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NUMERICAL METHOD FOR A SYSTEM OF CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS WITH NON-LOCAL BOUNDARY CONDITIONS

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Abstract. A class of systems of Caputo fractional differential equations with integral boundary conditions is considered. A numerical method based on a finite difference scheme on a uniform mesh is proposed. Supremum norm is used to derive an error estimate which is of order $\kappa - 1$, $1 < \kappa < 2$. Numerical examples are given which validate our theoretical results.

1. Introduction

Differential equations of fractional order have been the point of interest of many studies due to their frequent appearance in numerous applications in fluid mechanics, viscoelasticity, biology, physics and engineering. Systems of fractional differential equations have been extensively used in science and engineering applications over the past few decades to model complex real-world problems [3,4,14,17,19,26]. The Caputo definition of fractional order derivative is used in this paper. The Caputo fractional derivative of order $1 < \kappa < 2$ [6,14,19] of $v(x)$ is defined as

$$D_*^{\kappa} v(x) := \frac{1}{\Gamma(2-\kappa)} \int_0^x (x-s)^{1-\kappa} v''(s) ds.$$

It is unfortunate that many of the systems of fractional differential equations do not have precise analytic answers, consequently approximation and numerical techniques need to be applied for acquiring the solution of such systems. Various computational methods are used to solve systems of fractional differential equations for this purpose. For example, Adomian decomposition method [18,23], the variational iteration method [7], the differential transform method [8], the homotopy perturbation method [18], the homotopy analysis method [13], and the fractional natural decomposition method [20].

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Stynes and Gracia [25] proposed a finite difference method for approximating the fractional derivative two-point boundary value problem and demonstrated that it is of $O(N^{-(\kappa-1)})$, $1 < \kappa < 2$ convergent. The same authors [11] investigated Caputo fractional differential equations with Robin boundary value problems using central difference scheme for the convection term. Kopteva and Stynes [15] used a collocation method to solve the fractional differential equation that was transformed into a Volterra integral equation to obtain an approximate solution using a post-processing scheme. In [16], an upwind finite difference method is suggested to solve a fractional differential equation with integral boundary condition. Santra and Mohapatra [21,22] recently investigated fractional order Volterra integro-differential equations as well as time fractional order partial integro-differential equations of the Volterra type. To solve the linear/nonlinear systems of fractional differential equations, Daftardar-Gejji and Jafari [5,12] used the Adomian decomposition procedure, which provides numerical solution to any order with the desired precision. In [2], a multistep Adams method is constructed to solve the system of fractional differential equations.

Motivated by the works of [16,25,27], we discuss the following class of systems of fractional differential equations with integral boundary conditions:

$$\begin{cases} \mathcal{L}_1\bar{v}(x) = -D_*^\kappa v_1(x) + b_1(x)v_1'(x) + c_{11}v_1(x) + c_{12}v_2(x) = f_1(x), \\ \mathcal{L}_2\bar{v}(x) = -D_*^\kappa v_2(x) + b_2(x)v_2'(x) + c_{21}v_1(x) + c_{22}v_2(x) = f_2(x), \end{cases} \tag{1}$$

where $\bar{v}(x) = (v_1(x), v_2(x))$, $x \in D = (0,1)$, with integral boundary conditions

$$\begin{cases} \mathcal{B}_1 v_1(0) = v_1(0) - \alpha_1 v_1'(0) = g_1, \ \mathcal{B}_1 v_1(1) = v_1(1) - \int_0^1 a_1(x)v_1(x)dx = g_2, \\ \mathcal{B}_2 v_2(0) = v_2(0) - \alpha_2 v_2'(0) = g_3, \ \mathcal{B}_2 v_2(1) = v_2(1) - \int_0^1 a_2(x)v_2(x)dx = g_4, \end{cases} \tag{2}$$

where the functions $b_1(x)$, $b_2(x)$, $c_{11}(x)$, $c_{12}(x)$, $c_{21}(x)$, $c_{22}(x)$, $f_1(x)$, $f_2(x)$ are sufficiently smooth on $\bar{D} = [0,1]$. Here D_*^κ is the Caputo fractional derivative of order $1 < \kappa < 2$. We assume that

- (A_1) $b_i(x) \geq 0$, $i = 1,2$, $x \in \bar{D}$,
- (A_2) $c_{12}(x) \leq 0$, $c_{21}(x) \leq 0$, $c_{11}(x)+c_{12}(x) > 0$, $c_{21}(x)+c_{22}(x) > 0$, $x \in \bar{D}$,
- (A_3) $a_i(x)$ is nonnegative and $1 - \int_0^1 a_i(x)dx > 0$, $i = 1,2$,
- (A_4) $\alpha_i \geq \frac{1}{\kappa-1}$, $i = 1,2$.

All the above four conditions are necessary for comparision principle.

Notations.

1. $D^N = \{x_i\}_1^{N-1}$ and $\bar{D}^N = \{x_i\}_0^{N-1}$.
2. $C^{n,\delta}(0,1]$ denotes the space of functions $v \in C[0,1] \cap C^n(0,1]$ such that $|v^{(i)}(x)| \leq C(1 + x^{1-\delta-i})$ for $i = 0(1)n$ and $x \in (0,1]$ for all positive integer n and $\delta \in (-\infty, 1)$.
3. C denotes a generic positive constant independent of N. It should be noted that C can have different values at different places.
4. The maximum norm is defined as $||v(x)||_\infty = \max\limits_{x \in D} |v(x)|$.

The outline of this paper is as follows: In Section 2, maximum principle, stability result and bounds on the derivatives of the solution are proved. The numerical method is explained in Section 3. In the same section, the discrete maximum principle and mesh selection are discussed. The convergence analysis is discussed in Section 4. In Section 5, two numerical examples are presented to support our results which demonstrate the effectiveness of the numerical method presented. Conclusion of this paper is given in Section 6.

2. Maximum principle and derivative bounds

In this section, maximum principle and stability result for the system (1)-(2) are proved and we derive the estimates for the solution and its derivatives.

Theorem 2.1 (Maximum principle)**.** *Suppose that* $\bar{v}(x) = (v_1, v_2)$*,* $\bar{v} \in C^1(\bar{D}) \cap C^{2,\tau}(0,1]$ *satisfies* $\mathcal{B}_1 v_1(0) \geq 0$*,* $\mathcal{B}_1 v_1(1) \geq 0$*,* $\mathcal{B}_2 v_2(0) \geq 0$*,* $\mathcal{B}_2 v_2(1) \geq 0$*,* $\mathcal{L}_1 \bar{v}(x) \geq 0$ *and* $\mathcal{L}_2 \bar{v}(x) \geq 0$*,* $x \in D$*. Then* $\bar{v}(x) \geq 0$*,* $x \in \bar{D}$*.*

Proof. Define the test function $\bar{t}(x) = (t_1(x), t_2(x))$ as $t_1(x) = t_2(x) = 1 + \alpha_1 + \alpha_2 + x$. Then $\bar{t}(x) > 0$, $\forall x \in \bar{D}$, $\mathcal{B}_1 v_1(0) > 0$, $\mathcal{B}_1 v_1(1) > 0$, $\mathcal{B}_2 v_1(0) > 0$, $\mathcal{B}_1 v_2(1) > 0$, $\mathcal{L}_1(x) > 0$ and $\mathcal{L}_2(x) > 0$, $\forall x \in D$.

