Nonlinear Functional Analysis and Applications
Vol. 28, No. 1 (2023), pp. 81-93
ISSN: 1229-1595(print), 2466-0973(online)

https://doi.org/10.22771/nfaa.2023.28.01.05
http://nfaa.kyungnam.ac.kr/journal-nfaa

STUDY ON UNIFORMLY CONVEX AND UNIFORMLY STARLIKE MULTIVALENT FUNCTIONS ASSOCIATED WITH LIBERA INTEGRAL OPERATOR

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Abstract. By utilizing a certain Libera integral operator considered on analytic multivalent functions in the unit disk U. Using the hypergeometric function and the Libera integral operator, we included a new convolution operator that expands on some previously specified operators in U, which broadens the scope of certain previously specified operators. We introduced and investigated the properties of new subclasses of functions $f(z) \in A_p$ using this operator.

1. Introduction

Let A_p signify the class of all analytic multivalent functions of the form:

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p}, \ (p \in N := \{1,\ 2,\ 3,\ ...\},\ z \in U) \tag{1.1}$$

which are analytic in the unit disc $U := \{z \in: |z| < 1\}$. We denote by S the subclass of univalent functions $f(z)$ in A_p. For $(0 \leq \beta < p)$, we denote

[0]Received January 13, 2022. Revised April 30, 2022. Accepted May 23, 2022.
[0]2020 Mathematics Subject Classification: 30C45, 30C50.
[0]Keywords: Multivalent functions, uniformly k-starlike, uniformly k-convex functions, hypergeometric function, Libera integral operator.
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by $S_p^*(\beta)$ and $C_p(\beta)$ the subclasses of A_p consisting of all analytic functions which are, respectively, starlike of order β and convex of order β in U.

For functions $f(z)$ given by (1.1) and $g(z)$ given by

$$g(z) = z + \sum_{n=1}^{\infty} b_n z^n, \quad (z \in U), \tag{1.2}$$

the convolution (or Hadamard product), denoted by $f * g$ of the functions f and g is defined by

$$(f * g)(z) = z + \sum_{n=1}^{\infty} a_n b_n z^n = (g * f)(z) \qquad (z \in U). \tag{1.3}$$

In 1965, Libera [18] had studied an operator called the Libera integral operator $L : A \to A$ defined by:

$$L(z) = \frac{2}{z} \int_0^z f(t)\,dt = z + \sum_{n=1}^{\infty} \frac{2}{n+1} a_n z^n. \tag{1.4}$$

An integral operator was one such operator which has attracted many researchers. Later Kumar and Shukla [17], Bhoosnurmath and Swamy [8] and Noor and Noor [20] have studied certain types of integral operators. For more details about the properties of integral operators, one can refer [4], [5], [9], [10], [16], [19], [26] and [29].

In this paper, we introduce the operator $L_p : A_p \to A_p$ defined by

$$\begin{aligned} L_p(z) &= \frac{(p+1)^{\alpha}}{z^p \Gamma(\alpha)} \int_0^z \left(\log \frac{z^p}{t} \right)^{\alpha - 1} f(t)\,dt \\ &= z^p + \sum_{n=1}^{\infty} \left(\frac{p+1}{n+p+1} \right)^{\alpha} a_{n+p} z^{n+p}. \end{aligned} \tag{1.5}$$

When $p = 1$, equation (1.5) studied by [6], [7] and [16]. If $p = \alpha = 1$ we get back to Libera integral operator.

