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ON COVERING AND QUOTIENT MAPS FOR $\mathcal{I}^{\mathcal{K}}$-CONVERGENCE IN TOPOLOGICAL SPACES

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Abstract. In this article, we show that the family of all $\mathcal{I}^{\mathcal{K}}$-open subsets in a topological space forms a topology if $\mathcal{K}$ is a maximal ideal. We introduce the notion of $\mathcal{I}^{\mathcal{K}}$-covering map and investigate some basic properties. The notion of quotient map is studied in the context of $\mathcal{I}^{\mathcal{K}}$-convergence and the relationship between $\mathcal{I}^{\mathcal{K}}$-continuity and $\mathcal{I}^{\mathcal{K}}$-quotient map is established. We show that for a maximal ideal $\mathcal{K}$, the properties of continuity and preserving $\mathcal{I}^{\mathcal{K}}$-convergence of a function defined on X coincide if and only if X is an $\mathcal{I}^{\mathcal{K}}$-sequential space.

1. Introduction

The concept of usual convergence of a sequence in a space can be addressed by two fundamental conditions: first, if the sequence is eventually contained in each open neighborhood of a point and secondly, if the sequence possesses a co-finite tail that is convergent to the point. These equivalent notions for usual convergence have been the core idea for introducing some distinct notions like $\mathcal{I}$-convergence, $\mathcal{I}^*$-convergence and $\mathcal{I}^{\mathcal{K}}$-convergence in the context of an ideal topological space. The former two notions were introduced in the year 2000, by Kostyrko et al. [5] and the later was introduced by Macaz and Sleziak [9], as an extension of $\mathcal{I}^*$-convergence. In a nutshell, the theory of generalized convergence has attracted many researchers. For some remarkable contributions in this direction, refer to [2,3,5,6,15,16] for ideal convergence and [7] for G-convergence by Lin et al.

At the same time, mappings such as continuous maps, quotient maps and covering maps [1,8] have been utilised as vital tools in characterising different properties of associated parent topological spaces. In the context of statistical convergence, Renukadevi and Prakash studied some of their interrelationships [11,12]. Recently, some articles have focused on the investigation of different maps in an ideal topological space [10,17,19]. These developments along with

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the prolonged study of $\mathcal{I}^{\mathcal{K}}$-convergence in the last decade [2, 3, 13] anticipated an active role of mappings in the $\mathcal{I}^{\mathcal{K}}$-context, in characterizing the underlying spaces. So, as a relevant continuation to the above work, we take up the study of different mappings along with the more general non-axiomatised form of convergence, i.e., $\mathcal{I}^{\mathcal{K}}$-convergence as an extension to prior work.

The main purpose of this article is to study the $\mathcal{I}^{\mathcal{K}}$-covering map, the $\mathcal{I}^{\mathcal{K}}$-continuous map, the $\mathcal{I}^{\mathcal{K}}$-quotient map and similar maps. We investigate them to establish some properties of the spaces as well as correlations among the mappings.

For our topological terminologies, notations and results, one may refer to [4]. A family of subsets $\mathcal{I}$ of ω is said to be a proper non-trivial ideal ($\mathcal{I} \neq \phi$ and $\omega \notin \mathcal{I}$) if

(i) $I, J \in \mathcal{I}$, then $I \cup J \in \mathcal{I}$.
(ii) $J \subset I$ and $I \in \mathcal{I}$, then $J \in \mathcal{I}$.

For an ideal $\mathcal{I}$ in $P(\omega)$, we have $\mathcal{I}^\star := \{I \subset \omega : I^c \in \mathcal{I}\}$ and also, $\mathcal{I}^+$ comprises of all subsets of ω not in $\mathcal{I}$. An ideal $\mathcal{I}$ is said to be a maximal ideal on ω if $M \subset \omega$, then either $M \in \mathcal{I}$ or $M^c \in \mathcal{I}$.

Remark 1.1 ([13, Remark 1.2]). Two ideals $\mathcal{I}$ and $\mathcal{K}$ on a set S satisfy the ideality condition if and only if $S \neq I \cup K$ for all $I \in \mathcal{I}$, $K \in \mathcal{K}$.