We define,

$$\rho = \max\left\{\max_{x\in\bar{D}}\left(\frac{-v_1(x)}{t_1(x)}\right), \max_{x\in\bar{D}}\left(\frac{-v_2(x)}{t_2(x)}\right)\right\}.$$

Then there exists a point $x_* \in \bar{D}$ such that

$$\left(\frac{-v_1(x_*)}{t_1(x_*)}\right) = \rho \text{ or } \left(\frac{-v_2(x_*)}{t_2(x_*)}\right) = \rho \text{ or both.}$$

Also, we have $(\bar{v} + \rho\bar{t})(x) \geq 0$, $\forall x \in \bar{D}$. Suppose that theorem is not true. Then $\rho > 0$.

Case 1: $\left(\frac{-v_1}{t_1}\right)(x_*) = \rho$ and $x_* = 0$. Therefore $(v_1 + \rho t_1)$ attains its minimum at x_*. Then

$$0 < \mathcal{B}_1(v_1 + \rho t_1)(0) = (v_1 + \rho t_1)(0) - \alpha_1 (v_1 + \rho t_1)'(0) = 0.$$

Case 2: $\left(\frac{-v_1}{t_1}\right)(x_*) = \rho$ and $x_* \in D$. Therefore $(v_1 + \rho t_1)$ attains its minimum at x_*. Then

$$\begin{aligned} 0 < \mathcal{L}_1(\bar{v} + \rho\bar{t})(x_*) = & -D_*^{\kappa}(v_1 + \rho t_1)(x_*) + b_1(x_*)(v_1 + \rho t_1)'(x_*) \\ & + c_{11}(x_*)(v_1 + \rho t_1)(x_*) + c_{12}(x_*)(v_2 + \rho t_2)(x_*). \end{aligned}$$

By Lemma 3.3 of [1], we get $D_*^{\kappa}(v_1 + \rho t_1)(x_*) \geq 0$. Therefore

$$\mathcal{L}_1(\bar{v} + \rho\bar{t})(x_*) \leq 0.$$

Case 3: $\left(\frac{-v_1}{t_1}\right)(x_*) = \rho$ and $x_* = 1$. Therefore $(v_1 + \rho t_1)$ attains its minimum at x_*. Then

$$0 < \mathcal{B}_1(v_1 + \rho t_1)(1) = (v_1 + \rho t_1)(1) - \int_0^1 a_1(x)(v_1 + \rho t_1)(x)dx \leq 0.$$

In all the above three cases we arrived at a contradiction. Therefore $\rho \geq 0$. This proves that $v_1(x) \geq 0$. In the same way, we can prove that $v_2(x) \geq 0$. Hence $\bar{v}(x) \geq 0$, $x \in \bar{D}$ □

Theorem 2.2 (Stability result). *Let $\bar{v}(x)$ be the solution of* (1)-(2). *Then*

$$|v_i(x)| \leq C \max\{|\mathcal{B}_1 v_1(0)|, |\mathcal{B}_1 v_1(1)|, |\mathcal{B}_2 v_2(0)|, |\mathcal{B}_2 v_2(1)|, ||\mathcal{L}_1 \bar{v}||_D, ||\mathcal{L}_2 \bar{v}||_D\},$$

$\forall x \in \bar{D}$, $i = 1, 2$.

Proof. Let C be a constant. We define the barrier functions

$$\Phi_i^{\pm}(x) = CMt_i(x) \pm v_i(x), \ x \in \bar{D}, \ i = 1, 2,$$

where $M = \max\left\{\max_{i=1,2}\{|\mathcal{B}_i v_i(0)|\}, \ \max_{i=1,2}\{|\mathcal{B}_i v_i(1)|\}, \ \max_{i=1,2}\left\{\sup_{\xi \in D}|\mathcal{L}_i v_i(\xi)|\right\}\right\}$.
Then for $i = 1, 2$

$$\begin{aligned}\mathcal{B}_i \varphi_i^{\pm}(0) &= \varphi_i^{\pm}(0) - \alpha_i \varphi_i^{\pm'}(0)\\ &= CM(1 + \alpha_1 + \alpha_2 - \alpha_i) \pm (v_i(0) - \alpha_i v_i(0))\\ &\geq 0\end{aligned}$$

and

$$\begin{aligned}\mathcal{B}_i \varphi_i^{\pm}(1) &= \varphi_i^{\pm}(1) - \alpha_i \varphi_i^{\pm'}(1)\\ &= CM\left[t_i(1) - \int_0^1 a_i(x) t_i(x) dx\right] \pm \mathcal{B}_i v_i(1)\\ &\geq 0,\end{aligned}$$

by a suitable choice of C. Furthermore, we get

$$\begin{aligned}\mathcal{L}_1 \bar{\varphi}^{\pm}(x) &= -D_*^{\kappa} \varphi_1^{\pm} + b_1(x) \varphi_1^{\pm'} + c_{11}(x) \varphi_1^{\pm}(x) + c_{12}(x) \varphi_2^{\pm}(x)\\ &= CM\left[b_1(x) + (c_{11}(x) + c_{12}(x)) t_1(x)\right] \pm \mathcal{L}_1 v_1(x)\\ &\geq 0\end{aligned}$$

by a proper choice of C. Similarly, we can prove that $\mathcal{L}_2 \varphi_2^{\pm}(x) \geq 0$. By applying maximum principle, we obtain the required result. □

Theorem 2.3 (Derivative bounds). *Let $b_i(x), c_{ij}(x), f_i(x), a_i(x) \in C^{n,\tau}(0, 1]$, $i, j = 1, 2$ for some $n \in \mathbb{N}$ with $n \geq 2$ and $\tau \leq 2 - \kappa$. Then,* (1)-(2) *has a* *unique solution* $\bar{v}(x)$ *with* $\bar{v}(x) \in C^1(\bar{D}) \cap C^{n+1}(0,1]$ *and* $D_*^{\kappa}\bar{v}(x) \in C^{n,2-\kappa}(0,1]$. *Further, there exists a constant* C *such that*

$$|v_i^{(r)}(x)| \leq \begin{cases} C \text{ if } r = 0,1, \\ Cx^{\kappa-r} \text{ if } r = 2,3,\ldots,n+1. \end{cases}$$

