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COMMON FIXED POINT THEOREMS IN THE SETTING OF EXTENDED QUASI b-METRIC SPACES UNDER EXTENDED A-CONTRACTION MAPPINGS

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Abstract. In the setting of extended quasi b-metric spaces, we introduce a new concept called "extended A-contraction". We then use our concept to prove a common fixed point result for a pair of self mappings under a set of conditions. Also, we provide the concepts of extended B-contraction and extended R-contraction. We then establish a common fixed point under these new contractions. Our results generalize many existing result of fixed point in metric spaces. Furthermore, we give an example to illustrate and support our result.

1. Introduction

It is widely known that fixed point theory is an appealing mixture of pure and applied mathematical analysis used to explain some conditions in which mappings provide perfect solutions.

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Recently, fixed point theory has undergone a great deal of development, especially through the use of new generalizations of contraction [1, 9, 15, 24, 26] and also by enfeebling or extending the axioms that characterize a metric space to get many different spaces, such as quasi metric spaces [2, 12, 19], b-metric spaces [14, 18], extended b-metric spaces [8, 10, 13, 20, 25], extended quasi b-metric spaces [22], and so on.

Moreover, Shatanawi et al. obtained some fixed point results for a generalized ψ-weak contraction mappings [16], w-compatible mappings [17], cyclic mappings of Ω-distance [21], and Mizoguchi-Takahashi-type theorems in tvs-cone metric spaces [23].

In[6] Akram et al. defined "A-contraction" as a new proper class of Kannan's contractions, along with several other well-known contractions. This contraction was used by the authors to prove some fixed point theorems for self-mappings [3, 4, 5, 6, 7].

In this paper, we are interested in investigating the common fixed point theorem under extended A-contraction conditions in the setting of extended quasi b-metric space after proving that we give example to validate our results.

2. Preliminaries

Firstly, we present a definition and an example of extended quasi b-metric space which we require in the sequel.

Definition 2.1. ([11]) Let X be a nonempty set and $\theta : X \times X \longrightarrow [1, \infty)$. An extended b-metric is a function $d_\theta : X \times X \longrightarrow [0, \infty)$ such that for all $x, y, z \in X$, we have

(i) $d_\theta(x, y) = 0$ iff $x = y$;
(ii) $d_\theta(x, y) = d_\theta(y, x)$;
(iii) $d_\theta(x, z) \leq \theta(x, z)\big[d_\theta(x, y) + d_\theta(y, z)\big]$.

Then the pair (X, d_θ) is called an extended b-metric space.

(1) If d_θ satisfying just (1) and (3), then (X, d_θ) is called an extended quasi b-metric space.
(2) if d_θ satisfying just (1) and (3) and $\theta = s$, $s \geq 1$ be a given real number then (X, d_s) is called a quasi b-metric space.

Example 2.2. ([22]) Define $\theta : \{1, 2, 3\} \times \{1, 2, 3\} \to [1, \infty)$ by $\theta(x, y) = \max\{x, y\}$ and $d_\theta : \{1, 2, 3\} \times \{1, 2, 3\} \to [0, \infty)$ by $d_\theta(1, 1) = d_\theta(2, 2) = d_\theta(3, 3) = 0$, $d_\theta(1, 2) = d_\theta(1, 3) = d_\theta(2, 3) = 6$ and $d_\theta(2, 1) = d_\theta(3, 1) = d_\theta(3, 2) = 5$.
Then (X, d_θ) is an extended quasi b-metric space which is not an extended b-metric space.

Some fundamental concepts, like convergence, Cauchy sequence, and completeness in the setting of extended quasi b-metric space, are defined as follows.

Definition 2.3. ([22]) Let $\{x_n\}$ be a sequence in (X, d_θ). Then, the sequence $\{x_n\}$ converges to $x \in X$ if $\lim\limits_{n\to\infty} d_\theta(x_n, x) = \lim\limits_{n\to\infty} d_\theta(x, x_n) = 0$.

Definition 2.4. ([22]) Let (X, d_θ) be an extended quasi b-metric space. Then

(1) A sequence $\{x_n\}$ in X is said to be left-Cauchy, if for every $\epsilon > 0$ there exists a positive integer $N = N(\epsilon)$ such that $d_\theta(x_n, x_k) < \epsilon$ for all $n \geq k > N$.

(2) A sequence $\{x_n\}$ in X is said to be right-Cauchy, if for every $\epsilon > 0$ there exists a positive integer $N = N(\epsilon)$ such that $d_\theta(x_k, x_n) < \epsilon$ for all $n \geq k > N$.

