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GENERALIZED INTEGRAL TYPE F-CONTRACTION IN PARTIAL METRIC SPACES AND COMMON FIXED POINT

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Abstract. In this work, we study generalized integral type F-contractions in partial metric spaces and establish some common fixed point theorems. Also, we give some consequences of the established result. Our results extend and generalize several results from the existing literature.

1. Introduction

Fixed point theory is one of the most important topic in the development of nonlinear analysis. As it is well known, one of the most useful theorems in nonlinear analysis is the Banach contraction principle [7]. A mapping $T\colon \mathcal{M} \to \mathcal{M}$ where $(\mathcal{M}, d)$ is a metric space, is said to be a contraction if there exists

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$q \in [0, 1)$ such that for all $x, y \in \mathcal{M}$,

$$d(T(x), T(y)) \leq q\, d(x, y). \tag{1.1}$$

If the metric space $(\mathcal{M}, d)$ is complete then the mapping satisfying (1.1) has a unique fixed point and for every $u_0 \in \mathcal{M}$ a sequence $\{T^n u_0\}_{n\in\mathbb{N}}$ is convergent to the fixed point. Inequality (1.1) implies continuity of T. Many authors generalized this famous result in different ways. Afterwards, the crucial role of the principle in existence and uniqueness problems arising in mathematics has been realized which fact directed the researchers to extend and generalize the principle in many ways. Indeed, one of those ways is integral type contraction which was introduced by Branciari [8] in 2002 and proved a fixed point result for mappings defined on a complete metric space satisfying a general contractive type condition of integral type.

Matthews [14] introduced the concept of partial metric space as a part of the study of denotational semantics of dataflow networks [13, 14, 16, 23]. It is widely recognized that partial metric spaces play an important role in constructing models in the theory of computation. In partial metric spaces the distance of a point in the self may not be zero. Introducing partial metric space, Matthews extended the Banach contraction principle [7] and proved the fixed point theorem in this space.

In 2012, Wardowski [22] introduced a new type of contraction called F-contraction and proved a new fixed point theorem related to F-contraction and give an example to showing that the obtained extension is significant. Later, a large number of researchers have proved many results in this direction (for more details see the following articles: Acar *et al.* [2], Acar [3], Afassinou *et al.* [4], Ahmad *et al.*, [5], Kitkuan *et al.* [10], Mani *et al.* [12], Secelean [17], Shoaib *et al.* [18], Shukla and Radenovic [19], Tomar *et al.* [20], Younis *et al.* [24] and many others).

In this work, we study generalized integral type F-contraction in partial metric spaces and establish some common fixed point theorems. Our results extend and generalize several results in the existing literature.

2. Preliminaries

Now, we give some basic properties and results on the concept of partial metric space (PMS).

Definition 2.1. ([14]) Let $\mathcal{M}$ be a nonempty set and let $p\colon \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$ be a mapping such that for all $x, y, z \in \mathcal{M}$ the followings are satisfied:

(P1) $x = y$ if and only if $p(x, x) = p(x, y) = p(y, y)$,
(P2) $p(x, x) \leq p(x, y)$,

(P3) $p(x, y) = p(y, x)$,
(P4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

Then p is called a partial metric on $\mathcal{M}$ and the pair $(\mathcal{M}, p)$ is called a partial metric space (in short PMS).

Remark 2.2. It is clear that if $p(x, y) = 0$, then $x = y$. But, on the contrary $p(x, x)$ need not be zero.

Example 2.3. ([6]) Let $\mathcal{M} = \mathbb{R}^+$, where $\mathbb{R}^+ = [0, +\infty)$ and $p\colon \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$ given by $p(x, y) = \max\{x, y\}$ for all $x, y \in \mathbb{R}^+$. Then $(\mathbb{R}^+, p)$ is a partial metric space.

Example 2.4. ([6]) Let $\mathcal{M} = \{[a, b] : a, b \in \mathbb{R}, a \leq b\}$. Then $p\Big([a, b], [c, d]\Big) = \max\{b, d\} - \min\{a, c\}$ defines a partial metric p on $\mathcal{M}$.

Various applications of this space has been extensively investigated by many authors (see [11], [21] for details).

Remark 2.5. ([9]) Let $(\mathcal{M}, p)$ be a partial metric space.

(1) The function $d_m\colon \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$ defined as $d_m(x, y) = 2p(x, y) - p(x, x) - p(y, y)$ is a (usual) metric on $\mathcal{M}$ and $(\mathcal{M}, d_m)$ is a (usual) metric space.
(2) The function $d_s\colon \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$ defined as $d_s(x, y) = \max\{p(x, y) - p(x, x), p(x, y) - p(y, y)\}$ is a (usual) metric on $\mathcal{M}$ and $(\mathcal{M}, d_s)$ is a (usual) metric space.