Let Δ_p be defined as the function $\Delta_p(a, c; z)$ by

$$\Delta_p(a, c; z) = z^p + \sum_{n=1}^{\infty} \frac{(a)_n}{(c)_n} z^{n+p}, \tag{1.6}$$

for $c \neq 0, -1, -2, ...,$ *and* $a \in \mathbb{C} \setminus \{0\}$, $p \in N = 1, 2, 3, ...,$ where $(\lambda)_n$ is the Pochhammer symbol which is defined by

$$(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \lambda(\lambda + 1)...(\lambda + n - 1), \tag{1.7}$$

for $n = 1, 2, 3, ...,$ and $(\lambda)_0 = 1$. It should be noted that

$$\Delta_p (a, c; z) = z^p \, {}_2F_1 (a, 1, c; z), \tag{1.8}$$

where

$$F (a, 1, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (1)_n}{(c)_n (1)_n} z^n = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^n.$$

Corresponding to the function $\Delta_p (a, c; z)$, we define a new linear operator $\Omega_p (a, c) f (z)$ on A_p by the convolution product for $\Delta_p (a, c; z)$ and L_p given in (1.5), we obtain

$$\begin{aligned} \Omega_{p,\alpha}(a, c)f(z) &= (\Delta_p * L_p) f (z) \\ &= z^p + \sum_{n=2}^{\infty} \left(\frac{p+1}{n+p+1} \right)^{\alpha} \frac{(a)_n}{(c)_n} a_{n+p} z^{n+p} \end{aligned} \tag{1.9}$$

for $c \neq 0, -1, -2, ...,$ and $a \in C \setminus \{0\}, \, p \in N, \; \alpha \in N = 1, 2, 3, ...$

Using the definition of hypergeometric functions, the Hadamard product principle, and the definitions of the classes of uniformly k-starlike function $S^* (\beta, k)$ and the class of uniformly k-convex $C (\beta, k)$ function which are introduced and investigated by Gooodman [15], [16] and Rønning [25], [26], in this paper we will define new subclasses of multivalent hypergeometric functions $f \in A_p$ and study their properties.

Let $f \in A_p$ denote the subclass of A_p satisfying

$$\operatorname{Re}\left\{ \frac{z(\Omega_{p,\alpha} (a, c) f (z))' + \gamma z^2 (\Omega_{p,\alpha} (a, c) f (z))''}{(1-\gamma) \Omega_{p,\alpha} (a, c) f (z) + \gamma z (\Omega_{p,\alpha} (a, c) f (z))'} - \beta \right\} \tag{1.10}$$

$$> k \left| \frac{z(\Omega_{p,\alpha} (a, c) f (z))' + \gamma z^2 (\Omega_{p,\alpha} (a, c) f (z))''}{(1-\gamma) \Omega_{p,\alpha} (a, c) f (z) + \gamma z (\Omega_{p,\alpha} (a, c) f (z))'} - 1 \right|, \; z \in U,$$

where $-1 \leq \beta < 1, \; 0 \leq \gamma \leq 1, \, \alpha \in N$ and $k \geq 0$.

By appropriately specializing the values of $\alpha, \; \gamma, \; (a)$ and (c) the class given in (1.10) can be reduced to the class investigated by many researchers, see for example, [1], [2], [3], [10], [11], [12], [13], [14], [15], [21], [22], [23], [25], [27] and [28].

The primary goal of this paper is to investigate the coefficient bounds, extreme points, and radius of starlikeness for functions in the generalized class (1.10).

2. Characterization and other related properties

Our first conclusion provides a sufficient condition for $f(z) \in A_p$ which are analytic in U to be in $\Omega^k_{p,\alpha}(a,c,\beta,\gamma)$.

Theorem 2.1. *A function $f(z)$ of the form* (1.1) *is in $\Omega^k_{p,\alpha}(a,c,\beta,\gamma)$, if*

$$\sum_{n=1}^{\infty}[1+\gamma(n+p-1)][(k+1)(n+p)-(\beta+k)]\left(\frac{p+1}{n+p+1}\right)^{\alpha}\frac{(a)_n}{(c)_n}|a_{n+p}|$$
$$\leq(1-\beta)(1-\gamma+\gamma p)-(k+1)[p+p(p-1)\gamma-(1-\gamma+\gamma p)]. \quad (2.1)$$