Proof. From Theorem 2.1 it follows that the problem (1)-(2) has only the trivial solution $\bar{v} \equiv 0$ when $f_i \equiv 0$, $i = 1,2$ and $g_j \equiv 0$, $j = 1(1)4$. Applying the arguments as given in Theorem 2.2 of [16], we can prove the problem (1)-(2) has a unique solution $\bar{v}(x)$ with $\bar{v}(x) \in C^1(\bar{D})$ and $D_*^{\kappa}\bar{v}(x) \in C^{n,2-\kappa}$. Then, Theorem 3.4 of [25] yields the bounds on the derivative of the solution $\bar{v}(x)$. □

3. Discrete problem

Let N be a positive integer. The domain $\bar{D}$ is partitioned into N subintervals $0 = x_0 < x_1 < \cdots < x_N = 1$. Set a uniform mesh $x_j = jh$ with the step size $h = 1/N$ for $j = 0(1)N$.

Discretization of (1)-(2) is: Find $\bar{V}(x_j) = (V_1(x_j), V_2(x_j))$, $x_j \in \bar{D}^N$ such that

$$(3) \quad \begin{cases} \mathcal{L}_1^N \bar{V}(x_j) = -D_{*,L_2}^{\kappa} V_1(x_j) + b_1(x_j) D^- V_1(x_j) + c_{11}(x_j) V_1(x_j) + c_{12}(x_j) V_2(x_j), \\ \mathcal{L}_2^N \bar{V}(x_j) = -D_{*,L_2}^{\kappa} V_2(x_j) + b_2(x_j) D^- V_2(x_j) + c_{21}(x_j) V_1(x_j) + c_{22}(x_j) V_2(x_j), \end{cases}$$

$$(4) \quad \begin{cases} \mathcal{B}_1 V_1(x_0) = V_1(x_0) - \alpha_1 D^+ V_1(x_0) = g_1, \\ \mathcal{B}_1 V_1(x_N) = V_1(x_N) - \sum\limits_{j=1}^{N} \frac{a_1(x_{j-1})V_1(x_{j-1}) + a_1(x_j)V_1(x_j)}{2} h = g_2, \ \forall x_j \in \bar{D}^N, \\ \mathcal{B}_2 V_2(x_0) = V_2(x_0) - \alpha_2 D^+ V_2(x_0) = g_3, \\ \mathcal{B}_2 V_2(x_N) = V_2(x_N) - \sum\limits_{j=1}^{N} \frac{a_2(x_{j-1})V_2(x_{j-1}) + a_2(x_j)V_2(x_j)}{2} h = g_4, \ \forall x_j \in \bar{D}^N, \end{cases}$$

where

$$D^+ V_i(x_j) = \frac{V_i(x_{j+1}) - V_i(x_j)}{h}, \ D^- V_i(x_j) = \frac{V_i(x_j) - V_i(x_{j-1})}{h}, \ i = 1,2,$$

and D_{*,L_2}^{κ} is the L_2 discretization [24] of the Caputo fractional derivative given by

$$D_*^{\kappa} V_i(x_j) = \frac{1}{h^{\kappa}\Gamma(3-\kappa)} \sum_{l=0}^{j-1} w_{j-l} \left(V_i(x_{l+2}) - 2V_i(x_{l+1}) + V_i(x_l) \right)$$

with

$$w_{j-l} = (j-l)^{(2-\kappa)} - (j-l-1)^{(2-\kappa)}.$$

Theorem 3.1 (Discrete Maximum Principle)**.** *Assume that the mesh function* $\bar{\Psi}(x_j) = (\Psi_1(x_j), \Psi_2(x_j))$ *satisfies* $\mathcal{B}_1\Psi_1(x_0) \geq 0$, $\mathcal{B}_1\Psi_1(x_N) \geq 0$, $\mathcal{B}_2\Psi_2(x_0) \geq 0$, $\mathcal{B}_2\Psi_2(x_N) \geq 0$, $\mathcal{L}_1^N\bar{\Psi}(x_j) \geq 0$ *and* $\mathcal{L}_2^N\bar{\Psi}(x_j) \geq 0$, $\forall x_j \in D^N$. *Then* $\bar{\Psi}(x_j) \geq 0$, $\forall x_j \in \bar{D}^N$.

Proof. Let $E = (e_{jl})_{j,l=0}^{2N+1}$ denote the $(2N+2) \times (2N+2)$ matrix associated with the discretization of (3)-(4). The structure of the matrix E is as follows: the 0^{th} row of matrix is

$$e_{00} = 1 + \alpha_1 h^{-1},\ e_{01} = -\alpha_1 h^{-1},\ e_{0l} = 0 \ \text{ if } \ l = 2(1)2N+1,$$

the N^{th} row of matrix is

$$\begin{aligned} e_{N0} &= -\frac{a_1(x_0)}{2}h,\ e_{Nl} = -ha_1(x_l) \ \text{ if } \ l = 1(1)N-1, \\ e_{NN} &= 1 - \frac{a_1(x_N)}{2}h,\ e_{Nl} = 0 \ \text{ if } \ l = N+1(1)2N+1, \end{aligned}$$

the $(N+1)^{th}$ row of matrix is

$$\begin{aligned} e_{N+1,l} &= 0 \ \text{ if } \ l = 0(1)N,\ e_{N+1,N+1} = 1 + \alpha_2 h^{-1}, \\ e_{N+1,N+2} &= -\alpha_2 h^{-1},\ e_{N+1,l} = 0 \ \text{ if } \ l = N+3(1)2N+1, \end{aligned}$$

the $(2N+1)^{th}$ row of matrix is

$$\begin{aligned} e_{2N+1,l} &= 0 \ \text{ if } \ l = 0(1)N,\ e_{2N+1,N+1} = -\frac{a_2(x_0)}{2}h, \\ e_{2N+1,N+1+l} &= -ha_2(x_l) \ \text{ if } \ l = 1(1)N-1,\ e_{2N+1,2N+1} = 1 - \frac{a_2(x_N)}{2}h, \end{aligned}$$