Note also that each partial metric p on $\mathcal{M}$ generates a T_0 topology τ_p on $\mathcal{M}$, whose base is a family of open p-balls $\{B_p(x, \varepsilon) : x \in \mathcal{M}, \varepsilon > 0\}$ where $B_p(x, \varepsilon) = \{y \in \mathcal{M} : p(x, y) \leq p(x, x) + \varepsilon\}$ for all $x \in \mathcal{M}$ and $\varepsilon > 0$.

On a partial metric space the notions of convergence, the Cauchy sequence, completeness and continuity are defined as follows [13].

Definition 2.6. ([13]) Let $(\mathcal{M}, p)$ be a partial metric space. Then

(1) a sequence $\{x_n\}$ in $(\mathcal{M}, p)$ is said to be convergent to a point $x \in \mathcal{M}$ if $p(x, x) = \lim_{n\to\infty} p(x_n, x)$,
(2) a sequence $\{x_n\}$ is called a Cauchy sequence if $\lim_{m,n\to\infty} p(x_m, x_n)$ exists and is finite,
(3) $(\mathcal{M}, p)$ is said to be complete if every Cauchy sequence $\{x_n\}$ in $\mathcal{M}$ converges to a point $x \in \mathcal{M}$ with respect to τ_p. Furthermore,

$$\lim_{m,n\to\infty} p(x_m, x_n) = \lim_{n\to\infty} p(x_n, x) = p(x, x),$$

(4) A mapping $S\colon \mathcal{M} \to \mathcal{M}$ is said to be continuous at $y_0 \in \mathcal{M}$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$S\Big(B_p(y_0,\delta)\Big) \subset B_p\Big(S(y_0),\varepsilon\Big).$$

Definition 2.7. ([15]) Let $(\mathcal{M},p)$ be a partial metric space. Then

(1) a sequence $\{x_n\}$ in $(\mathcal{M},p)$ is called 0-Cauchy if $\lim_{m,n\to\infty} p(x_m,x_n) = 0$,

(2) $(\mathcal{M},p)$ is said to be 0-complete if every 0-Cauchy sequence $\{x_n\}$ in $\mathcal{M}$ converges to a point $x \in \mathcal{M}$, such that $p(x,x) = 0$.

Lemma 2.8. ([13, 14]) *Let $(\mathcal{M},p)$ be a partial metric space. Then*

(a1) *a sequence $\{x_n\}$ in $(\mathcal{M},p)$ is a Cauchy sequence if and only if it is a Cauchy sequence in the metric space $(\mathcal{M},d_m)$,*

(a2) *$(\mathcal{M},p)$ is complete if and only if the metric space $(\mathcal{M},d_m)$ is complete,*

(a3) *a subset E of a partial metric space $(\mathcal{M},p)$ is closed if a sequence $\{x_n\}$ in E such that $\{x_n\}$ converges to some $x \in \mathcal{M}$, then $x \in E$.*

Lemma 2.9. ([1]) *Assume that $x_n \to u$ as $n \to \infty$ in a partial metric space $(\mathcal{M},p)$ such that $p(u,u) = 0$. Then $\lim_{n\to\infty} p(x_n,y) = p(u,y)$ for every $y \in \mathcal{M}$.*

Definition 2.10. ([22]) Let $F\colon [0,\infty) \to \mathbb{R}$ be a mapping satisfying:

(**F1**) F is strictly increasing, that is, for all $\alpha,\beta \in [0,\infty)$ such that $\alpha < \beta$, $F(\alpha) < F(\beta)$.

(**F2**) For each sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of positive numbers $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$.

(**F3**) There exists $k \in (0,1)$ such that $\lim_{\alpha\to\ 0^+} \alpha^k F(\alpha) = 0$.

We denote by $\mathcal{F}$, the set of all functions satisfying the conditions (**F1**)-(**F3**).