Proof. Suppose that (2.1) is true for $-1 \leq \beta < 1$, $0 \leq \gamma \leq 1$, $\alpha \in N$ and $k \geq 0$, in order to prove that $f \in \Omega^k_{p,\alpha}(a,c,\beta,\gamma)$. It suffices to show that (1.10) is bounded by $1-\beta$, that is,

$$k\left|\frac{z(\Omega_{p,\alpha}(a,c)(a,c)f(z))'+\gamma z^2(\Omega_{p,\alpha}(a,c)f(z))''}{(1-\gamma)\Omega_{p,\alpha}(a,c)f(z)+\gamma z(\Omega_{p,\alpha}(a,c)f(z))'}-1\right|$$
$$-\operatorname{Re}\left\{\frac{z(\Omega_{p,\alpha}(a,c)f(z))'+\gamma z^2(\Omega_{p,\alpha}(a,c)f(z))''}{(1-\gamma)\Omega_{p,\alpha}(a,c)f(z)+\gamma z(\Omega_{p,\alpha}(a,c)f(z))'}-1\right\}\leq 1-\beta.$$

We have

$$\begin{aligned}
&k\left|\frac{z(\Omega_{p,\alpha}(a,c)f(z))'+\gamma z^2(\Omega_{p,\alpha}(a,c)f(z))''}{(1-\gamma)\Omega_{p,\alpha}(a,c)f(z)+\gamma z(\Omega_{p,\alpha}(a,c)f(z))'}-1\right| \\
&\quad-\operatorname{Re}\left\{\frac{z(\Omega_{p,\alpha}(a,c)f(z))'+\gamma z^2(\Omega_{p,\alpha}(a,c)f(z))''}{(1-\gamma)\Omega_{p,\alpha}(a,c)f(z)+\gamma z(\Omega_{p,\alpha}(a,c)f(z))'}-1\right\} \\
&\leq (1+k)\left|\frac{z(\Omega_{p,\alpha}(a,c)f(z))'+\gamma z^2(\Omega_{p,\alpha}(a,c)f(z))''}{(1-\gamma)\Omega_{p,\alpha}(a,c)f(z)+\gamma z(\Omega_{p,\alpha}(a,c)f(z))'}-1\right| \\
&\leq (1+k)\left(\frac{M+N}{Q}\right),
\end{aligned}$$

where

$$M=[(p+p(p-1)\gamma)-(1-\gamma+\gamma p)],$$

$$N=\sum_{n=1}^{\infty}[1+\gamma(n+p-1)](n+p-1)\left(\frac{p+1}{n+p+1}\right)\frac{(a)_n}{(c)_n}|a_{n+p}|$$

and

$$Q=(1-\gamma+\gamma p)-\sum_{n=1}^{\infty}[1+\gamma(n+p-1)]\left(\frac{p+1}{n+p+1}\right)\frac{(a)_n}{(c)_n}|a_{n+p}|.$$

The above mentioned expression is bound by $(1-\beta)$

$$\sum_{n=1}^{\infty}\left[1+\gamma\left(n+p-1\right)\right]\left[\left(k+1\right)\left(n+p\right)-\left(\beta+k\right)\right]\left(\frac{p+1}{n+p+1}\right)^{\alpha}\frac{(a)_n}{(c)_n}\left|a_{n+p}\right|$$

$$\leq\left(1-\beta\right)\left(1-\gamma+\gamma p\right)-\left[p+p\left(p-1\right)\gamma-\left(1-\gamma+\gamma p\right)\right]\left(k+1\right)$$

and hence the proof is complete. □

Corollary 2.2. *If $f\in\Omega_{p,\alpha}^{k}\left(a,c,\beta,\gamma\right)$, then*

$$a_{n+p}\leq\frac{\left(1-\beta\right)\left(n+p+1\right)^{\alpha}(c)_n\left(1-\gamma+\gamma p\right)-\left(k+1\right)\left[M\right]}{\left(1+\gamma\left(n+p-1\right)\right)\left(p+1\right)^{\alpha}\left[\left(k+1\right)\left(n+p\right)-\left(\beta+k\right)\right]\left(a\right)_n},\qquad(2.2)$$