Lemma 1.2 ([19, Lemma 3.6]). *Let $P \subset X$, where X a topological space and $\mathcal{I}$ be an ideal. Then the following are equivalent for a sequence $\{a_n\}$ in X.*

(1) *P is an $\mathcal{I}$-open subset of X.*
(2) *$\{n \in \omega : a_n \in P\} \notin \mathcal{I}$, if $\mathcal{I} - \lim a_n = a \in P$.*
(3) *$|\{n \in \omega : a_n \in P\}| = \omega$, if $\mathcal{I} - \lim a_n = a \in P$.*

In Section 2, we obtain a condition on the ideal such that the family of $\mathcal{I}^{\mathcal{K}}$-open subsets of a topological space forms a topology (Remark 2.10). In Section 3, we introduce $(\mathcal{I}^{\mathcal{K}}, \mathcal{J}^{\mathcal{L}})$-covering map defined on a topological space for given ideals $\mathcal{I}, \mathcal{K}, \mathcal{J}, \mathcal{L}$ in ω and with the assumption of ideality condition among ideals, some properties are obtained. In Section 4, we show that for a maximal ideal $\mathcal{K}$, the properties of continuity and preserving $\mathcal{I}^{\mathcal{K}}$-convergence of a function defined on X coincides if and only if X is an $\mathcal{I}^{\mathcal{K}}$-sequential space (Theorem 4.6). In the same section, we define an $(\mathcal{I}^{\mathcal{K}}, \mathcal{J}^{\mathcal{L}})$-quotient map and obtain several properties. The role of maximality of ideals is investigated to establish a relationship among different mappings (Theorem 4.15).

Unless mentioned specifically, X, Y are arbitrary topological spaces and $\mathcal{I}$, $\mathcal{K}$, $\mathcal{J}$ and $\mathcal{L}$ are proper ideals on the set of natural numbers ω.

2. Some preliminary results

In this section, we review some basic results in $\mathcal{I}^{\mathcal{K}}$-convergence and obtain a few new ones. We begin with some prior discussions in the context of $\mathcal{I}^{\mathcal{K}}$-convergence as follows: a sequence $\{a_n\}$ in X is $\mathcal{I}^{\mathcal{M}}$-convergent to $a \in X$ if there exists a set $A \in \mathcal{I}^*$ such that the sequence $\{b_n\}_{n\in\omega}$ defined as:

$$b_n = \begin{cases} a_n, & n \in A \\ a, & n \notin A \end{cases}$$

is $\mathcal{M}$-convergent to a, where $\mathcal{M}$ is an ideal convergence mode.

If $\mathcal{M} = \mathcal{K}$, then $\{a_n\}$ is $\mathcal{I}^{\mathcal{K}}$-convergent [9] to a point $a \in X$. In particular, if $\mathcal{I} = \mathcal{K}$, it gives the condition of $\mathcal{I}$-convergence [5] for the sequence $\{a_n\}$ in X. Also, for $\mathcal{K} = Fin$ (the ideal containing all the finite subsets of ω), we say that $\{a_n\}$ is $\mathcal{I}^*$-convergent [5] to $a \in X$.

Reader may refer to [5] for definitions of $\mathcal{I}$-convergence and $\mathcal{I}^*$-convergence.

Lemma 2.1 ([9, Lemma 2.1])**.** *If $\mathcal{I}$ and $\mathcal{K}$ are two ideals on a set S and $q : S \to X$ is a function such that $\mathcal{K} - \lim q = a$, then $\mathcal{I}^{\mathcal{K}} - \lim q = a$.*

Definition 2.2 ([13, Definition 3.1])**.** Let X be a topological space and $O, F \subseteq X$. Then

(1) O is said to be $\mathcal{I}^{\mathcal{K}}$-open if no sequence in $X \setminus O$ has an $\mathcal{I}^{\mathcal{K}}$-limit in O. Otherwise, for each sequence $\{a_n : n \in \omega\} \subseteq X \setminus O$ with $a_n \to_{\mathcal{I}^{\mathcal{K}}} a \in X$, then $a \in X \setminus O$.

(2) A subset $F \subseteq X$ is said to be $\mathcal{I}^{\mathcal{K}}$-closed if $X \setminus F$ is $\mathcal{I}^{\mathcal{K}}$-open in X.