Definition 2.11. ([22]) A mapping $T\colon \mathcal{M} \to \mathcal{M}$ is said to be an F-contraction if there exists $\tau > 0$ such that

$$\forall\, x,y \in \mathcal{M},\ \big\{(d(Tx,Ty)) > 0 \Rightarrow \tau + F(d(Tx,Ty)) \le F(d(x,y))\big\}. \tag{2.1}$$

Theorem 2.12. ([22]) *Let $(\mathcal{M},d)$ be a complete metric space and let $T\colon \mathcal{M} \to \mathcal{M}$ be an F-contraction. Then T has a unique fixed point in $\mathcal{M}$.*

Example 2.13. ([22]) Let $F\colon [0,\infty) \to \mathbb{R}$ be given by $F(\alpha) = \ln\alpha$. Then F satisfies $(F1)$-$(F3)$. Each mapping satisfying (2.1) is an F-contraction such that

$$d(Tx,Ty) \le e^{-\tau}\, d(x,y), \tag{2.2}$$

for all $x, y \in \mathcal{M}$, $Tx \neq Ty$. It is clear that for $x, y \in \mathcal{M}$ such that $Tx = Ty$ the inequality $d(Tx, Ty) \leq e^{-\tau}\, d(x, y)$ also holds, that is, T is a Banach contraction [7].

3. Main results

In this section, we shall prove some unique common fixed point theorems in the setting of complete partial metric spaces via generalized integral type F-contraction.

Theorem 3.1. *Let $(\mathcal{M}, p)$ be a complete partial metric space and let $S, T\colon \mathcal{M} \to \mathcal{M}$ be two self-mappings. Suppose that there exist $F \in \mathcal{F}$ and $\tau > 0$ such that for all $x, y \in \mathcal{M}$ satisfying $p(Sx, Ty) > 0$, the following holds:*

$$\tau + F\Big(\int_0^{p(Sx,Ty)} \phi(t)dt\Big) \leq F\Big(\int_0^{\mathcal{M}(x,y)} \phi(t)dt\Big), \tag{3.1}$$

where

$$\mathcal{M}(x, y) = \max\Big\{p(x, y), \frac{1}{2}[p(x, Sx) + (y, Ty)], \frac{1}{3}[p(x, y) + p(x, Ty) + p(y, Sx)]\Big\}$$

and $\phi\colon [0, \infty) \to [0, \infty)$ is a Lebesgue integrable mapping which is summable on each compact subset of $[0, \infty)$, nonnegative and for each $\varepsilon > 0$

$$\int_0^{\varepsilon} \phi(t)dt > 0, \tag{3.2}$$

and F is continuous. Then S and T have a unique common fixed point in $\mathcal{M}$.

Proof. Let $y_0 \in \mathcal{M}$ be an arbitrary point. Define a sequence $\{y_n\}$ for $n \geq 0$ by

$$y_{2n+1} = Sy_{2n} \text{ and } y_{2n+2} = Ty_{2n+1}. \tag{3.3}$$

Step I. Now, we have to prove that $p(y_{n+1}, y_n) \to 0$ as $n \to \infty$. By equation (3.1), we have

$$\begin{aligned} \tau + F\Big(\int_0^{p(y_{2n+1},y_{2n})} \phi(t)dt\Big) &= \tau + F\Big(\int_0^{p(Sy_{2n},Ty_{2n-1})} \phi(t)dt\Big) \\ &\leq F\Big(\int_0^{\mathcal{M}(y_{2n},y_{2n-1})} \phi(t)dt\Big), \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} \mathcal{M}(y_{2n}, y_{2n-1}) = \max\Big\{&p(y_{2n}, y_{2n-1}), \frac{1}{2}[p(y_{2n}, Sy_{2n}) + p(y_{2n-1}, Ty_{2n-1})], \\ &\frac{1}{3}[p(y_{2n}, y_{2n-1}) + p(y_{2n}, Ty_{2n-1}) + p(y_{2n-1}, Sy_{2n})]\Big\} \end{aligned}$$

$$\begin{aligned} &= \max\Big\{p(y_{2n}, y_{2n-1}), \frac{1}{2}[p(y_{2n}, y_{2n+1}) + p(y_{2n-1}, y_{2n})], \\ &\quad \frac{1}{3}[p(y_{2n}, y_{2n-1}) + p(y_{2n}, y_{2n}) + p(y_{2n-1}, y_{2n+1})]\Big\} \\ &\leq \max\Big\{p(y_{2n-1}, y_{2n}), \frac{1}{2}[p(y_{2n+1}, y_{2n}) + (y_{2n-1}, y_{2n})], \\ &\quad \frac{1}{3}[p(y_{2n-1}, y_{2n}) + p(y_{2n}, y_{2n}) + p(y_{2n-1}, y_{2n}) \\ &\quad + p(y_{2n+1}, y_{2n}) - p(y_{2n}, y_{2n})]\Big\}. \end{aligned} \tag{3.5}$$