(3) A sequence $\{x_n\}$ in (X, d_θ) is said to be Cauchy, if it is left-Cauchy and right-Cauchy.

Definition 2.5. ([22]) Let (X, d_θ) be an extended quasi b-metric space. Then

(1) (X, d_θ) is said to be left-complete, if every left-Cauchy sequence is convergent in X.

(2) (X, d_θ) is said to be right-complete, if every right-Cauchy sequence is convergent in X.

(3) (X, d_θ) is said to be complete, if every Cauchy sequence is convergent in X.

Now, the following definitions and results related to A-contraction are required in the main results.

Definition 2.6. ([6]) Let $\mathbb{R}_+$ denote the set of all non-negative real numbers and A be the set of all functions $\tau : \mathbb{R}_+^3 \to \mathbb{R}_+$ satisfying:

(i) τ is continuous on the set $\mathbb{R}_+^3$ (with respect to the Euclidean metric on $\mathbb{R}^3$).

(ii) $\zeta \leq \kappa\eta$, for some $\kappa \in [0, 1)$ whenever $\zeta \leq \tau(\zeta, \eta, \eta)$, $\zeta \leq \tau(\eta, \zeta, \eta)$ or $\zeta \leq \tau(\eta, \eta, \zeta)$ for $\zeta, \eta \in \mathbb{R}_+$.

Definition 2.7. ([6]) On a metric space (X, d), a self-mapping T is called A-contraction if for all $x, y \in X$ and some $\tau \in A$,

$$d(Tx, Ty) \leq \tau(d(x, y), d(x, Tx), d(y, Ty)).$$

Definition 2.8. ([6]) On a metric space (X, d), a self-mapping T is called

(1) B-contraction if there exists a $\beta \in [0, 1)$ such that for all $x, y \in X$,

$$d(Tx, Ty) \leq \beta \max\{d(x, Tx), d(y, Ty)\}. \tag{2.1}$$

(2) R-contraction if there exist non-negative numbers $\kappa_1, \kappa_2, \kappa_3$ satisfying $\kappa_1 + \kappa_2 + \kappa_3 \leq 1$ such that for all $x, y \in X$,

$$d(Tx, Ty) \leq \kappa_1 d(Tx, x) + \kappa_2 d(Ty, y) + \kappa_3 d(x, y). \tag{2.2}$$

Akram et al. [6] investigated comparison of an A-contraction with B-contraction (2.1) and R-contraction (2.2), they get

Theorem 2.9. ([6]) *Every B-contraction and R-contraction in a metric space are an A-contraction.*

3. Main results

Definition 3.1. Let (X, d_θ) be an extended quasi b-metric space. We say that the pair (T, S) of mappings T and S is an extended A-contraction if there exists $\alpha \in (0, 1)$ such that for all $x, y \in X$, we have

$$d_\theta(Tx, Sy) \leq \tau\Big(\alpha\theta(x, y)d_\theta(x, y), \alpha\theta(Tx, x)d_\theta(Tx, x), \alpha\theta(Sy, y)d_\theta(Sy, y)\Big)$$

and

$$d_\theta(Sy, Tx) \leq \tau\Big(\alpha\theta(y, x)d_\theta(y, x), \alpha\theta(x, Tx)d_\theta(x, Tx), \alpha\theta(y, Sy)d_\theta(y, Sy)\Big)$$

for some $\tau \in$ A provided that τ is non-decreasing on its variables.

Theorem 3.2. *Let $S, T : X \to X$ be two continuous mappings on a complete extended quasi b-metric space (X, d_θ) such that d_θ continuous in its variables. Assume the pair (T, S) is an extended A-contraction, where $\alpha \in (0, 1)$ and θ is bounded by $\frac{1}{\alpha}$. Then, the pair of the mappings T and S has a common fixed point x^*, that is, $Tx^* = x^* = Sx^*$, provided that $\kappa < \alpha$, where κ is the constant satisfies condition* (ii) *of the definition A.*

Proof. Let x_0 be a given point in X. Put $x_1 = Tx_0$ and $x_2 = Sx_1$. Continuing in the same way, we construct a sequence $\{x_n\}$ in X such that

$$x_{2n+1} = Tx_{2n} \text{ and } x_{2n+2} = Sx_{2n+1}, \ \forall n \in N \cup \{0\}.$$