Lemma 2.3 ([13, Observation 3.3])**.** *Let $\mathcal{M}_1, \mathcal{M}_2$ be two convergence modes in a topological space X such that $\mathcal{M}_1$-convergence implies $\mathcal{M}_2$-convergence. Then $O \subseteq X$ is $\mathcal{M}_2$-open implies that O is $\mathcal{M}_1$-open.*

Proposition 2.4 ([13, Proposition 2.1])**.** *Let X be a topological space and $f : S \to X$ be a function (generalized sequence). Let $\mathcal{I}, \mathcal{K}$ be two ideals on S such that $\mathcal{I} \cup \mathcal{K}$ is an ideal. Then*

(i) $\mathcal{I}^{\mathcal{K}^*} - \lim f(s) = a$ *if and only if* $(\mathcal{I} \cup \mathcal{K})^* - \lim f(s) = a$.

(ii) $\mathcal{I}^{\mathcal{K}} - \lim f(s) = a \implies \mathcal{I} \cup \mathcal{K} - \lim f(s) = a$.

Lemma 2.5. *Let P be a subset of a topological space X. Then, the following are equivalent for a sequence $\{a_n\}$ in X.*

(1) *P is an $\mathcal{I}^{\mathcal{K}}$-open subset of X.*

(2) $\{n \in \omega : x_n \in P\} \notin \mathcal{K}$, *if* $\mathcal{I}^{\mathcal{K}} - \lim a_n = a \in P$.

(3) $|\{n \in \omega : x_n \in P\}| = \omega$, *if* $\mathcal{I}^{\mathcal{K}} - \lim a_n = a \in P$.

Proof. (1) $\implies$ (2) It follows from the definition of $\mathcal{I}^{\mathcal{K}}$-open subsets.

(2) $\implies$ (3) Since $Fin \subset \mathcal{K}$, it follows that $|\{n \in \omega : a_n \in P\}| = \omega$.

(3) $\implies$ (1) Contrapositively, let P be not $\mathcal{I}^{\mathcal{K}}$-open. Then $X \setminus P$ is not $\mathcal{I}^{\mathcal{K}}$-closed, and there is a sequence $\{a_n\} \subseteq X \setminus P$ with $\mathcal{I}^{\mathcal{K}} - \lim a_n = a \in P$. This is a contradiction to our assumption. $\square$

Theorem 2.6. *In a topological space, arbitrary union of $\mathcal{I}^{\mathcal{K}}$-open sets is $\mathcal{I}^{\mathcal{K}}$-open.*

Proof. Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be an arbitrary family of $\mathcal{I}^{\mathcal{K}}$-open sets. Let $\{a_n\}$ be a sequence in X such that $\mathcal{I}^{\mathcal{K}} - \lim a_n = a \in \bigcup_{\alpha\in\Lambda} U_\alpha$. Then there exists an element $\alpha \in \Lambda$ for which $a_n \to_{\mathcal{I}^{\mathcal{K}}} a \in U_\alpha$. So, by Proposition 2.8, there exists $K \in \mathcal{K}$ such that $\{a_n : n \in \omega \setminus K\} \in U_\alpha$. Thus, $\{a_n : n \in \omega \setminus K\} \in \bigcup_{\alpha\in\Lambda} U_\alpha$. Thus, $\bigcup_{\alpha\in\Lambda} U_\alpha$ is an $\mathcal{I}^{\mathcal{K}}$-open set. $\square$

Definition 2.7. For a given ideal $\mathcal{I}$ of ω and a sequence $\{a_n\}$ in X, the set $\{n \in \omega : a_n \in \omega \setminus I,\ I \in \mathcal{I}\}$ is said to be an $\mathcal{I}$-tail of X.

Proposition 2.8. *Let $\mathcal{K}$ be a maximal ideal of ω and $P \subset X$. Then P is $\mathcal{I}^{\mathcal{K}}$-open if and only if each $\mathcal{I}^{\mathcal{K}}$-convergent X-valued sequence converging to a point in P has a $\mathcal{K}$-tail in P.*

Proof. Consider P to be $\mathcal{I}^{\mathcal{K}}$-open and $\{a_n\}$ be $\mathcal{I}^{\mathcal{K}}$-convergent to $a \in P$. Then we have $E = \{n \in \omega : a_n \in P\} \notin \mathcal{K}$. Since $\mathcal{K}$ is maximal, therefore, $\omega \setminus E = \{n \in \omega : a_n \in P\} \in \mathcal{K}^*$. Thus, $\{a_n : n \in \omega \setminus E\} \in P$.