If $\max\Big\{p(y_{2n-1}, y_{2n}), p(y_{2n+1}, y_{2n})\Big\} = p(y_{2n+1}, y_{2n})$, then it follows from (3.4)

$$\tau + F\Big(\int_0^{p(y_{2n+1},y_{2n})} \phi(t)dt\Big) \leq F\Big(\int_0^{p(y_{2n+1},y_{2n})} \phi(t)dt\Big), \tag{3.6}$$

which is a contradiction (as $\tau > 0$). Thus,

$$\max\Big\{p(y_{2n-1}, y_{2n}), p(y_{2n+1}, y_{2n})\Big\} = p(y_{2n-1}, y_{2n}). \tag{3.7}$$

From equation (3.4), we have

$$F\Big(\int_0^{p(y_{2n+1},y_{2n})} \phi(t)dt\Big) \leq F\Big(\int_0^{p(y_{2n-1},y_{2n})} \phi(t)dt\Big) - \tau. \tag{3.8}$$

Continuing in the same way, we obtain

$$F\Big(\int_0^{p(y_{2n-1},y_{2n})} \phi(t)dt\Big) \leq F\Big(\int_0^{p(y_{2n-2},y_{2n-1})} \phi(t)dt\Big) - \tau. \tag{3.9}$$

Using (3.8) and (3.9), we get

$$\begin{aligned} F\Big(\int_0^{p(y_{2n+1},y_{2n})} \phi(t)dt\Big) &\leq F\Big(\int_0^{p(y_{2n},y_{2n-1})} \phi(t)dt\Big) - \tau \\ &\leq F\Big(\int_0^{p(y_{2n-1},y_{2n-2})} \phi(t)dt\Big) - 2\tau \\ &\leq \cdots \\ &\leq F\Big(\int_0^{p(y_1,y_0)} \phi(t)dt\Big) - (2n)\tau. \end{aligned} \tag{3.10}$$

Then, it follows $\lim_{n\to\infty} F\Big(\int_0^{p(y_{n+1},y_n)} \phi(t)dt\Big) = -\infty$. By $F \in \mathcal{F}$ and (**F2**), we have

$$\lim_{n\to\infty} p(y_{n+1}, y_n) = 0. \tag{3.11}$$

Step II. Now, we have to show that $\{y_n\}$ is a p-Cauchy sequence. Put $t_n = p(y_{n+1}, y_n)$, $n = 0, 1, 2, \ldots$. By $F \in \mathcal{F}$ and (**F3**), there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} (t_n)^k F(t_n) = 0. \tag{3.12}$$

By (3.10), we have

$$\Big(p(y_{2n+1}, y_{2n})\Big)^k \Big[F\Big(\int_0^{p(y_{2n+1},y_{2n})} \phi(t)dt\Big) - F\Big(\int_0^{p(y_1,y_0)} \phi(t)dt\Big)\Big]$$

$$\leq \; -(2n)\Big(p(y_{2n+1}, y_{2n})\Big)^k \tau \leq 0. \tag{3.13}$$

Using the above inequality and (3.12), we get

$$\lim_{n\to\infty} n\Big(p(y_{n+1}, y_n)\Big)^k = 0. \tag{3.14}$$

Therefore, there exists a positive integer $n_1 \in \mathbb{N}$ such that

$$n\Big(p(y_{n+1}, y_n)\Big)^k < 1$$

for all $n > n_1$, or

$$p(y_{n+1}, y_n) < \frac{1}{n^{1/k}}. \tag{3.15}$$

Let $m, n \in \mathbb{N}$ with $m > n > n_1$, using $(P4)$ (triangular inequality), we have

$$\begin{aligned} p(y_n, y_m) &\leq p(y_n, y_{n+1}) + p(y_{n+1}, y_{n+2}) + \cdots + p(y_{m-1}, y_m) \\ &\quad -[p(y_{n+1}, y_{n+1}) + p(y_{n+2}, y_{n+2}) + \cdots + p(y_{m-1}, y_{m-1})] \\ &\leq p(y_n, y_{n+1}) + p(y_{n+1}, y_{n+2}) + \cdots + p(y_{m-1}, y_m) \\ &= \sum_{i=n}^{m-1} p(y_{i+1}, y_i) \leq \sum_{i=n}^{\infty} p(y_{i+1}, y_i) \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}. \end{aligned} \tag{3.16}$$

Since $k \in (0, 1)$, the series $\sum_{i=n}^{\infty} \left(\frac{1}{i^{1/k}}\right)$ is convergent, so

$$\lim_{n,m\to\infty} p(y_n, y_m) = 0. \tag{3.17}$$

Thus $\{y_n\}$ is a Cauchy sequence in $(\mathcal{M}, p)$. Therefore, $\{y_n\}$ is a Cauchy sequence in $(\mathcal{M}, d_m)$. Since $(\mathcal{M}, p)$ is complete partial metric space, by Lemma 2.8, $(\mathcal{M}, d_m)$ is also complete. Thus, there exists a $u \in \mathcal{M}$ such that $\lim_{n\to\infty} y_n$