the $j^{th}(j = 1(1)N-1)$ row of matrix satisfies

$$\begin{aligned} e_{j0} &= \frac{-w_j}{h^\kappa \Gamma(3-\kappa)} - \eta_{j1}\frac{b_1(x_1)}{h}, \\ e_{j1} &= \frac{-w_{j-1} + 2w_j}{h^\kappa \Gamma(3-\kappa)} - \eta_{j2}\frac{b_1(x_2)}{h} + \eta_{j1}\left(\frac{b_1(x_1)}{h} + c_{11}(x_1)\right), \\ e_{jl} &= \frac{-w_{j-l} + 2w_{j-l+1} - w_{j-l+2}}{h^\kappa \Gamma(3-\kappa)} + \frac{b_1(x_j)}{h}(\eta_{jl} - \eta_{j,l+1}) \\ &\quad + \eta_{jl} c_{11}(x_j), \\ e_{j,N+l} &= c_{12}(x_j),\ e_{N+1+j,l-1} = c_{21}(x_j), \\ e_{N+1+j,N+1} &= \frac{-w_j}{h^\kappa \Gamma(3-\kappa)} - \eta_{j1}\frac{b_2(x_1)}{h}, \\ e_{N+1+j,N+2} &= \frac{-w_{j-1} + 2w_j}{h^\kappa \Gamma(3-\kappa)} - \eta_{j2}\frac{b_2(x_2)}{h} + \eta_{j1}\left(\frac{b_2(x_1)}{h} + c_{22}(x_1)\right), \\ e_{N+1+j,N+1+l} &= \frac{-w_{j-l} + 2w_{j-l+1} - w_{j-l+2}}{h^\kappa \Gamma(3-\kappa)} \\ &\quad + \frac{b_2(x_j)}{h}\left(\eta_{N+1+j,N+1+l} - \eta_{N+1+j,N+2+l}\right) \\ &\quad + (\eta_{N+1+j,N+1+l}) c_{22}(x_j), \end{aligned}$$

where

$$\eta_{jl} = \begin{cases} 1 \text{ if } j = l, \\ 0 \text{ otherwise}, \end{cases}$$

$$w_j = \begin{cases} j^{(2-\kappa)} - (j-1)^{(2-\kappa)}, \ j = 1, 2, 3, \ldots, \\ 0, \ j \leq 0. \end{cases}$$

From Section 4.1 of [25] and Theorem 3.1 of [16], we get

$$\begin{aligned}
&e_{00}, e_{N+1,N+1} > 0, \ e_{01}, e_{N+1,N+2} < 0, \\
&e_{j0}, e_{N+1+j,N+1} < 0 \text{ for } j = 1(1)N-1, \\
&e_{j1}, e_{N+1+j,N+2} > 0 \text{ for } j = 3(1)N-1, \\
&e_{jj}, e_{N+1+j,N+1+j} > 0 \text{ for } j = 1(1)N-1, \\
&e_{jl}, e_{N+1+j,N+1+l} < 0 \text{ for } j = 1(1)N-1, \text{ and } l = 2, 3, \ldots, j-2, j-1, j+1.
\end{aligned}$$

From the assumption (A2), we obtain

$$e_{j,N+l}, e_{N+1+j,l-1} \geq 0.$$

Next, we show that the matrix E is an M-matrix, i.e., all the off diagonal entries of E are non-positive and there exists a vector $\vec{u} \geq 0$ such that $E\vec{u} > 0$ ([10]).

We define the matrix E' as

$$E' = T^{(N-1)} T^{(N-2)} \cdots T^{(1)} E, \tag{5}$$

where $T^{(k)} := (t_{jl}^{(k)})_{j,l=0}^{2N+1}$ is an elementary matrix with

$$t_{jl}^{(k)} := \eta_{jl} - \frac{e_{k0}}{e_{00}} \eta_{jk} \eta_{l0} - \frac{e_{N+1+k,N+1}}{e_{N+1,N+1}} \eta_{j,N+1+k} \eta_{l,N+1}.$$

By the construction of E',

$$\begin{aligned}
&e'_{jl} = e_{jl} \text{ for } l > 1 \text{ and } \forall j, \\
&e'_{j0} = 0 \text{ for } j = 1, 2, \ldots, N-1, \\
&e'_{j1} = e_{j1} + e_{N+1+j,N+2} + \frac{\alpha_1 h^{-1}}{1 + \alpha_1 h^{-1}} e_{j0} + \frac{\alpha_2 h^{-1}}{1 + \alpha_2 h^{-1}} e_{N+1+j,N+1},
\end{aligned}$$

we have for $j = 1(1)N-1$,

$$\begin{aligned}
\sum_{l=0}^{2N+1} e'_{jl} &= -\frac{e_{j0}}{1 + \alpha_1 h^{-1}} + \sum_{l=0}^{2N+1} e_{jl} \\
&= -\frac{e_{j0}}{1 + \alpha_1 h^{-1}} + c_{11}(x_j) + c_{12}(x_j) > 0,
\end{aligned}$$

$$\sum_{l=0}^{2N+1} e'_{N+1+j,l} = -\frac{e_{N+1+j,N+1}}{1 + \alpha_2 h^{-1}} + \sum_{l=0}^{2N+1} e_{N+1+j,l}$$

$$= -\frac{e_{N+1+j,N+1}}{1+\alpha_2 h^{-1}} + c_{21}(x_j) + c_{22}(x_j) > 0.$$

Using Theorem 4.4 of [25] and Theorem 3.1 of [16], we get E' as an M-matrix and $(E')^{-1}$ exists with $(E')^{-1} \geq 0$. Then (5) implies that

$$E^{-1} = (E')^{-1} T^{(N-1)} T^{(N-2)} \cdots T^{(1)}$$

exists and $E^{-1} \geq 0$. By the hypothesis, we get $\bar{\Psi}(x_j) \geq 0$, $\forall x_j \in \bar{D}$, which concludes the proof. $\square$

4. Convergence analysis

We provide an error estimate for the discrete problem (3)-(4) in this section.

Theorem 4.1. *Let $\bar{v}(x_j)$ be the solution of* (1)-(2) *and $\bar{V}(x_j)$ be the solution of the discrete problem* (3)-(4) *on the mesh D^N. Then, by the hypothesis of* (1) *and the hypothesis of Theorem 2.3, we have*

$$|V_i^N(x_j) - v_i(x_j)| \leq CN^{-(\kappa-1)} \text{ for all } j = 0(1)N,\ i = 1, 2.$$

Proof. Let $z_j = V_i^N(x_j) - v_i(x_j)$.