Conversely, suppose that P is not $\mathcal{I}^{\mathcal{K}}$-open. Then, for a sequence $\{a_n\}$ in $X \setminus P$ such that $\mathcal{I}^{\mathcal{K}} - \lim a_n = a \in P$, we have that $\{n \in \omega : a_n \in P\} \in \mathcal{K}^*$. It is clear that $\{n \in \omega : a_n \notin P\} \in \mathcal{K}$. Consider, $E = \{n \in \omega : a_n \notin P\}$, then $\{a_n : n \in \omega \setminus E\} \in P$. But $a_n \in X \setminus P$, so, E must be equal to ω. This is a contradiction to the fact that $\mathcal{K}$ is a proper ideal. Hence, P is $\mathcal{I}^{\mathcal{K}}$-open. $\square$

Theorem 2.9. *In a topological space X, if $\mathcal{J}$ be a maximal ideal and P, Q are two $\mathcal{I}^{\mathcal{J}}$-open subsets of X, then $P \cap Q$ is $\mathcal{I}^{\mathcal{J}}$-open.*

Proof. Let $\{a_n\}$ be a sequence in X such that $\mathcal{I}^{\mathcal{J}} - \lim a_n = a \in P \cap Q$. Then, $a_n \to_{\mathcal{I}^{\mathcal{J}}} a \in P$ and $a_n \to_{\mathcal{I}^{\mathcal{J}}} a \in Q$ simultaneously. By Proposition 2.8, there exist $E, F \in \mathcal{J}$ such that the corresponding $\mathcal{J}$-tails which are $\{a_n : n \in \omega \setminus E\}$ and $\{a_n : n \in \omega \setminus F\}$ belong to P and Q, respectively. Thus, for $E \cup F \in \mathcal{J}$, the $\mathcal{J}$-tail of $\{a_n\}$, that is, $\{a_n : n \in \omega \setminus E \cup F\} \in P \cap Q$. Hence, by Proposition 2.8, $P \cap Q$ is $\mathcal{I}^{\mathcal{J}}$-open. $\square$

Remark 2.10. If $\mathcal{J}$ be a maximal ideal, then

(i) Consider the set $\tau_{\mathcal{I}^{\mathcal{J}}}$: the collection of all $\mathcal{I}^{\mathcal{J}}$-open subsets of X, then from Theorems 2.9 and 2.6, we observe that $(X, \tau_{\mathcal{I}^{\mathcal{J}}})$, is a topological space. $(X, \tau_{\mathcal{I}^{\mathcal{J}}})$ is said to be the $\mathcal{I}^{\mathcal{J}}$-sequential coreflection of the space X and we denote it by $\mathcal{I}^{\mathcal{J}}$-$sX$.

(ii) The $\mathcal{I}^{\mathcal{J}}$-sequential coreflection of a space X is an $\mathcal{I}^{\mathcal{J}}$-sequential space: for each $A \subset X$, by Proposition 2.8, A is $\mathcal{I}^{\mathcal{J}}$-open in $\mathcal{I}^{\mathcal{J}}$-$sX$ if and only if A is $\mathcal{I}^{\mathcal{J}}$-open in X if and only if A is open in $\mathcal{I}^{\mathcal{J}}$-$sX$. Thus, $\mathcal{I}^{\mathcal{J}}$-$sX$ is an $\mathcal{I}^{\mathcal{J}}$-Sequential space.

(iii) X is an $\mathcal{I}^{\mathcal{J}}$-Sequential space if and only if $\mathcal{I}^{\mathcal{J}}$-$sX = X$.

Definition 2.11. $P \subset X$ is said to be an $\mathcal{I}^{\mathcal{K}}$-sequential neighborhood of a point $a \in X$ whenever a sequence $\{a_n\}_{n\in\omega}$ is $\mathcal{I}^{\mathcal{K}}$-convergent to a, the sequence $\{a_n\}_{n\in\omega}$ possesses an $\mathcal{I}$-tail which is $\mathcal{K}$-eventually in P, i.e., there exists $M \in \mathcal{I}^*$ with $\{n \in M : a_n \notin P\} \in \mathcal{K}$.