Now, we shall prove that $\{x_n\}$ is a left-Cauchy sequence. To this end, we have

$$d_\theta(x_{2n+1}, x_{2n}) \quad = \quad d_\theta(Tx_{2n}, Sx_{2n-1}).$$

By using extended A-contraction condition, we get

$$\begin{aligned} d_\theta(Tx_{2n}, Sx_{2n-1}) \leq \tau\Big(&\alpha\theta(x_{2n}, x_{2n-1})d_\theta(x_{2n}, x_{2n-1}), \\ &\alpha\theta(Tx_{2n}, x_{2n})d_\theta(Tx_{2n}, x_{2n}), \\ &\alpha\theta(Sx_{2n-1}, x_{2n-1})d_\theta(Sx_{2n-1}, x_{2n-1})\Big). \end{aligned}$$

Therefore

$$d_\theta(x_{2n+1}, x_{2n}) \le \tau\Big(\alpha\theta(x_{2n}, x_{2n-1})d_\theta(x_{2n}, x_{2n-1}),\\ \alpha\theta(x_{2n+1}, x_{2n})d_\theta(x_{2n+1}, x_{2n}),\\ \alpha\theta(x_{2n}, x_{2n-1})d_\theta(x_{2n}, x_{2n-1})\Big).$$

Since θ is bounded by $\frac{1}{\alpha}$ and τ is non-decreasing in its variables, we get

$$d_\theta(x_{2n+1}, x_{2n}) \quad \le \quad \tau\Big(d_\theta(x_{2n}, x_{2n-1}), d_\theta(x_{2n+1}, x_{2n}), d_\theta(x_{2n}, x_{2n-1})\Big).$$

Let $\zeta = d_\theta(x_{2n+1}, x_{2n})$ and $\eta = d_\theta(x_{2n}, x_{2n-1})$. Then $\zeta \le \tau(\eta, \zeta, \eta)$, as a result there is $\kappa \in [0, 1)$ such that $\zeta \le \kappa\eta$. Hence

$$d_\theta(x_{2n+1}, x_{2n}) \le \kappa d_\theta(x_{2n}, x_{2n-1}).$$

holds for all $n \in N$. Thus we obtain

$$d_\theta(x_{2n+1}, x_{2n}) \le \kappa d_\theta(x_{2n}, x_{2n-1}) \le \kappa^2 d_\theta(x_{2n-1}, x_{2n-2}) \le \ldots \le \kappa^{2n} d_\theta(x_1, x_0).$$

Thus

$$d_\theta(x_{2n+1}, x_{2n}) \le \kappa^{2n} d_\theta(x_1, x_0).$$

From here, one can easily show that $\{x_n\}$ is a left-Cauchy sequence.

Now, on a similar manner for right-Cauchy sequence, we have

$$d_\theta(x_{2n}, x_{2n+1}) \quad = \quad d_\theta(Sx_{2n-1}, Tx_{2n}).$$

By using extended A-contraction, we obtain

$$d_\theta(Sx_{2n-1}, Tx_{2n}) \le \tau\Big(\alpha\theta(x_{2n-1}, x_{2n})d_\theta(x_{2n-1}, x_{2n}),\\ \alpha\theta(x_{2n}, Tx_{2n})d_\theta(x_{2n}, Tx_{2n}),\\ \alpha\theta(x_{2n-1}, Sx_{2n-1})d_\theta(x_{2n-1}, Sx_{2n-1})\Big).$$

Since θ is bounded by $\frac{1}{\alpha}$ and τ is non-decreasing in its variables, we get

$$d_\theta(x_{2n}, x_{2n+1}) \quad \le \quad \tau\Big(d_\theta(x_{2n-1}, x_{2n}), d_\theta(x_{2n}, x_{2n+1}), d_\theta(x_{2n-1}, x_{2n})\Big).$$

Now, let $\zeta = d_\theta(x_{2n}, x_{2n+1})$, $\eta = d_\theta(x_{2n-1}, x_{2n})$. Then $\zeta \le \tau(\eta, \zeta, \eta)$. By Definition 2.6, there is $\kappa \in [0, 1)$ such that $\zeta \le \kappa\eta$. So we have

$$d_\theta(x_{2n}, x_{2n+1}) \le \kappa d_\theta(x_{2n-1}, x_{2n}).$$

holds for all $n \in \mathbf{N}$. Consequently, we have

$$d_\theta(x_{2n}, x_{2n+1}) \le \kappa d_\theta(x_{2n-1}, x_{2n}) \le \kappa^2 d_\theta(x_{2n-2}, x_{2n-1}) \le \ldots \le \kappa^{2n} d_\theta(x_0, x_1).$$

So

$$d_\theta(x_{2n}, x_{2n+1}) \le \kappa^{2n} d_\theta(x_0, x_1).$$

From here, one can easily show that $\{x_n\}$ is a right-Cauchy sequence. Consequently $\{x_n\}$ is a Cauchy sequence.