The truncation error is defined as $\vec{z} = (z_0, z_1, \ldots, z_{2N+1})^T$. Then,

$$\begin{aligned}
z_0 &= \alpha_1\Big(v_1'(x_0) - D^+ v_1(x_0)\Big),\\
z_{N+1} &= \alpha_2\Big(v_2'(x_0) - D^+ v_2(x_0)\Big),\\
z_N &= -\int_{x_0}^{x_N} a_1(x)v_1(x)dx + \sum_{j=1}^{N} \frac{a_1(x_{j-1})v_1(x_{j-1}) + a_1(x_j)v_1(x_j)}{2} h,\\
z_{2N+1} &= -\int_{x_0}^{x_N} a_2(x)v_2(x)dx + \sum_{j=1}^{N} \frac{a_2(x_{j-1})v_2(x_{j-1}) + a_2(x_j)v_2(x_j)}{2} h,\\
z_j &= (Ev_1)_j - f_1(x_j),\\
z_{N+1+j} &= (Ev_2)_j - f_2(x_j) \text{ for } j = 1(1)N-1.
\end{aligned}$$

First, we show that the bounds for the error function $|z_j|$.
Case (i): If $j = 0$, then

$$\begin{aligned}
|z_j| &= \frac{\alpha_1}{h}\left|\int_0^h \left(\int_0^t v_1''(s)\ ds\right) dt\right|\\
&\leq CN^{-(\kappa-1)},
\end{aligned}$$

from Theorem 2.3.
Case (ii): If $j = N$, we utilise Theorem 2.3 to deduce that

$$\begin{aligned}
|z_j| &\leq Ch^3\Big(v_1''(\lambda_1) + \cdots + v_1''(\lambda_N)\Big)\\
&\leq CN^{-(\kappa-1)},
\end{aligned}$$

where $x_{j+1} \leq \lambda_j \leq x_j$.

Case (iii): If $j = 2(1)N-1$

$$\begin{aligned} z_j &= (D_*^\kappa v_1(x_j) - D_{*,L_2}^\kappa v_1(x_j)) + b_1(x_j)(v_1'(x_j) - D^- v_1(x_j)) \\ &= T_1(x_j) + T_2(x_j). \end{aligned}$$

These two terms are bounded separately. By using mean value theorem for integrals, for some $\rho_1 \in (x_l, x_{l+1})$, we obtain

$$T_1(x_j) = \sum_{l=0}^{j=1} \frac{h^{2-\kappa} w_{j-l}}{\Gamma(3-\kappa)} \left[v_1''(\rho_1) - \frac{v_1(x_{l+2}) - 2v_1(x_{l+1}) + v_1(x_j)}{h^2} \right].$$

For $l \geq 1$, we can conclude that

$$|T_1(x_j)| \leq C \sum_{l=0}^{j-1} \frac{w_{j-l} l^{\kappa-3}}{\Gamma(3-\kappa)},$$

by again using mean value theorem and the bound on v_1'''. Now, applying Lemma 4.5 of [25], we get

$$|T_1(x_j)| \leq C j^{1-\kappa}.$$

If $l = 0$, then

$$\begin{aligned} |T_1(x_j)| \leq & \left| \frac{1}{\Gamma(2-\kappa)} \int_0^h (x_j - s)^{1-\kappa} v_1''(s)\ ds \right| \\ & + \left| \frac{w_j}{h^\kappa \Gamma(3-\kappa)} [v_1(x_2) - 2v_1(x_1) + v_1(x_0)] \right|. \end{aligned}$$

Using Theorem 2.3 and mean value theorem, we get

$$\begin{aligned} |T_1(x_j)| &\leq C \frac{(j-1)^{1-\kappa}}{(\kappa-1)\Gamma(2-\kappa)} + C \frac{w_j}{\Gamma(3-\kappa)} \\ &\leq C(j-1)^{1-\kappa}. \end{aligned}$$

Next,

$$\begin{aligned} |T_2(x_j)| &= \frac{1}{h} \left| \int_{s=x_{j-1}}^{x_j} \left(\int_{t=x_j}^{s} v_1''(t)\ dt \right) ds \right| \\ &\leq C h x_{j-1}^{\kappa-2}. \end{aligned}$$

Hence, $hj \leq 1$ and $\kappa > 1$ imply that

$$|T_2(x_j)| \leq C(j-1)^{\kappa-1}.$$

Combining the bounds of $T_1(x_j)$ and $T_2(x_j)$ we obtain

$$|z_j| \leq C(j-1)^{1-\kappa} \text{ for } j = 2(1)N-1.$$

Case (iv): If $j = 1$, then using the definition of Euler's Beta function, we have

$$|T_1(x_1)| \leq \left| \frac{1}{\Gamma(2-\kappa)} \int_{x_0}^{x_1} (x_1 - s)^{1-\kappa} v_1''(s)\ ds \right|$$

$$
\begin{aligned}
&+\left|\frac{d_1}{h^\kappa\Gamma(3-\kappa)}[v_1(x_2)-2v_1(x_1)+v_1(x_0)\right|\\
&\leq \frac{C\Gamma(2-\kappa)\Gamma(\kappa-1)}{\Gamma(2-\kappa)}+Cd_1\\
&\leq C.
\end{aligned}
$$

Now,

$$
\begin{aligned}
|T_2(x_1)| &= |b_1(x_1)|\left|\frac{v_1(x_1)-v_1(x_0)}{h}-v_1'(x_1)\right|\\
&\leq C,
\end{aligned}
$$

where we have used mean value theorem. Adding these bounds, we get $|z_1|\leq C$. Likewise, we can demonstrate for $|z_j|,\ j=N+1(1)2N+1$.

We conclude in all four cases that

$$
|z_j|\leq\begin{cases} CN^{-(\kappa-1)} \text{ if } j=0,N+1,\\ C \text{ if } j=1,N+2,\\ C(j-1)^{1-\kappa} \text{ if } j=2(1)N-1,N+3(1)2N,\\ CN^{-(\kappa-1)} \text{ if } j=N,2N.\end{cases} \tag{6}
$$

Define $z' := (z_0, z_1', z_2', \ldots, z_{2N}', z_{2N+1}')^T$. Thus,

$$
z_j' = z_j + \frac{z_0}{1+\alpha_1 h^{-1}}\left[\frac{w_j}{h^\kappa\Gamma(3-\kappa)}+\eta_{j1}\frac{b_1(x_1)}{h}\right], \tag{7}
$$

$$
z_{N+1+j}' = z_{N+1+j} + \frac{z_{N+1}}{1+\alpha_2 h^{-1}}\left[\frac{w_j}{h^\kappa\Gamma(3-\kappa)}+\eta_{j1}\frac{b_2(x_1)}{h}\right]. \tag{8}
$$

Multiplying j^{th} equation of (7) and (8) by

$$
\frac{(1+\alpha_1h^{-1})h^\kappa\Gamma(3-\kappa)}{w_j} \text{ and } \frac{(1+\alpha_2h^{-1})h^\kappa\Gamma(3-\kappa)}{w_j}, \text{ respectively}
$$

we have for $j=1(1)N-1$

$$
\begin{aligned}
|\vec{z}_j| &= \left|\frac{(1+\alpha_1h^{-1})h^\kappa\Gamma(3-\kappa)}{w_j}z_j+\left[1+\eta_{j1}\frac{b_1(x_1)}{h}\cdot\frac{h^\kappa\gamma(3-\kappa)}{w_j}\right]z_0\right|\\
&\leq CN^{-(\kappa-1)},\\
|\vec{z}_{N+1+j}| &= \left|\frac{(1+\alpha_2h^{-1})h^\kappa\Gamma(3-\kappa)}{w_j}z_{N+1+j}\right.\\
&\qquad\left.+\left[1+\eta_{j1}\frac{b_2(x_1)}{h}\cdot\frac{h^\kappa\gamma(3-\kappa)}{w_j}\right]z_{N+1}\right|\\
&\leq CN^{-(\kappa-1)}.
\end{aligned}
$$

The desired result follows from (6). $\square$

5. Numerical exemplification

TABLE 1. Evaluated maximum and uniform errors and their rates of convergence of Example 5.1 for different values of κ and N.