$= u$ and $\lim_{n\to\infty} d_m(y_n, u) = 0$. Moreover, by Definition 2.6 (3) and equation (3.17), we have

$$p(u,u) = \lim_{n\to\infty} p(y_n,u) = \lim_{n,m\to\infty} p(y_n,y_m) = 0. \tag{3.18}$$

Step III. Now, we shall show that u is a common fixed point of S and T. Using given contractive condition (3.1) for $x = y_{2n}$ and $y = u$, we have

$$\begin{aligned} \tau + F\Big(\int_0^{p(y_{2n+1},Tu)} \phi(t)dt\Big) &= \tau + F\Big(\int_0^{p(Sy_{2n},Tu)} \phi(t)dt\Big) \\ &\le F\Big(\int_0^{\mathcal{M}(y_{2n},u)} \phi(t)dt\Big), \end{aligned} \tag{3.19}$$

where

$$\begin{aligned} \mathcal{M}(y_{2n},u) &= \max\Big\{p(y_{2n},u), \frac{1}{2}[p(y_{2n},Sy_{2n}) + p(u,Tu)], \\ &\qquad \frac{1}{3}[p(y_{2n},u) + p(y_{2n},Tu) + p(u,Sy_{2n})]\Big\} \\ &= \max\Big\{p(y_{2n},u), \frac{1}{2}[p(y_{2n},y_{2n+1}) + p(u,Tu)], \\ &\qquad \frac{1}{3}[p(y_{2n},u) + p(y_{2n},Tu) + p(u,y_{2n+1})]\Big\}. \end{aligned} \tag{3.20}$$

Passing to the limit as $n \to \infty$ in (3.20) and using (3.18), we obtain

$$\mathcal{M}(y_{2n},u) \to \max\Big\{0, \frac{p(u,Tu)}{2}, \frac{p(u,Tu)}{3}\Big\} = \frac{p(u,Tu)}{2} < p(u,Tu). \tag{3.21}$$

Now, using (3.19) and (3.21), we get

$$\tau + F\Big(\int_0^{p(y_{2n+1},Tu)} \phi(t)dt\Big) \le F\Big(\int_0^{p(u,Tu)} \phi(t)dt\Big). \tag{3.22}$$

Passing to the limit as $n \to \infty$ in (3.22) and using continuity of F, we obtain

$$\tau + F\Big(\int_0^{p(u,Tu)} \phi(t)dt\Big) \le F\Big(\int_0^{p(u,Tu)} \phi(t)dt\Big),$$

which is a contradiction since $\tau > 0$. Thus, we have $Tu = u$. This shows that u is a fixed point of T. By similar fashion we can show that $Su = u$. Hence u is a common fixed point of S and T.

Step IV. We now show uniqueness of common fixed point. Assume that u' is another common fixed point of S and T, that is, $Su' = u' = Tu'$ with

$u \neq u'$. From the given contractive condition (3.1), we have

$$\begin{aligned}\tau + F\Big(\int_0^{p(u,u')} \phi(t)dt\Big) &= F\Big(\int_0^{p(Su,Tu')} \phi(t)dt\Big) \\ &\leq F\Big(\int_0^{\mathcal{M}(u,u')} \phi(t)dt\Big), \end{aligned} \tag{3.23}$$

where

$$\begin{aligned}\mathcal{M}(u,u') &= \max\Big\{p(u,u'), \frac{1}{2}[p(u,Su)+p(u',Tu')], \\ &\qquad \frac{1}{3}[p(u,u')+p(u,Tu')+p(u',Su)]\Big\} \\ &= \max\Big\{p(u,u'), \frac{1}{2}[p(u,u)+p(u',u')], \\ &\qquad \frac{1}{3}[p(u,u')+p(u,u')+p(u',u)]\Big\}. \end{aligned} \tag{3.24}$$

Using condition $(P3)$ and (3.18), we get

$$\mathcal{M}(u,u') \to \max\Big\{p(u,u'), 0, p(u,u')\Big\} = p(u,u'). \tag{3.25}$$

From (3.23) and (3.25), we obtain

$$\tau + F\Big(\int_0^{p(u,u')} \phi(t)dt\Big) \leq F\Big(\int_0^{p(u,u')} \phi(t)dt\Big), \tag{3.26}$$

which is a contradiction since $\tau > 0$. Thus, we have $u = u'$. This shows that the common fixed point of S and T is unique. This completes the proof. □