Since X is a complete space, $\{x_n\}$ converges to some $x^* \in X$. So, we obtain

$$\lim_{n\to\infty} d_\theta(x_{2n}, x^*) = \lim_{n\to\infty} d_\theta(x^*, x_{2n}) = 0 = d_\theta(x^*, x^*)$$

and

$$\lim_{n\to\infty} d_\theta(x_{2n-1}, x^*) = \lim_{n\to\infty} d_\theta(x^*, x_{2n-1}) = 0 = d_\theta(x^*, x^*).$$

Now, to show that $Tx^* = x^*$, we have

$$d_\theta(Tx^*, x_{2n}) = d_\theta(Tx^*, Sx_{2n-1}).$$

Apply contraction condition

$$\begin{aligned} d_\theta(Tx^*, Sx_{2n-1}) \leq \tau\Big(&\alpha\theta(x^*, x_{2n-1})d_\theta(x^*, x_{2n-1}), \alpha\theta(Tx^*, x^*)d_\theta(Tx^*, x^*), \\ &\alpha\theta(Sx_{2n-1}, x_{2n-1})d_\theta(Sx_{2n-1}, x_{2n-1})\Big) \\ = \tau\Big(&\alpha\theta(x^*, x_{2n-1})d_\theta(x^*, x_{2n-1}), \alpha\theta(Tx^*, x^*)d_\theta(Tx^*, x^*), \\ &\alpha\theta(x_{2n}, x_{2n-1})d_\theta(x_{2n}, x_{2n-1})\Big). \end{aligned}$$

Since θ is bounded by $\frac{1}{\alpha}$ and τ is non-decreasing in its variables, we get

$$d_\theta(Tx^*, x_{2n}) \leq \tau\Big(d_\theta(x^*, x_{2n-1}), d_\theta(Tx^*, x^*), d_\theta(x_{2n}, x_{2n-1})\Big).$$

Letting $n \longrightarrow \infty$ in above inequality, the continuity of d_θ and τ implies that

$$\begin{aligned} d_\theta(Tx^*, x^*) &\leq \tau\Big(d_\theta(x^*, x^*), d_\theta(Tx^*, x^*), d_\theta(x^*, x^*)\Big) \\ &= \tau\Big(0, d_\theta(Tx^*, x^*), 0\Big). \end{aligned}$$

By Definition 2.6, there is $\kappa \in [0, 1)$ such that

$$d_\theta(Tx^*, x^*) \leq \kappa 0 = 0,$$

which implies that x^* is a fixed point of T.

For continuous mapping S, we have

$$d_\theta(Sx^*, x^*) = \lim_{n\to\infty} d_\theta(Sx^*, x_{2n}) = \lim_{n\to\infty} d_\theta(Sx^*, Sx_{2n-1}) = d_\theta(Sx^*, Sx^*) = 0.$$

This implies that x^* is a fixed point of S.

For the sake of the uniqueness, suppose that there exist another common fixed point of T and S say y^*. Thus, we have

$$d_\theta(x^*, y^*) = d_\theta(Tx^*, Sy^*).$$

By contraction condition, we have

$$\begin{aligned}
d_\theta(Tx^*, Sy^*) &\leq \tau\Big(\alpha\theta(x^*, y^*)d_\theta(x^*, y^*), \alpha\theta(Tx^*, x^*)d_\theta(Tx^*, x^*), \\
&\qquad \alpha\theta(Sy^*, y^*)d_\theta(Sy^*, y^*)\Big) \\
&\leq \tau\Big(d_\theta(x^*, y^*), d_\theta(x^*, x^*), d_\theta(y^*, y^*)\Big) \\
&= \tau\Big(d_\theta(x^*, y^*), 0, 0\Big).
\end{aligned}$$

Let $\zeta = d_\theta(x^*, y^*)$ and $\eta = 0$. By Definition 2.6, we get $\zeta \leq \tau(\zeta, \eta, \eta)$ for some $\kappa \in [0, 1)$, thus $\zeta \leq \kappa\eta$. Therefore

$$d_\theta(x^*, y^*) \leq \kappa 0 = 0.$$

So $x^* = y^*$. It follows that x^* is a unique common fixed point in X, This completes the proof. □

Example 3.3. Let $X = [0, 1]$, define a mapping $\theta : X \times X \to [1, \infty)$, by $\theta(x, y) = 1 + |x - y|$ and define $d_\theta : X \times X \to [0, \infty)$ by $d_\theta(x, y) = |x - y|$. Then (X, d_θ) is a complete extended quasi b-metric space.