Remark 2.12. If $\mathcal{J}$ is a maximal ideal of ω and $P \subset X$, then P is $\mathcal{I}^{\mathcal{J}}$-open if and only if P is an $\mathcal{I}^{\mathcal{J}}$-sequential neighborhood of each of its points.

Lemma 2.13. *Let X be a topological space and $P, Q \subset X$. Then*

(1) *If $P \subset Q$, then P is an $\mathcal{I}^{\mathcal{K}}$-sequential neighborhood of $a \implies Q$ is an $\mathcal{I}^{\mathcal{K}}$-sequential neighborhood of a.*
(2) *If P, Q are $\mathcal{I}^{\mathcal{K}}$-sequential neighborhoods of a, then $P \cap Q$ is an $\mathcal{I}^{\mathcal{K}}$-sequential neighborhood of a.*

Proof. We prove (2). Let us assume that P, Q are two $\mathcal{I}^{\mathcal{K}}$-sequential neighborhoods of a in X. Consider $a \in (P \cap Q)$ with $a_n \to_{\mathcal{I}^{\mathcal{K}}} a \in P \cap Q$. Then $a_n \to_{\mathcal{I}^{\mathcal{K}}} a \in P$ and $a_n \to_{\mathcal{I}^{\mathcal{K}}} a \in Q$. So, there exist $M_1, M_2 \in \mathcal{I}^*$ such that $\{n \in M_1 : a_n \notin P\} \in \mathcal{K}$ and $\{n \in M_2 : a_n \notin Q\} \in \mathcal{K}$. Now, for $M = M_1 \cap M_2 \in \mathcal{I}^*$, $\{n \in M : a_n \notin (P \cap Q)\} \subset \{n \in M_1 : a_n \notin P\} \cap \{n \in M_2 : a_n \notin Q\} \in \mathcal{K}$. In essence, $P \cap Q$ is an $\mathcal{I}^{\mathcal{K}}$-sequential neighborhood of a. □

Proposition 2.14. *Let $\mathcal{J}$ be a maximal ideal of ω and $P \subset X$. If P is not an $\mathcal{I}^{\mathcal{J}}$-sequentially neighborhood of a, then there exists a sequence $\{a_n\}$ in $X \setminus P$ such that $a_n \to_{\mathcal{I}^{\mathcal{J}}} a$.*

Proof. Suppose that P is not an $\mathcal{I}^{\mathcal{J}}$-sequentially neighborhood of a. Then there exists a sequence $\{a_n\}$ in X such that $a_n \to_{\mathcal{I}^{\mathcal{J}}} a$. So, there exists $M \in \mathcal{I}^*$ for which $\{n \in M : a_n \notin P\} \notin \mathcal{J}$. By maximality of $\mathcal{J}$, $\{n \in M : a_n \notin P\} \in \mathcal{J}^* \implies \{n \in \omega : a_n \notin P\} \notin \mathcal{J}^* \implies \{n \in \omega : a_n \in P\} \in \mathcal{J}$. Consider $J = \{n \in \omega : a_n \in P\}$. Since $\mathcal{J}$ is proper, $P \neq X$. Let $b \in X \setminus P$. Define a sequence $\{b_n\}$ such that $b_n = b$, if $n \in J$ and $b_n = a_n$ if $n \in \omega \setminus J$. But then $b_n \in X \setminus P$ and $b_n \to_{\mathcal{I}^{\mathcal{J}}} b$. This is a contradiction. □

The following operators on P are called $\mathcal{I}^{\mathcal{J}}$-closure and $\mathcal{I}^{\mathcal{J}}$-interior of $P \subset X$.

$[P]_{\mathcal{I}^{\mathcal{J}}} = \{a \in X :$ there exists a sequence $\{a_n\}$ in P with $a_n \to_{\mathcal{I}^{\mathcal{J}}} a\}$.

$(P)_{\mathcal{I}^{\mathcal{J}}} = \{a \in X : P$ is an $\mathcal{I}^{\mathcal{J}}$-sequential neighborhood of $a\}$.