$n\geq 1$, where $-1\leq\beta<1$, $0\leq\gamma\leq 1$, $\alpha\in N$ and $k\geq 0$. The equality (2.1) *holds for the function*

$$f_n\left(z\right)=z^p+\frac{\left(n+p+1\right)^{\alpha}(c)_n\left(1-\beta\right)\left(1-\gamma+\gamma p\right)-\left(1+k\right)\left[M\right]}{\left(p+1\right)^{\alpha}\left(1+\gamma\left(n+p-1\right)\right)\left[\left(n+p\right)\left(1+k\right)-\left(\beta+k\right)\right]\left(a\right)_n}z^{n+p},\qquad(2.3)$$

$$\left(n\geq 1,\ z\in U\right).$$

The following is the growth and distortion property for function f in the class $\Omega_{p,\alpha}^{k}\left(a,c,\beta,\gamma\right)$.

Theorem 2.3. *If the function $f\left(z\right)$ defined by* (1.1) *is in the class $\Omega_{p,\alpha}^{k}\left(a,c,\beta,\gamma\right)$, then for $0\leq|z|=r<1$, we have*

$$r^p-\frac{\left(1-\beta\right)\left(p+2\right)^{\alpha}\left(1-\gamma+\gamma p\right)-\left(k+1\right)\left[M\right]r^{p+1}}{\left(1+\gamma p\right)\left(p+1\right)^{\alpha}\left[\left(k+1\right)\left(1+p\right)-\left(\beta+k\right)\right]}\leq\left|f\left(z\right)\right|\leq r^p+\frac{\left(1-\beta\right)\left(p+2\right)^{\alpha}\left(1-\gamma+\gamma p\right)-\left(k+1\right)\left[M\right]r^{p+1}}{\left(1+\gamma p\right)\left(p+1\right)^{\alpha}\left[\left(k+1\right)\left(1+p\right)-\left(\beta+k\right)\right]}$$

and

$$pr^{p-1}-\frac{\left(1-\beta\right)\left(p+2\right)^{\alpha}\left(1-\gamma+\gamma p\right)-\left(k+1\right)\left[M\right]r^{p}}{\left(1+\gamma\left(n+p-1\right)\right)\left(p+1\right)^{\alpha-1}\left[\left(k+1\right)\left(1+p\right)-\left(\beta+k\right)\right]}\leq\left|f'\left(z\right)\right|\leq pr^{p-1}+\frac{\left(1-\beta\right)\left(p+2\right)^{\alpha}\left(1-\gamma+\gamma p\right)-\left(k+1\right)\left[M\right]r^{p}}{\left(1+\gamma\left(n+p-1\right)\right)\left(p+1\right)^{\alpha-1}\left[\left(k+1\right)\left(1+p\right)-\left(\beta+k\right)\right]}.$$

Proof. Since $f \in \Omega_{p,\alpha}^{k}(a,c,\beta,\gamma)$, Theorem 2.1 readily yields the inequality

$$\sum_{n=1}^{\infty} a_{n+p} \leq \frac{(1-\beta)(p+2)^{\alpha}(1-\gamma+\gamma p)-(k+1)[M]}{(p+1)^{\alpha}(1+\gamma p)[(k+1)(1+p)-(\beta+k)]},\ n \geq 1. \tag{2.4}$$

As a result, for $0 \leq |z| = r < 1$ and using (2.4), we obtain

$$\begin{aligned} |f(z)| &\leq |z^p| + \sum_{n=1}^{\infty} a_n \left|z^{n+p}\right| \leq r^p + r^{p+1}\sum_{n=1}^{\infty} a_{n+p} \\ &\leq r^p + \frac{(1-\beta)(p+2)^{\alpha}(1-\gamma+\gamma p)-(k+1)[M]\, r^{p+1}}{(1+\gamma p)(p+1)^{\alpha}[(k+1)(1+p)-(\beta+k)]} \end{aligned}$$

and

$$\begin{aligned} |f(z)| &\geq |z^p| - \sum_{n=1}^{\infty} a_n \left|z^{n+p}\right| \geq r^p - r^{p+1}\sum_{n=1}^{\infty} a_{n+p} \\ &\geq r^p - \frac{(1-\beta)(p+2)^{\alpha}(1-\gamma+\gamma p)-(k+1)[M]\, r^{p+1}}{(1+\gamma p)(p+1)^{\alpha}[(k+1)(1+p)-(\beta+k)]}. \end{aligned}$$