As well, define the mappings $S, T : X \to X$ by $S(y) = \frac{y}{3}$ and $T(x) = \frac{\sin(x)}{2}$. Then, we have

$$d_\theta(Tx, Sy) = \Big|\frac{\sin(x)}{2} - \frac{y}{3}\Big|.$$

Take $y = 3x$, for some $x \in X$, we obtain

$$d_\theta(Tx, Sy) = \Big|\frac{\sin(x)}{2} - x\Big|.$$

From Figure 1, we find that

$$\begin{aligned}
d_\theta(Tx, Sy) &\leq \frac{2}{3}|x - 3x| \\
&= \frac{2}{3}d_\theta(x, y).
\end{aligned}$$

Now, define $\tau : \mathbb{R}_+^3 \to \mathbb{R}_+$ by $\tau(\zeta, \eta, \varrho) = \frac{1}{4}(\zeta + \eta + \varrho)$ for all $\zeta, \eta, \varrho \in \mathbb{R}_+$. It is clear that τ is a well-define, continuous and non-decreasing function on $\mathbb{R}_+^3$ and $\zeta \leq \frac{2}{3}\eta$, whenever $\zeta \leq \tau(\zeta, \eta, \eta)$ or $\zeta \leq \tau(\eta, \zeta, \eta)$ or $\zeta \leq \tau(\eta, \eta, \zeta)$ for all $\zeta, \eta \in \mathbb{R}_+$. Since, we have $d_\theta(Tx, Sy) \leq \frac{2}{3}d_\theta(x, y)$, from Definition 2.6 for

τ, we observe that

$$\begin{aligned} d_\theta(Tx,Sy) &\le \tau\Big(d_\theta(x,y), d_\theta(Tx,Sy), d_\theta(x,y)\Big) \\ &= \tau\Big(|x-y|, \Big|\frac{\sin(x)}{2} - \frac{y}{3}\Big|, |x-y|\Big) \\ &= \tau\Big(|x-y|, \Big|\frac{\sin(x)}{2} - \frac{3x}{3}\Big|, \Big|\frac{3x}{3} - y\Big|\Big) \\ &= \tau\Big(|x-y|, \Big|\frac{\sin(x)}{2} - x\Big|, \Big|\frac{y}{3} - y\Big|\Big) \\ &= \tau\Big(d_\theta(x,y), d_\theta(Tx,x), d_\theta(Sy,y)\Big). \end{aligned}$$

Let $\alpha \in (0,1)$. Since θ is bounded by $\frac{1}{\alpha}$, we get

$$d_\theta(Tx,Sy) \le \tau\Big(\alpha\theta(x,y)d_\theta(x,y), \alpha\theta(Tx,x)d_\theta(Tx,x), \alpha\theta(Sy,y)d_\theta(Sy,y)\Big).$$

On a similar manner, we can get

$$d_\theta(Sy,Tx) \le \tau\Big(\alpha\theta(y,x)d_\theta(y,x), \alpha\theta(x,Tx)d_\theta(x,Tx), \alpha\theta(y,Sy)d_\theta(y,Sy)\Big).$$

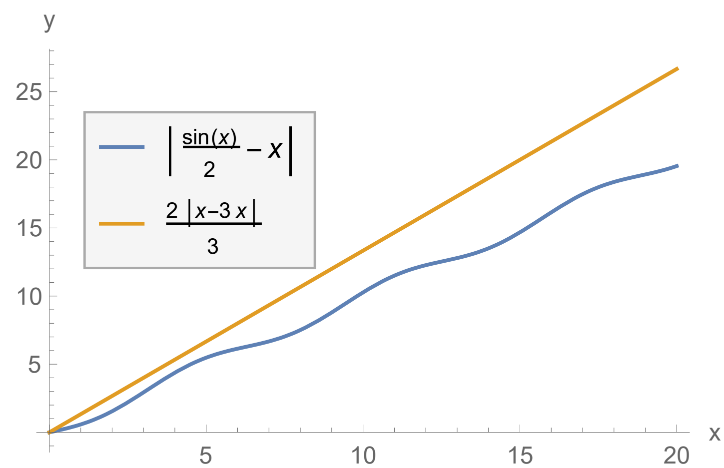

FIGURE 1. Comparison between $\left|\frac{\sin(x)}{2} - x\right|$ and $\frac{2}{3}|x - 3x|$.