κ/N		64	128	256	512	1024	2048	4096	8192
1.1	V_1	1.291e-01	1.289e-01	1.185e-01	9.860e-02	7.339e-02	4.873e-02	2.927e-02	1.631e-02
		3.144e-03	1.202e-01	2.662e-01	4.260e-01	5.907e-01	7.354e-01	8.436e-01	
	V_2	1.430e-01	1.411e-01	1.288e-01	1.065e-01	7.887e-02	5.214e-02	3.120e-02	1.734e-02
		1.978e-02	1.305e-01	2.749e-01	4.335e-01	5.971e-01	7.406e-01	8.473e-01	
1.3	V_1	7.070e-02	4.156e-02	2.288e-02	1.207e-02	6.208e-03	3.150e-03	1.586e-03	7.963e-04
		7.662e-01	8.611e-01	9.228e-01	9.591e-01	9.789e-01	9.893e-01	9.946e-01	
	V_2	7.050e-02	4.105e-02	2.244e-02	1.178e-02	6.047e-03	3.063e-03	1.542e-03	7.736e-04
		7.800e-01	8.712e-01	9.293e-01	9.628e-01	9.809e-01	9.903e-01	9.9519-01	
1.5	V_1	5.060e-02	2.728e-02	1.420e-02	7.254e-03	3.666e-03	1.842e-03	9.238e-04	4.625e-04
		8.912e-01	9.414e-01	9.695e-01	9.845e-01	9.922e-01	9.962e-01	9.981e-01	
	V_2	5.125e-02	2.748e-02	1.426e-02	7.269e-03	3.669e-03	1.843e-03	9.237e-04	4.623e-04
		8.990e-01	9.464e-01	9.725e-01	9.862e-01	9.932e-01	9.967e-01	9.985e-01	
1.7	V_1	4.035e-02	2.110e-02	1.080e-02	5.464e-03	2.747e-03	1.377e-03	6.895e-04	3.449e-04
		9.350e-01	9.664e-01	9.831e-01	9.918e-01	9.962e-01	9.983e-01	9.993e-01	
	V_2	4.156e-02	2.167e-02	1.107e-02	5.593e-03	2.810e-03	1.408e-03	7.044e-04	3.522e-04
		9.392e-01	9.692e-01	9.850e-01	9.930e-01	9.971e-01	9.990e-01	9.999e-01	
1.9	V_1	3.449e-02	1.776e-02	9.015e-03	4.540e-03	2.277e-03	1.140e-03	5.703e-04	2.846e-04
		9.573e-01	9.786e-01	9.896e-01	9.952e-01	9.981e-01	9.994e-01	1.002e+00	
	V_2	3.607e-02	1.855e-02	9.406e-03	4.733e-03	2.372e-03	1.187e-03	5.935e-04	2.965e-04
		9.592e-01	9.801e-01	9.908e-01	9.962e-01	9.989e-01	1.000e+00	1.001e+00	
	E_1^N	1.291e-01	1.289e-01	1.185e-01	9.860e-02	7.339e-02	4.873e-02	2.927e-02	1.631e-02
	r_1^N	3.144e-03	1.202e-01	2.662e-01	4.260e-01	5.907e-01	7.354e-01	8.436e-01	
	E_2^N	1.430e-01	1.411e-01	1.288e-01	1.065e-01	7.887e-02	5.214e-02	3.120e-02	1.734e-02
	r_2^N	1.978e-02	1.305e-01	2.749e-01	4.335e-01	5.971e-01	7.406e-01	8.473e-01	

In this section, the theoretical results obtained in the preceding section are tested experimentally. For the following two test problems, the errors and rate of convergence for the proposed numerical method are provided. For our numerical solution, we use the double mesh principle [9] to estimate the error and compute the order of convergence.

Example 5.1.

$$\begin{cases} -D_*^\kappa v_1(x) + \left(\frac{x^3}{2}+1\right) v_1'(x) + 6v_1(x) - 2v_2(x) = 2, \ x \in D, \\ -D_*^\kappa v_2(x) + \left(\frac{x}{2}+1\right) v_2'(x) - 2v_1(x) + 6v_2(x) \ = 1.2, \ x \in D, \end{cases}$$

with the boundary conditions

$$\begin{cases} v_1(0) - v_1'(0) = 1, \ v_1(1) - \int_0^1 \frac{x}{3} v_1(x)dx = 2, \\ v_2(0) - v_2'(0) = 1, \ v_2(1) - \int_0^1 \frac{x}{2} v_2(x)dx = 2. \end{cases}$$

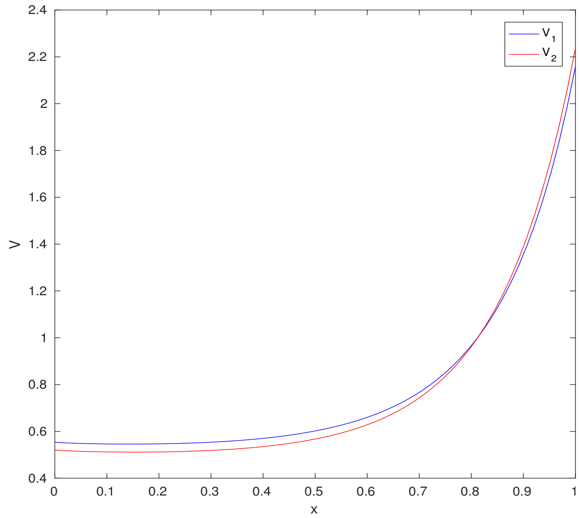

FIGURE 1. Approximate solutions of Example 5.1 for $N = 2^6$ and $\kappa = 1.3$.