We also obtain the following from Theorem 2.1

$$\begin{aligned} f'(z) &= pz^{p-1} \\ &\quad + \frac{(n+p+1)^{\alpha}(n+p)(c)_n(1-\beta)(1-\gamma+\gamma p)-(k+1)[M]}{(1+\gamma(n+p-1))(p+1)^{\alpha}[(k+1)(n+p)-(\beta+k)](a)_n} z^{n+p-1} \end{aligned}$$

and

$$\sum_{n=1}^{\infty}(n+p)\, a_{n+p} \leq \frac{(p+2)^{\alpha}(1-\beta)(1-\gamma+\gamma p)-(1+k)[M]}{(p+1)^{\alpha-1}(1+\gamma(n+p-1))[(1+p)(1+k)-(\beta+k)]}.$$

Hence, we have

$$\begin{aligned} \left|f'(z)\right| &\leq \left|pz^{p-1}\right| + \sum_{n=1}^{\infty}(n+p)\, a_{n+p}\left|z^{n+p-1}\right| \\ &\leq pr^{p-1} + r^p \sum_{n=1}^{\infty}(n+p)\, a_{n+p} \\ &\leq pr^{p-1} + \frac{(1-\beta)(p+2)^{\alpha}(1-\gamma+\gamma p)-(k+1)[M]\, r^p}{(1+\gamma(n+p-1))(p+1)^{\alpha-1}[(k+1)(1+p)-(\beta+k)]} \end{aligned}$$

and

$$\begin{aligned}\left|f'(z)\right| &\geq \left|pz^{p-1}\right| - \sum_{n=1}^{\infty}(n+p)\,a_{n+p}\left|z^{n+p-1}\right| \\ &\geq pr^{p-1} - r^{p}\sum_{n=1}^{\infty}(n+p)\,a_{n+p} \\ &\geq pr^{p-1} - \frac{(1-\beta)(p+2)^{\alpha}(1-\gamma+\gamma p)-(k+1)[M]\,r^{p}}{(1+\gamma(n+p-1))(p+1)^{\alpha-1}[(k+1)(1+p)-(\beta+k)]}.\end{aligned}$$

The proof of Theorem 2.3 is now complete. □

The following theorems provide the radii of starlikeness and convexity for the class $\Omega_{p}^{k}(a,c,\beta,\gamma)$.

Theorem 2.4. *If the function f in (1.1) belongs to the class $\Omega_{p,\alpha}^{k}(a,c,\beta,\gamma)$, then f is starlike of order $\delta\,(0\leq\delta<1)$ in the disc $|z|=r_1$, where*

$$r_1 = \inf_{n\geq 1}\left(\frac{(2-p-\delta)(1+\gamma(n+p-1))[(n+p)(1+k)-(\beta+k)]}{(n+p-\delta)(1-\beta)(1-\gamma+\gamma p)-(1+k)[M]}\right)^{\frac{1}{n}}.$$

For the function $f_n(z)$ provided by (2.3), the result is sharp.

Proof. Since $f(z)$ is starlike of order $\delta\,(0\leq\delta<1)$, we have

$$\mathrm{Re}\left\{\frac{z(\Omega_{p,\alpha}(a,c)\,f(z))'}{\Omega_{p,\alpha}(a,c)\,f(z)}\right\} > \delta.$$

That is

$$\left|\frac{z(\Omega_{p,\alpha}(a,c)\,f(z))'}{\Omega_{p,\alpha}(a,c)\,f(z)} - 1\right| \leq 1-\delta.$$