Corollary 3.4. *Let $T : X \to X$ be a continuous mapping on a complete extended quasi b-metric space (X, d_θ), where d_θ is continuous in its variables.*

Assume there exist $\alpha \in (0,1)$ and $\tau \in A$ such that

$$d_\theta(Tx,Ty) \leq \tau\Big(\alpha\theta(x,y)d_\theta(x,y), \alpha\theta(Tx,x)d_\theta(Tx,x), \alpha\theta(Ty,y)d_\theta(Ty,y)\Big)$$

and

$$d_\theta(Ty,Tx) \leq \tau\Big(\alpha\theta(y,x)d_\theta(y,x), \alpha\theta(x,Tx)d_\theta(x,Tx), \alpha\theta(y,Ty)d_\theta(y,Ty)\Big)$$

hold for all $x,y \in X$. If τ is non-decreasing on its variables and θ is bounded by $\frac{1}{\alpha}$, then the mapping T has a unique fixed point in X provided that $\kappa < \alpha$, where κ is the constant satisfies condition (ii) *of the definition A.*

Proof. It follows from Theorem 3.2 by taking $S = T$. □

Corollary 3.5. *Let $T : X \to X$ be a continuous mapping on a complete quasi b-metric space (X, d_s). Assume there exist $\alpha \in (0,1)$ and $\tau \in A$ such that*

$$d_s(Tx,Ty) \leq \tau\Big(\alpha s d_s(x,y), \alpha s d_s(Tx,x), \alpha s d_s(Ty,y)\Big)$$

and

$$d_s(Ty,Tx) \leq \tau\Big(\alpha s d_s(y,x), \alpha s d_s(x,Tx), \alpha s d_s(y,Ty)\Big)$$

hold for all $x,y \in A$. If τ is non-decreasing on its variables and $s \leq \frac{1}{\alpha}$, then the mapping T has a unique fixed point in X provided that $\kappa < \alpha$, where κ is the constant satisfies condition (ii) *of the definition A.*

Proof. It follows from Theorem 3.2 by taking $\theta = s$, $s \geq 1$ and $S = T$. □

Definition 3.6. A pair (T,S) of the self-mappings T and S in frame of an extended quasi b-metric space (X, d_θ) is said to be an

(1) extended B-contraction, if there exist $\beta \in [0,1)$ and $\alpha \in (0,1)$ such that for all $x,y \in X$,

$$d_\theta(Tx,Sy) \leq \beta \max\{\alpha\theta(Tx,x)d_\theta(Tx,x), \alpha\theta(Sy,y)d_\theta(Sy,y)\} \tag{3.1}$$

and

$$d_\theta(Sy,Tx) \leq \beta \max\{\alpha\theta(x,Tx)d_\theta(x,Tx), \alpha\theta(y,Sy)d_\theta(y,Sy)\}. \tag{3.2}$$

(2) extended R-contraction, if there exist non-negative numbers $\kappa_1, \kappa_2, \kappa_3$ satisfying $\kappa_1 + \kappa_2 + \kappa_3 \leq 1$, $\alpha \in (0,1)$ such that for all $x,y \in X$,

$$\begin{aligned} d_\theta(Tx,Sy) \leq\ & \kappa_1\alpha\theta(x,y)d_\theta(x,y) + \kappa_2\alpha\theta(Tx,x)d_\theta(Tx,x) \\ & + \kappa_3\alpha\theta(Sy,y)d_\theta(Sy,y) \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} d_\theta(Sy,Tx) \leq\ & \kappa_1\alpha\theta(y,x)d_\theta(y,x) + \kappa_2\alpha\theta(x,Tx)d_\theta(x,Tx) \\ & + \kappa_3\alpha\theta(y,Sy)d_\theta(y,Sy). \end{aligned} \tag{3.4}$$

Proposition 3.7. *Every extended B-contraction and extended R-contraction in an extended quasi b-metric space are an extended A-contraction.*

Proof. First, assume that the pair (T, S) is an extended B-contraction. Then there exist $\beta \in [0, 1)$ and $\alpha \in (0, 1)$ such that for all $x, y \in X$, we have

$$d_\theta(Tx, Sy) \leq \beta \max\{\alpha\theta(Tx, x)d_\theta(Tx, x), \alpha\theta(Sy, y)d_\theta(Sy, y)\}$$

and

$$d_\theta(Sy, Tx) \leq \beta \max\{\alpha\theta(x, Tx)d_\theta(x, Tx), \alpha\theta(y, Sy)d_\theta(y, Sy)\}.$$

Define $\tau : \mathbb{R}^3 \to \mathbb{R}_+$ by $\tau(\zeta, \eta, \varrho) = \beta \max\{\eta, \varrho\}$ where $\beta \in [0, 1)$. In addition

(i) If $\zeta \leq \tau(\zeta, \eta, \eta)$, then $\zeta \leq \beta \max\{\eta, \eta\} = \beta\eta$. Here, we take $\kappa = \beta \in [0, 1)$.