Theorem 3.2. *Let $(\mathcal{M}, p)$ be a complete partial metric space and let $S, T\colon \mathcal{M} \to \mathcal{M}$ be two self-mappings. Suppose that there exist $F \in \mathcal{F}$ and $\tau > 0$ such that for all $x, y \in \mathcal{M}$ satisfying $p(Sx, Ty) > 0$, the following holds:*

$$\tau + F\Big(\int_0^{p(Sx,Ty)} \phi(t)dt\Big) \leq F\Big(\int_0^{\mathcal{M}(x,y)} \phi(t)dt\Big),$$

where $\mathcal{M}(x,y)$ and ϕ are as in Theorem 3.1. If the following hold:

(i) *S or T is continuous or*

(ii *F is continuous.*

Then S and T have a unique common fixed point in $\mathcal{M}$.

Proof. Let $y_0 \in \mathcal{M}$ be an arbitrary point. Define a sequence $\{y_n\}$ for $n \geq 0$ by

$$y_{2n+1} = Sy_{2n} \text{ and } y_{2n+2} = Ty_{2n+1}.$$

Step I. We have to prove that $p(y_{n+1}, y_n) \to 0$ as $n \to \infty$. The proof is similar to that Step I of Theorem 3.1.

We will now prove that S and T have a common fixed point. Since $(\mathcal{M}, p)$ is complete partial metric space, from Lemma 2.8, $(\mathcal{M}, d_m)$ is also complete. Thus, there exists a $u \in \mathcal{M}$ such that $\lim_{n\to\infty} y_n = u$ and $\lim_{n\to\infty} d_m(y_n, u) = 0$. Moreover, by Definition 2.6 (3) and equation (3.17), we have

$$p(u,u) = \lim_{n\to\infty} p(y_n, u) = \lim_{n,m\to\infty} p(y_n, y_m) = 0.$$

Now, we consider the following two cases.

Case 1. Suppose S is continuous. Then $u = \lim_{n\to\infty} y_n = \lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} Sy_{2n} = Su$. Thus u is a fixed point of S.

Now, we have to prove that u is a fixed point of T. On the contrary, we assume that $Tu \neq u$. From (3.1), we have

$$\tau + F\Big(\int_0^{p(Sy_{2n}, Tu)} \phi(t)dt\Big) \le F\Big(\int_0^{\mathcal{M}(y_{2n}, u)} \phi(t)dt\Big), \tag{3.27}$$

where

$$\begin{aligned} \mathcal{M}(y_{2n}, u) &= \max\Big\{p(y_{2n}, u), \frac{1}{2}[p(y_{2n}, Sy_{2n}) + p(u, Tu)], \\ &\qquad \frac{1}{3}[p(y_{2n}, u) + p(y_{2n}, Tu) + p(u, Sy_{2n})]\Big\} \\ &= \max\Big\{p(y_{2n}, u), \frac{1}{2}[p(y_{2n}, y_{2n+1}) + p(u, Tu)], \\ &\qquad \frac{1}{3}[p(y_{2n}, u) + p(y_{2n}, Tu) + p(u, y_{2n+1})]\Big\}. \end{aligned} \tag{3.28}$$

Taking the limit as $n \to \infty$ in (3.28) and using (3.18), we obtain

$$\mathcal{M}(y_{2n}, u) \to \max\Big\{0, \frac{p(u,Tu)}{2}, \frac{p(u,Tu)}{3}\Big\} = \frac{p(u,Tu)}{2} < p(u,Tu). \tag{3.29}$$

Thus, from (3.27) and (3.29), we get

$$\tau + F\Big(\int_0^{p(u,Tu)} \phi(t)dt\Big) \le F\Big(\int_0^{p(u,Tu)} \phi(t)dt\Big),$$

which is a contradiction since $\tau > 0$. Thus, we have $Tu = u$.

Similarly, we find the same result when T is continuous.