Now, for $|z|=r_1$, we have

$$\begin{aligned}&\left|\frac{z(\Omega_{p,\alpha}(a,c)\,f(z))'}{\Omega_{p,\alpha}(a,c)\,f(z)} - 1\right| \\ &\quad= \left|\frac{(p-1)\,z^{p} + \sum_{n=1}^{\infty}(n+p-1)\left(\frac{p+1}{n+p+1}\right)^{\alpha}\frac{(a)_n}{(c)_n}\,a_{n+p}\,z^{n+p}}{z^{p} + \sum_{n=1}^{\infty}\left(\frac{p+1}{n+p+1}\right)^{\alpha}\frac{(a)_n}{(c)_n}\,a_{n+p}\,z^{n+p}}\right|\end{aligned}$$

$$\leq \frac{(p-1)|z|^p + \sum_{n=1}^{\infty}(n+p-1)\left(\frac{p+1}{n+p+1}\right)^{\alpha}\frac{(a)_n}{(c)_n}\,|a_{n+p}|\,|z|^{n+p}}{|z|^p + \sum_{n=1}^{\infty}\left(\frac{p+1}{n+p+1}\right)^{\alpha}\frac{(a)_n}{(c)_n}\,|a_{n+p}|\,|z|^{n+p}}$$

$$\leq \frac{(p-1) + \sum_{n=1}^{\infty}(n+p-1)\left(\frac{p+1}{n+p+1}\right)^{\alpha}\frac{(a)_n}{(c)_n}\,|a_{n+p}|\,|z|^{n}}{1 - \sum_{n=1}^{\infty}\left(\frac{p+1}{n+p+1}\right)^{\alpha}\frac{(a)_n}{(c)_n}\,|a_{n+p}|\,|z|^{n}}. \tag{2.5}$$

Hence (2.5) holds true if

$$(p-1) + \sum_{n=1}^{\infty}(n+p-1)\left(\frac{p+1}{n+p+1}\right)^{\alpha}\frac{(a)_n}{(c)_n}\,|a_{n+p}|\,|z|^{n}$$

$$\leq (1-\delta)\left(1 - \sum_{n=1}^{\infty}\left(\frac{p+1}{n+p+1}\right)^{\alpha}\frac{(a)_n}{(c)_n}\,|a_{n+p}|\,|z|^{n}\right) \tag{2.6}$$

or

$$\sum_{n=1}^{\infty}\frac{(n+p-\delta)}{(2-p-\delta)}\left(\frac{p+1}{n+p+1}\right)^{\alpha}\frac{(a)_n}{(c)_n}\,|a_{n+p}|\,|z|^{n} \leq 1. \tag{2.7}$$

With the help of (2.2) and (2.7), it is indeed correct to say that

$$\sum_{n=1}^{\infty}\frac{(n+p-\delta)}{(2-p-\delta)}\left(\frac{p+1}{n+p+1}\right)^{\alpha}\frac{(a)_n}{(c)_n}\,|z|^{n}$$

$$\leq \frac{(1+\gamma(n+p-1))(p+1)^{\alpha}[(k+1)(n+p)-(\beta+k)](a)_n}{(1-\beta)(n+p+1)^{\alpha}(c)_n(1-\gamma+\gamma p)-(k+1)[M]}. \tag{2.8}$$

Solving (2.8) for $|z| = r_1$, we obtain

$$|z| \leq \left(\frac{(2-p-\delta)(1+\gamma(n+p-1))[(n+p)(1+k)-(\beta+k)]}{(n+p-\delta)(1-\beta)(1-\gamma+\gamma p)-(1+k)[M]}\right)^{\frac{1}{n}},\ n \geq 1.$$

By observing that the function $f(z)$, given by (2.3), is indeed an extremal function for the assertion (2.1), Thus Theorem 2.4 is proved. □

Theorem 2.5. *If the function f given by* (1.1) *is in the class* $\Omega^{k}_{p,\alpha}(a,c,\beta,\gamma)$, *then it is convex of order* $\delta\,(0 \leq \delta < 1)$ *in the disc* $|z| = r_2$, *where*

$$r_2 = \inf_{n\geq 1}\left(\frac{(1+\gamma(n+p-1))(2-p-\delta)[(n+p)(1+k)-(\beta+k)]}{(1-\beta)(n+p-\delta)(n+p)(1-\gamma+\gamma p)-(1+k)[M]}\right)^{\frac{1}{n}}.$$