(ii) If $\zeta \leq \tau(\eta, \zeta, \eta)$, then $\zeta \leq \beta \max\{\zeta, \eta\}$. Because $\beta < 1$, we conclude that $\max\{\zeta, \eta\} = \eta$. So $\zeta \leq \kappa\eta$. Here, we take $\kappa = \beta \in [0, 1)$.

(iii) If $\zeta \leq \tau(\eta, \eta, \zeta)$, then $\zeta \leq \beta \max\{\eta, \zeta\}$. Because $\beta < 1$, we conclude that $\max\{\zeta, \eta\} = \eta$. So $\zeta \leq \kappa\eta$. Here, we take $\kappa = \beta \in [0, 1)$.

Then, we get $\tau \in A$.

Case 1. Taking $\zeta = \alpha\theta(x, y)d_\theta(x, y)$, $\eta = \alpha\theta(Tx, x)d_\theta(Tx, x)$ and $\varrho = \alpha\theta(Sy, y)d_\theta(Sy, y)$, for all $x, y \in X$, $\alpha \in (0, 1)$, then we get

$$\begin{aligned}&\beta \max\{\alpha\theta(Tx, x)d_\theta(Tx, x), \alpha\theta(Sy, y)d_\theta(Sy, y)\}\\ &\quad = \tau\Big(\alpha\theta(x, y)d_\theta(x, y), \alpha\theta(Tx, x)d_\theta(Tx, x), \alpha\theta(Sy, y)d_\theta(Sy, y)\Big).\end{aligned}$$

On a similar manner:

Case 2. Taking $\zeta = \alpha\theta(y, x)d_\theta(y, x)$, $\eta = \alpha\theta(x, Tx)d_\theta(x, Tx)$ and $\varrho = \alpha\theta(y, Sy)d_\theta(y, Sy)$, for all $x, y \in X$,$\alpha \in (0, 1)$, then we get

$$\begin{aligned}&\beta \max\{\alpha\theta(x, Tx)d_\theta(x, Tx), \alpha\theta(y, Sy)d_\theta(y, Sy)\}\\ &\quad = \tau\Big(\alpha\theta(y, x)d_\theta(y, x), \alpha\theta(x, Tx)d_\theta(x, Tx), \alpha\theta(y, Sy)d_\theta(y, Sy)\Big).\end{aligned}$$

Consequently, every extended B-contraction is an extended A-contraction.

Next, assume that the pair (T, S) is an extended R-contraction. Then there exist non-negative numbers $\kappa_1, \kappa_2, \kappa_3$ with $\kappa_1 + \kappa_2 + \kappa_3 \leq 1$, $\alpha \in (0, 1)$ such that for all $x, y \in X$.

$$\begin{aligned}d_\theta(Tx, Sy) \leq &\kappa_1\alpha\theta(x, y)d_\theta(x, y) + \kappa_2\alpha\theta(Tx, x)d_\theta(Tx, x)\\ &+ \kappa_3\alpha\theta(Sy, y)d_\theta(Sy, y)\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}d_\theta(Sy, Tx) \leq &\kappa_1\alpha\theta(y, x)d_\theta(y, x) + \kappa_2\alpha\theta(x, Tx)d_\theta(x, Tx)\\ &+ \kappa_3\alpha\theta(y, Sy)d_\theta(y, Sy).\end{aligned} \tag{3.6}$$

Define $\tau : \mathbb{R}^3 \to \mathbb{R}_+$ by $\tau(\zeta, \eta, \varrho) = \kappa_1\zeta + \kappa_2\eta + \kappa_3\varrho$. Then τ is continuous. Moreover, we have

(i) $\zeta \leq \tau(\zeta, \eta, \eta) = \kappa_1\zeta + \kappa_2\eta + \kappa_3\eta$ implies $(1-\kappa_1)\zeta \leq (\kappa_2+\kappa_3)\eta$ and so $\zeta \leq \kappa\eta$, $\kappa = \frac{\kappa_2+\kappa_3}{1-\kappa_1} \in [0, 1)$.