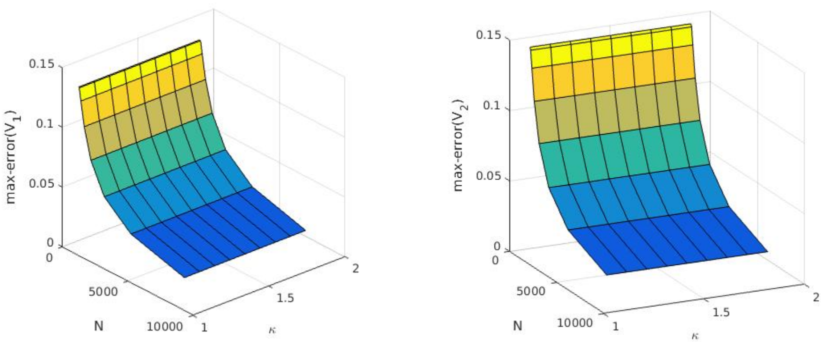

FIGURE 2. Error plot of Example 5.1.

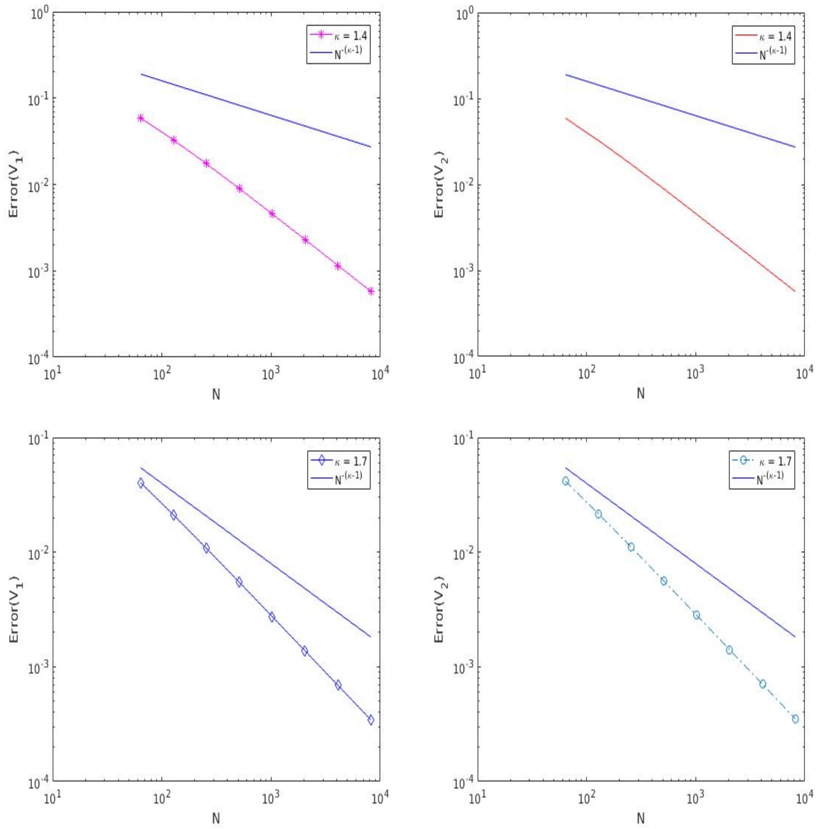

FIGURE 3. Loglog plot of Example 5.1.

Example 5.2.

$$\begin{cases} -D_*^{\kappa} v_1(x) + \left(\frac{x^3}{2}+1\right) v_1'(x) + 6v_1(x) - 2v_2(x) = e^x, \ x \in D, \\ -D_*^{\kappa} v_2(x) + \left(\frac{x}{3}+1\right) v_2'(x) - 2v_1(x) + 6v_2(x) \ = \cos x, \ x \in D, \end{cases}$$

with the boundary conditions

$$\begin{cases} v_1(0) - \left(\frac{1}{\kappa-1}\right) v_1'(0) = 0, \ v_1(1) - \int_0^1 \frac{x}{2} v_1(x)dx = 1, \\ v_2(0) - \left(\frac{1}{\kappa-1}\right) v_2'(0) = 0, \ v_2(1) - \int_0^1 \frac{x}{2} v_2(x)dx = 2. \end{cases}$$

TABLE 2. Evaluated maximum and uniform errors and their rates of convergence of Example 5.2 for different values of κ and N.

κ/N		64	128	256	512	1024	2048	4096	8192
1.1	V_1	5.073e-02	4.791e-02	4.088e-02	3.127e-02	2.146e-02	1.333e-02	7.651e-03	4.145e-03
		8.264e-02	2.286e-01	3.866e-01	5.432e-01	6.862e-01	8.017e-01	8.841e-01	
	V_2	1.149e-01	1.064e-01	8.806e-02	6.447e-02	4.193e-02	2.473e-02	1.360e-02	7.168e-03
		1.108e-01	2.733e-01	4.498e-01	6.206e-01	7.615e-01	8.620e-01	9.247e-01	
1.3	V_1	1.909e-02	1.046e-02	5.506e-03	2.827e-03	1.433e-03	7.214e-04	3.619e-04	1.812e-04
		8.675e-01	9.265e-01	9.615e-01	9.804e-01	9.902e-01	9.951e-01	9.976e-01	
	V_2	7.922e-02	4.539e-02	2.454e-02	1.280e-02	6.544e-03	3.309e-03	1.664e-03	8.343e-04
		8.033e-01	8.872e-01	9.389e-01	9.681e-01	9.837e-01	9.918e-01	9.959e-01	
1.5	V_1	1.530e-02	7.954e-03	4.057e-03	2.048e-03	1.029e-03	5.157e-04	2.581e-04	1.291e-04
		9.437e-01	9.712e-01	9.858e-01	9.931e-01	9.968e-01	9.986e-01	9.995e-01	
	V_2	5.934e-02	3.172e-02	1.643e-02	8.365e-03	4.220e-03	2.119e-03	1.062e-03	5.316e-04
		9.036e-01	9.490e-01	9.738e-01	9.869e-01	9.935e-01	9.969e-01	9.985e-01	
1.7	V_1	1.354e-02	6.938e-03	3.509e-03	1.764e-03	8.841e-04	4.423e-04	2.211e-04	1.105e-04
		9.653e-01	9.831e-01	9.922e-01	9.968e-01	9.990e-01	9.999e-01	1.001e+00	
	V_2	4.896e-02	2.549e-02	1.301e-02	6.574e-03	3.302e-03	1.654e-03	8.279e-04	4.140e-04
		9.411e-01	9.701e-01	9.853e-01	9.931e-01	9.970e-01	9.989e-01	9.998e-01	
1.9	V_1	1.258e-02	6.399e-03	3.224e-03	1.617e-03	8.097e-04	4.048e-04	2.023e-04	1.013e-04
		9.760e-01	9.887e-01	9.952e-01	9.984e-01	1.000e+00	1.000e+00	1.000e-01	
	V_2	4.296e-02	2.207e-02	1.118e-02	5.626e-03	2.820e-03	1.411e-03	7.056e-04	3.526e-04
		9.609e-01	9.808e-01	9.910e-01	9.962e-01	9.988e-01	1.000e+00	1.000e+00	
	E_1^N	5.073e-02	4.791e-02	4.088e-02	3.127e-02	2.146e-02	1.333e-02	7.651e-03	4.145e-03
	r_1^N	8.264e-02	2.286e-01	3.866e-01	5.432e-01	6.862e-01	8.017e-01	8.841e-01	
	E_2^N	1.149e-01	1.064e-01	8.806e-02	6.447e-02	4.193e-02	2.473e-02	1.360e-02	7.168e-03
	r_2^N	1.108e-01	2.733e-01	4.498e-01	6.206e-01	7.615e-01	8.620e-01	9.247e-01	