For a maximal ideal $\mathcal{J}$, it is clear that $[\phi]_{\mathcal{I}^{\mathcal{J}}} = \phi$. From the definitions, it is observed that the hierarchy $P^o \subset (P)_{\mathcal{I}^{\mathcal{J}}} \subset P \subset [P]_{\mathcal{I}^{\mathcal{J}}} \subset \bar{P}$ holds.

Lemma 2.15. *Consider $\mathcal{J}$ to be a maximal ideal and P, $Q \subset X$. Then*

(i) $[P]_{\mathcal{I}^{\mathcal{J}}} = X \setminus (X \setminus P)_{\mathcal{I}^{\mathcal{J}}}$.
(ii) *If $P \subset Q$, then $(Q)_{\mathcal{I}^{\mathcal{J}}} \subset (P)_{\mathcal{I}^{\mathcal{J}}}$ and $[Q]_{\mathcal{I}^{\mathcal{J}}} \subset [P]_{\mathcal{I}^{\mathcal{J}}}$.*
(iii) $(P \cap Q)_{\mathcal{I}^{\mathcal{J}}} = (P)_{\mathcal{I}^{\mathcal{J}}} \cap (Q)_{\mathcal{I}^{\mathcal{J}}}$ *and* $[P \cup Q]_{\mathcal{I}^{\mathcal{J}}} = [P]_{\mathcal{I}^{\mathcal{J}}} \cup [Q]_{\mathcal{I}^{\mathcal{J}}}$.

Proof. Let $\mathcal{J}$ be a maximal ideal and P, $Q \subset X$.

(i) Suppose that $a \in X \setminus (X \setminus P)_{\mathcal{I}^{\mathcal{J}}}$. Then $a \notin (X \setminus P)_{\mathcal{I}^{\mathcal{J}}}$, i.e., $(X \setminus P)$ is not an $\mathcal{I}^{\mathcal{J}}$-sequential neighborhood of a. Then by Proposition 2.14, there exists a sequence $\{a_n\}$ in P such that $a_n \to_{\mathcal{I}^{\mathcal{J}}} a$, that means $a \in [P]_{\mathcal{I}^{\mathcal{J}}}$. Conversely, let $a \in [P]_{\mathcal{I}^{\mathcal{J}}}$, then there exists a sequence $\{a_n\}$ in P such that $a_n \to_{\mathcal{I}^{\mathcal{J}}} a$.

That is, $\{n \in \omega : a_n \in X \setminus P\} = \phi$. But then $(X \setminus P)$ is not an $\mathcal{I}^{\mathcal{J}}$-sequential neighborhood of a. So, $a \notin (X \setminus P)_{\mathcal{I}^{\mathcal{J}}}$. Thus, $a \in X \setminus (X \setminus P)_{\mathcal{I}^{\mathcal{J}}}$.

(ii) Since $P \subset Q$ in X, then by Lemma 2.13(1), we obtain that $(Q)_{\mathcal{I}^{\mathcal{J}}} \subset (P)_{\mathcal{I}^{\mathcal{J}}}$ and $[Q]_{\mathcal{I}^{\mathcal{J}}} \subset [P]_{\mathcal{I}^{\mathcal{J}}}$.

(iii) Since $(P \cap Q) \subset P$ and $(P \cap Q) \subset Q$, then $(P \cap Q)_{\mathcal{I}^{\mathcal{J}}} \subset (P)_{\mathcal{I}^{\mathcal{J}}}$ and $(P \cap Q)_{\mathcal{I}^{\mathcal{J}}} \subset (Q)_{\mathcal{I}^{\mathcal{J}}}$. Then $(P \cap Q)_{\mathcal{I}^{\mathcal{J}}} \subset (P)_{\mathcal{I}^{\mathcal{J}}} \cap (Q)_{\mathcal{I}^{\mathcal{J}}}$. Again, suppose that $a \in (P)_{\mathcal{I}^{\mathcal{J}}} \cap (Q)_{\mathcal{I}^{\mathcal{J}}}$. So, $a \in (P)_{\mathcal{I}^{\mathcal{J}}}$ and $a \in (Q)_{\mathcal{I}^{\mathcal{J}}}$. Hence, P and Q both are $\mathcal{I}^{\mathcal{J}}$-sequential neighborhoods of a in X. Again, by Lemma 2.13(2), we have $P \cap Q$ is an $\mathcal{I}^{\mathcal{J}}$-sequential neighborhood of a in X. Thus, $a \in (P \cap Q)_{\mathcal{I}^{\mathcal{J}}}$.