(ii) $\zeta \leq \tau(\eta, \zeta, \eta) = \kappa_1\eta + \kappa_2\zeta + \kappa_3\eta$ implies $(1-\kappa_2)\zeta \leq (\kappa_1+\kappa_3)\eta$ and so $\zeta \leq \kappa\eta$, where $\kappa = \frac{\kappa_1+\kappa_3}{1-\kappa_2} \in [0, 1)$.

(iii) Similarly, $\zeta \leq \tau(\eta, \eta, \zeta) = \kappa_1\eta + \kappa_2\eta + \kappa_3\zeta$ implies that $(1-\kappa_3)\zeta \leq (\kappa_1+\kappa_2)\eta$ and so $\zeta \leq \kappa\eta$ where $\kappa = \frac{\kappa_1+\kappa_2}{1-\kappa_3} \in [0, 1)$.

So, we conclude that $\tau \in A$.

Case 1. Taking $\zeta = \alpha\theta(x, y)d_\theta(x, y)$, $\eta = \alpha\theta(Tx, x)d_\theta(Tx, x)$ and $\varrho = \alpha\theta(Sy, y)d_\theta(Sy, y)\Big)$, for all $x, y \in X$ and $\alpha \in (0, 1)$, then we get

$$\begin{aligned}\kappa_1\alpha\theta(x, y)d_\theta(x, y) &+ \kappa_2\alpha\theta(Tx, x)d_\theta(Tx, x) + \kappa_3\alpha\theta(Sy, y)d_\theta(Sy, y)\\ &= \tau\Big(\alpha\theta(x, y)d_\theta(x, y), \alpha\theta(Tx, x)d_\theta(Tx, x), \alpha\theta(Sy, y)d_\theta(Sy, y)\Big).\end{aligned}$$

Case 2. Taking $\zeta = \alpha\theta(y, x)d_\theta(y, x)$, $\eta = \alpha\theta(x, Tx)d_\theta(x, Tx)$ and $\varrho = \alpha\theta(y, Sy)d_\theta(y, Sy)\Big)$, for all $x, y \in X$ and $\alpha \in (0, 1)$, then we obtain

$$\begin{aligned}\kappa_1\alpha\theta(y, x)d_\theta(y, x) &+ \kappa_2\alpha\theta(x, Tx)d_\theta(x, Tx) + \kappa_3\alpha\theta(y, Sy)d_\theta(y, Sy)\\ &= \tau\Big(\alpha\theta(y, x)d_\theta(y, x), \alpha\theta(x, Tx)d_\theta(x, Tx), \alpha\theta(y, Sy)d_\theta(y, Sy)\Big).\end{aligned}$$

Hence, we conclude that every extended R-contraction is an extended A-contraction. This completes the proof. □

Corollary 3.8. *Let the pair (S, T) of self-mappings T and S on a complete extended quasi b- metric space (X, d_θ) be an extended B-contraction, where d_θ is continuous in its variables, $\alpha \in (0, 1)$ and θ is bounded by $\frac{1}{\alpha}$. Then T and S have a unique common fixed point in X provided that $\beta < \alpha$, where β is the constant that satisfies condition* (1) *of Definition* 3.6.

Proof. From Proposition 3.7, we know that every extended B-contraction is an extended A-contraction. Hence by Theorem 3.2, we get the desired result. □

Corollary 3.9. *Let the pair (S, T) of self-mappings T and S on a complete extended quasi b- metric space (X, d_θ) be an extended R-contraction, where d_θ is continuous in its variables, $\alpha \in (0, 1)$ and θ is bounded by $\frac{1}{\alpha}$. Then T and S have a unique common fixed point in X provided that $\kappa < \alpha$, where $\kappa = \max\left\{\frac{\kappa_1+\kappa_2}{1-\kappa_3}, \frac{\kappa_1+\kappa_3}{1-\kappa_2}, \frac{\kappa_2+\kappa_3}{1-\kappa_1}\right\}$ such that κ_1, κ_2 and κ_3 are the constants that satisfy condition* (2) *of Definition* 3.6.

Proof. From Proposition 3.7, we know that every extended R-contraction is an extended A-contraction. Hence by Theorem 3.2 we have the desired result. $\square$

4. Conclusion

Due to the large number of the active research papers and the significant development of metric spaces in parallel with the generalization of different type of contractions, we have introduced a new concept called "extended A-contraction" in the context of extended quasi b-metric spaces. Next, we made a generalization of the main result presented in [6] by M. Akram et al. Also, we have provided an example to illustrate and support our main result.

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