Case 2. Now, we suppose that F is continuous. We can assume that there exists a positive integer $m_1 \in \mathbb{N}$ such that $Sy_{n+1} \neq u$ (that is, $p(y_n, u) > 0$) for all $n \geq m_1$. Then from equation (3.1), we have

$$\tau + F\Big(\int_0^{p(Su, Ty_{2n+1})} \phi(t)dt\Big) \le F\Big(\int_0^{\mathcal{M}(u, y_{2n+1})} \phi(t)dt\Big), \tag{3.30}$$

where

$$\begin{aligned}\mathcal{M}(u, y_{2n+1}) &= \max\Big\{p(u, y_{2n+1}), \frac{1}{2}[p(u, Su) + p(y_{2n+1}, Ty_{2n+1})], \\ &\quad \frac{1}{3}[p(u, y_{2n+1}) + p(u, Ty_{2n+1}) + p(y_{2n+1}, Su)]\Big\} \\ &= \max\Big\{p(u, y_{2n+1}), \frac{1}{2}[p(u, Su) + p(y_{2n+1}, y_{2n+2})], \\ &\quad \frac{1}{3}[p(u, y_{2n+1}) + p(u, y_{2n+2}) + p(y_{2n+1}, Su)]\Big\}. \quad (3.31)\end{aligned}$$

Taking the limit as $n \to \infty$ in (3.31) and using (3.18), we obtain

$$\mathcal{M}(u, y_{2n+1}) \to \max\Big\{0, \frac{p(u, Su)}{2}, \frac{p(u, Su)}{3}\Big\} = \frac{p(u, Su)}{2} < p(u, Su). \quad (3.32)$$

Now, from (3.30) and (3.32), we have

$$\tau + F\Big(\int_0^{p(Su, Ty_{2n+1})} \phi(t)dt\Big) \le F\Big(\int_0^{p(u, Su)} \phi(t)dt\Big). \quad (3.33)$$

Since F is continuous, passing to the limit as $n \to \infty$ in (3.33), we get

$$\tau + F\Big(\int_0^{p(Su, u)} \phi(t)dt\Big) \le F\Big(\int_0^{p(u, Su)} \phi(t)dt\Big),$$

which is a contradiction since $\tau > 0$. Therefore, we have $u = Su$ and u is a fixed point of S.

Now, we show that u is a fixed point of T. Using equation (3.1), we have

$$\begin{aligned}\tau + F\Big(\int_0^{p(u, Tu)} \phi(t)dt\Big) &= \tau + F\Big(\int_0^{p(Su, Tu)} \phi(t)dt\Big) \\ &\le F\Big(\int_0^{\mathcal{M}(u, u)} \phi(t)dt\Big), \quad (3.34)\end{aligned}$$

where

$$\begin{aligned}\mathcal{M}(u, u) &= \max\Big\{p(u, u), \frac{1}{2}[p(u, Su) + p(u, Tu)], \\ &\quad \frac{1}{3}[p(u, u) + p(u, Tu) + p(u, Su)]\Big\} \\ &= \max\Big\{p(u, u), \frac{1}{2}[p(u, u) + p(u, Tu)], \\ &\quad \frac{1}{3}[p(u, u) + p(u, Tu) + p(u, u)]\Big\}.\end{aligned}$$

Using equation (3.18), we have

$$\mathcal{M}(u, u) \to \max\Big\{0, \frac{p(u, Tu)}{2}, \frac{p(u, Tu)}{3}\Big\} = \frac{p(u, Tu)}{2} < p(u, Tu). \quad (3.35)$$

From (3.34) and (3.35), we obtain

$$\tau + F\Big(\int_0^{p(u,Tu)} \phi(t)dt\Big) \leq \Big(\int_0^{p(u,Tu)} \phi(t)dt\Big),$$

which is a contradiction since $\tau > 0$. Hence, $u = Tu$. Thus u is a common fixed point of S and T.

The proof of uniqueness of the common fixed point is same as that of Theorem 3.1. This completes the proof. □

4. Consequences of theorem 3.1 and theorem 3.2

Corollary 4.1. *Let $(\mathcal{M}, p)$ be a complete partial metric space and let $T\colon \mathcal{M} \to \mathcal{M}$ be a self-mapping. Suppose that there exist $F \in \mathcal{F}$ and $\tau > 0$ such that for all $x, y \in \mathcal{M}$ satisfying $p(Tx, Ty) > 0$, the following holds:*

$$\tau + F\Big(\int_0^{p(Tx,Ty)} \phi(t)dt\Big) \leq F\Big(\int_0^{\mathcal{M}_1(x,y)} \phi(t)dt\Big),$$

where

$$\mathcal{M}_1(x, y) = \max\Big\{p(x, y), \frac{1}{2}[p(x, Tx)+(y, Ty)], \frac{1}{3}[p(x, y)+p(x, Ty)+p(y, Tx)]\Big\},$$

and $\phi\colon [0, \infty) \to [0, \infty)$ is a Lebesgue integrable mapping which is summable on each compact subset of $[0, \infty)$, nonnegative and for each $\varepsilon > 0$

$$\int_0^{\varepsilon} \phi(t)dt > 0,$$

and F is continuous. Then T has a unique fixed point in $\mathcal{M}$.

