered us with more applicable algorithms involving a sequence of classical resolvents of a corresponding sequence of maximal monotone set-valued mappings. In this paper, we established an equivalence between the graph convergence of $H(\cdot, \cdot)$ -monotone operators and convergence of Cayley operators which can be further exploited to solve a variety of variational inclusion problems. Using the concept of graph convergence, we developed a perturbed iterative algorithm to approximate the solution of a system of Cayley inclusions. Our results generalize most of the results on variational inclusion problems involving classical resolvents to the case of generalized resolvents.

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