For the function $f_n(z)$ *provided by* (2.3), *the result is sharp.*

Proof. Using the method used in the proof of Theorem 2.4, we can demonstrate that

$$\begin{aligned}&\left|\frac{z(\Omega_{p,\alpha}(a,c)f(z)f(z))''}{(\Omega_{p,\alpha}(a,c)f(z)f(z))'}\right|\\ &\quad\leq\frac{p(p-1)+\sum\limits_{n=1}^{\infty}\frac{(n+p)(n+p-1)(p+1)}{n+p+1}\frac{(a)_n}{(c)_n}\left|a_{n+p}\right|\left|z\right|^n}{p-\sum\limits_{n=1}^{\infty}\frac{(n+p)(p+1)}{n+p+1}\frac{(a)_n}{(c)_n}\left|a_{n+p}\right|\left|z\right|^n}\\ &\quad\leq 1-\delta. \end{aligned}\tag{2.9}$$

We can show from (2.1) that (2.9) is true if

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{(n+p-\delta)(n+p)(p+1)}{(2-p-\delta)(n+p+1)}\frac{(a)_n}{(c)_n}\left|z\right|^n\\ &\quad\leq\frac{(p+1)(1+\gamma(n+p-1))\left[(n+p)(k+1)-(\beta+k)\right](a)_n}{(1-\beta)(c)_n(n+p+1)(1-\gamma+\gamma p)-(k+1)[M]}.\end{aligned}\tag{2.10}$$

When we solve (2.10) for $|z|=r_2$, we obtain

$$|z|\leq\left(\frac{(1+\gamma(n+p-1))(2-p-\delta)\left[(k+1)(n+p)-(\beta+k)\right]}{(1-\beta)(n+p-\delta)(n+p)(1-\gamma+\gamma p)-(k+1)[M]}\right)^{\frac{1}{n}}.$$

Sharpness of the result follows by setting

$$\begin{aligned}&f_n(z)\\ &\quad=z^p+\frac{(n+p+1)^{\alpha}(c)_n(1-\beta)(1-\gamma+\gamma p)-(1+k)[M]}{(p+1)^{\alpha}(1+\gamma(n+p-1))\left[(n+p)(1+k)-(\beta+k)\right](a)_n}z^{n+p},\end{aligned}$$

$(n\geq 1,\ z\in U)$. This completes the proof. □

The following result is a linear combination of several functions of the type (1.9).

Theorem 2.6. *Let*

$$f_1(z)=z\tag{2.11}$$

and

$$\begin{aligned}&f_n(z)\\ &\quad=z^p+\frac{(1-\beta)(n+p+1)^{\alpha}(1-\gamma+\gamma p)-(k+1)[M](c)_n}{(1+\gamma(n+p-1))(p+1)^{\alpha}\left[(k+1)(n+p)-(\beta+k)\right](a)_n}z^{n+p},\end{aligned}\tag{2.12}$$

then $f \in \Omega_{p,\alpha}^{k}(a,c,\beta,\gamma)$ *if and only if it is possible to express it in the following way:*

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z), \tag{2.13}$$

where $\lambda_n \geq 0$ *and* $\sum_{n=1}^{\infty} \lambda_n = 1.$

Proof. Suppose $f(z)$ can be written as in (2.14). Then

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} \lambda_n f_n(z) \\ &= z^p + \frac{(n+p+1)^{\alpha}(1-\beta)(1-\gamma+\gamma p) - (1+k)[M](c)_n \lambda_n}{(p+1)^{\alpha}(1+\gamma(n+p-1))[(n+p)(1+k)-(\beta+k)](a)_n} z^{n+p}. \end{aligned}$$