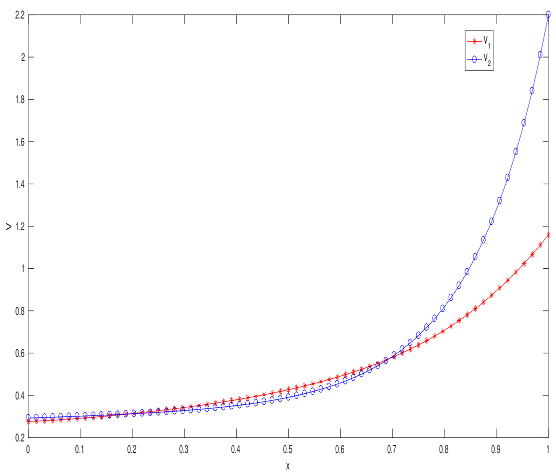

FIGURE 4. Approximate solutions of Example 5.2 for $N = 2^6$ and the value of $\kappa = 1.3$.

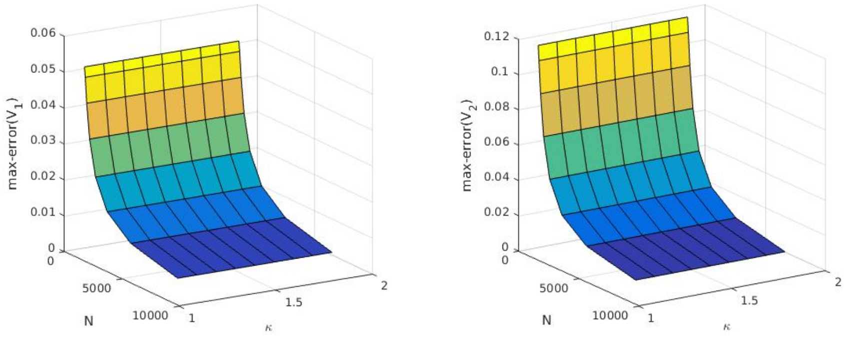

FIGURE 5. Error plot of Example 5.2.

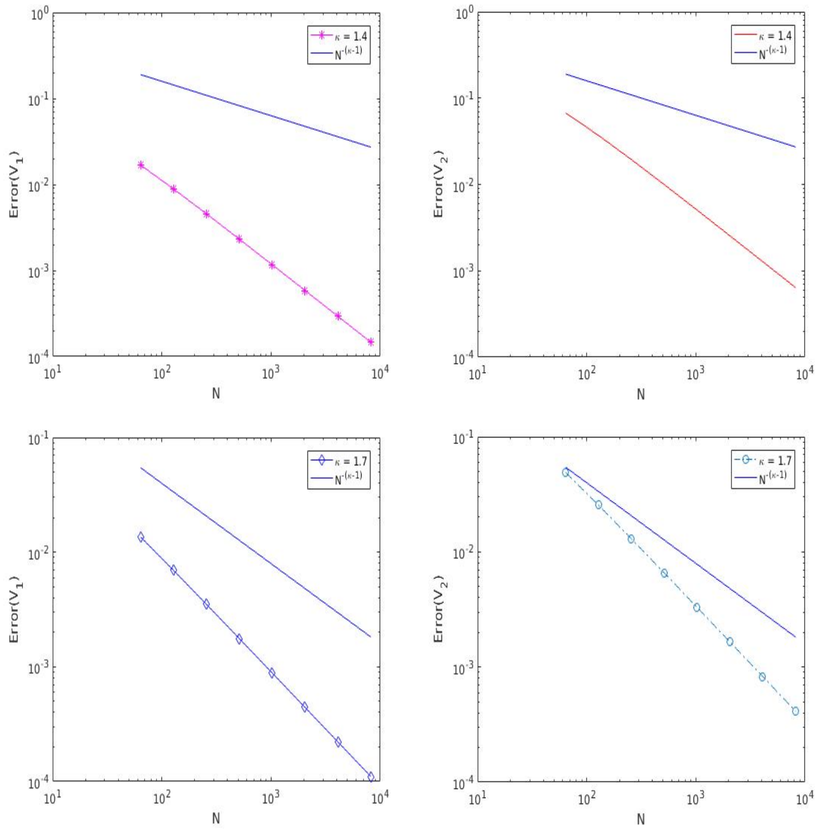

FIGURE 6. Loglog plot of Example 5.2.

The exact solution to these examples is not known. The rate of convergence is calculated using the double mesh principle which is defined as

$$E_\kappa^N = \max_{x_i \in \bar{D}^N} |V^N(x_i) - V^{2N}(x_i)| \text{ and } E^N = \max_\kappa E_\kappa^N,$$

where $V^N(x_i)$ and $V^{2N}(x_i)$ are computed solutions with N and $2N$ mesh points. The rate of convergence is obtained from

$$r_\kappa^N = \log_2\left(\frac{E_\kappa^N}{E_\kappa^{2N}}\right) \text{ and } r^N = \log_2(\frac{E^N}{E^{2N}}).$$

E_1^N and E_2^N in Tables 1 and 2 denote the maximum errors of V_1 and V_2 and r_1^N and r_2^N denote the order of convergence with respect to V_1 and V_2, respectively. The numerical solutions are depicted in Figure 1 and Figure 4. The error plots of Examples 5.1 and 5.2 are given in Figure 2 and Figure 5. Figures 3 and 6 depict the loglog plot of the maximum pointwise errors for the solutions V_1 and V_2 of Examples 5.1 and 5.2.

6. Conclusion

We used the finite difference scheme on a uniform mesh to solve a class of systems of fractional differential boundary value problems with integral boundary conditions. Two examples are provided to demonstrate the validity of the numerical method we have proposed. We have proved that our numerical method is of the order of $\kappa - 1$, $1 < \kappa < 2$, as seen in Table 1 and Table 2. The error plot in Figures 2 and 5 shows that as N increases, the maximum errors decrease. The loglog plot in Figures 3 and 6 validates our theoretical error bound which is of $O(N^{-(\kappa-1)})$. We are working on a numerical method for the class of fractional differential equations with integral boundary conditions for Caputo fractional derivative of various orders.

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