Corollary 4.2. *Let $(\mathcal{M}, p)$ be a complete partial metric space and let $T\colon \mathcal{M} \to \mathcal{M}$ be a self-mapping. Suppose that there exist $F \in \mathcal{F}$ and $\tau > 0$ such that for all $x, y \in \mathcal{M}$ satisfying $p(Tx, Ty) > 0$, the following holds:*

$$\tau + F\Big(\int_0^{p(Tx,Ty)} \phi(t)dt\Big) \leq F\Big(\int_0^{p(x,y)} \phi(t)dt\Big),$$

where $\phi\colon [0, \infty) \to [0, \infty)$ is a Lebesgue integrable mapping which is summable on each compact subset of $[0, \infty)$, nonnegative and for each $\varepsilon > 0$

$$\int_0^{\varepsilon} \phi(t)dt > 0,$$

and F is continuous. Then T has a unique fixed point in $\mathcal{M}$.

Corollary 4.3. *Let $(\mathcal{M}, p)$ be a complete partial metric space and let $T\colon \mathcal{M} \to \mathcal{M}$ be a self-mapping. Suppose that there exist $F \in \mathcal{F}$ and $\tau > 0$ such that for all $x, y \in \mathcal{M}$ satisfying $p(Tx, Ty) > 0$, the following holds:*

$$\tau + F\Big(\int_0^{p(Tx,Ty)} \phi(t)dt\Big) \leq F\Big(\int_0^{\frac{1}{2}[p(x,Tx)+(y,Ty)]} \phi(t)dt\Big),$$

where $\phi\colon [0,\infty) \to [0,\infty)$ is a Lebesgue integrable mapping which is summable on each compact subset of $[0,\infty)$, nonnegative and for each $\varepsilon > 0$

$$\int_0^{\varepsilon} \phi(t)dt > 0,$$

and F is continuous. Then T has a unique fixed point in $\mathcal{M}$.

Corollary 4.4. *Let $(\mathcal{M}, p)$ be a complete partial metric space and $S, T\colon \mathcal{M} \to \mathcal{M}$ be two self-mappings. Suppose that there exist $F \in \mathcal{F}$ and $\tau > 0$ such that for all $x, y \in \mathcal{M}$ satisfying $p(Sx, Ty) > 0$, the following holds:*

$$\tau + F\Big(\int_0^{p(Sx,Ty)} \phi(t)dt\Big) \leq F\Big(\int_0^{p(x,y)} \phi(t)dt\Big),$$

where ϕ is as in Theorem 3.1. If the following hold:

(i) *S or T is continuous or*
(ii) *F is continuous.*

Then S and T have a unique common fixed point in $\mathcal{M}$.

We give an example to support the result.

Example 4.5. Let $\mathcal{M} = [0,1]$ and $p(x,y) = \max\{x,y\}$ for all $x, y \in \mathcal{M}$. Then $(\mathcal{M}, p)$ complete partial metric space. Let $S, T\colon \mathcal{M} \to \mathcal{M}$ and $\phi\colon (0,\infty) \to (0,\infty)$ be defined by $S(x) = x$, $T(x) = 0$ and $\phi(t) = 2t$ for all $t \geq 0$. Let $F\colon [0,\infty) \to \mathbb{R}$ be given by $F(\alpha) = \ln \alpha$. Then all conditions of Theorem 3.1 and the contractive condition (3.1) are satisfied for some $\tau > 0$ and for $p(x,y) > 0$.

If $x > y$, then we have

$$\begin{aligned} \tau + F\Big(\int_0^{p(Sx,Ty)} \phi(t)dt\Big) &= \tau + \ln(x^2) \leq ln\,(x^2) \\ &= F\Big(\int_0^{\mathcal{M}(x,y)} \phi(t)dt\Big). \end{aligned}$$

If $x < y$, then we have

$$\begin{aligned} \tau + F\Big(\int_0^{p(Sx,Ty)} \phi(t)dt\Big) &= \tau + \ln(x^2) < \tau + \ln(y^2) \\ &\leq \ln(y^2) = F\Big(\int_0^{\mathcal{M}(x,y)} \phi(t)dt\Big). \end{aligned}$$

Hence $0 \in \mathcal{M}$ is a common fixed point of S and T.

5. Conclusion

In this article, we establish some unique common fixed point theorems via generalized integral type F-contraction in the setting of complete partial metric spaces and give some consequences as corollaries of the established results. Also we give an example to support the result. The results obtained in this article generalize and extend several results from the existing literature.

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