Since

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{(1+\gamma(n+p-1))(p+1)^{\alpha}[(n+p)(1+k)-(\beta+k)](a)_n}{(1-\beta)(n+p+1)^{\alpha}(1-\gamma+\gamma p)-(1+k)[M](c)_n} \\ &\quad \times \frac{((1-\beta)n+p+1)^{\alpha}(1-\gamma+\gamma p)-(1+k)[M](c)_n \lambda_n}{(1+\gamma(n+p-1))(p+1)^{\alpha}[(n+p)(1+k)-(\beta+k)](a)_n} \\ &= \sum_{n=1}^{\infty} \lambda_n = 1 - \lambda_1 < 1. \end{aligned}$$

It follows from Theorem 2.1 that the function $f \in \Omega_{p,\alpha}^{k}(a,c,\beta,\gamma)$.

Conversely, let us assume that $f \in \Omega_{p,\alpha}^{k}(a,c,\beta,\gamma)$. Since

$$a_{n+p} \leq \frac{(1-\beta)(n+p+1)^{\alpha}(1-\gamma+\gamma p)-(1+k)[M](c)_n}{(1+\gamma(n+p-1))(p+1)^{\alpha}[(n+p)(1+k)-(\beta+k)](a)_n}, n \geq 1.$$

Setting

$$\lambda_n = \frac{(1+\gamma(n+p-1))(p+1)^{\alpha}[(k+1)(n+p)-(\beta+k)](a)_n}{(1-\beta)(n+p+1)^{\alpha}(1-\gamma+\gamma p)-(k+1)[M](c)_n} a_{n+p}, n \geq 1$$

and

$$\lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n.$$

It follows that $f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z)$. Thus, the theorem is proved. □

Theorem 2.7. *The class* $\Omega_{p,\alpha}^{k}(a,c,\beta,\gamma)$ *is closed under convex linear combinations.*

Proof. Assume that the functions $f_1(z)$ and $f_2(z)$ are defined by

$$f_j(z) = z + \sum_{n=1}^{\infty} a_{n+p,j} z^{n+p}, \quad (a_{n+p,j} \geq 0,\ j = 1,2;\ z \in U),$$

which belongs to the class $\Omega_{p,\alpha}^{k}(a, c, \beta, \gamma)$. Setting

$$f(z) = \mu f_1(z) + (1-\mu) f_2(z), \quad 0 \leq \mu \leq 1. \tag{2.14}$$

We may deduce from (2.14) that

$$f(z) = z + \sum_{n=2}^{\infty} \{\mu a_{n,1} + (1-\mu) a_{n,2}\} z^n, \ (0 \leq \mu \leq 1;\ z \in U).$$

In view of Theorem 2.1, we may conclude that

$$\begin{aligned}
&\sum_{n=1}^{\infty} [1 + \gamma(n+p-1)] [(k+1)(n+p) - (\beta+k)] \\
&\qquad \times \left(\frac{p+1}{n+p+1}\right)^{\alpha} \frac{(a)_n}{(c)_n} \{\mu a_{n,1} + (1-\mu) a_{n,2}\} \\
&= \mu \sum_{n=1}^{\infty} [1 + \gamma(n+p-1)] [(k+1)(n+p) - (\beta+k)] \\
&\qquad \times \left(\frac{p+1}{n+p+1}\right)^{\alpha} \frac{(a)_n}{(c)_n} a_{n,1} \\
&\quad + (1-\mu) \sum_{n=1}^{\infty} [1 + \gamma(n+p-1)] [(k+1)(n+p) - (\beta+k)] \\
&\qquad \times \left(\frac{p+1}{n+p+1}\right)^{\alpha} \frac{(a)_n}{(c)_n} a_{n,2} \\
&\leq \mu (1-\beta)(1-\gamma+\gamma p) - [M](k+1) \\
&\quad + (1-\mu)(1-\beta)(1-\gamma+\gamma p) - [M](k+1) \\
&= (1-\beta)(1-\gamma+\gamma p) - [M](k+1).
\end{aligned}$$

This completes the proof. □

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