Now, consider $X \setminus P$ and $X \setminus Q$ instead of P and Q, respectively, then we obtain $((X \setminus P) \cap (X \setminus Q))_{\mathcal{I}^{\mathcal{J}}} = (X \setminus P)_{\mathcal{I}^{\mathcal{J}}} \cap (X \setminus Q)_{\mathcal{I}^{\mathcal{J}}}$. Therefore, we have $[P \cup Q]_{\mathcal{I}^{\mathcal{J}}} = X \setminus (X \setminus (P \cup Q)_{\mathcal{I}^{\mathcal{J}}}) = X \setminus ((X \setminus P) \cap (X \setminus Q))_{\mathcal{I}^{\mathcal{J}}} = X \setminus ((X \setminus P)_{\mathcal{I}^{\mathcal{J}}} \cap (X \setminus Q)_{\mathcal{I}^{\mathcal{J}}}) = (X \setminus (X \setminus P)_{\mathcal{I}^{\mathcal{J}}}) \cap (X \setminus (X \setminus Q)_{\mathcal{I}^{\mathcal{J}}}) = [P]_{\mathcal{I}^{\mathcal{J}}} \cup [Q]_{\mathcal{I}^{\mathcal{J}}}$. □

3. $\mathcal{I}^{\mathcal{K}}$-covering property

The concept of covering map has recently been introduced as an ideal sequence covering map or an $\mathcal{I}$-sequence covering map [10, 18] in the context of ideal convergence. In this section, we define the notion of $(\mathcal{I}^{\mathcal{K}}, \mathcal{J}^{\mathcal{L}})$-sequence covering map extending the idea of ideal sequence covering map.

Definition 3.1. Let X and Y be two topological spaces and let $\mathcal{I}$, $\mathcal{K}$, $\mathcal{J}$ and $\mathcal{L}$ are proper ideals. Then $q : X \to Y$ is said to be an $(\mathcal{I}^{\mathcal{K}}, \mathcal{J}^{\mathcal{L}})$-sequence covering map if for a sequence $\{b_n\}$ such that $b_n \to_{\mathcal{J}^{\mathcal{L}}} b \in Y$, then there exists $a_n \in q^{-1}(b_n)$ for each $n \in \omega$ such that $a_n \to_{\mathcal{I}^{\mathcal{K}}} a \in q^{-1}(b)$.

These are generalizations of the covering maps defined in [10, 18].

Proposition 3.2. *Let X, Y and Z be topological spaces. Then*

(a) *If $p : X \to Y$ is an $(\mathcal{I}^{\mathcal{K}}, \mathcal{M}^{\mathcal{N}})$-sequence covering map and $q : Y \to Z$ is an $(\mathcal{M}^{\mathcal{N}}, \mathcal{J}^{\mathcal{L}})$-sequence covering map, then $q \circ p$ is an $(\mathcal{I}^{\mathcal{K}}, \mathcal{J}^{\mathcal{L}})$-sequence covering map.*

(b) *Finite product of $(\mathcal{I}^{\mathcal{K}}, \mathcal{J}^{\mathcal{L}})$-sequence covering maps is an $(\mathcal{I}^{\mathcal{K}}, \mathcal{J}^{\mathcal{L}})$-sequence covering map.*

(c) *Restriction of an $(\mathcal{I}^{\mathcal{K}}, \mathcal{J}^{\mathcal{L}})$-sequence covering map is an $(\mathcal{I}^{\mathcal{K}}, \mathcal{J}^{\mathcal{L}})$-sequence covering map.*

Proof. (a) Let us consider a sequence $c_n \to_{\mathcal{I}^{\mathcal{J}}} c$ in Z. By the $(\mathcal{M}^{\mathcal{N}}, \mathcal{J}^{\mathcal{L}})$-sequence covering property of q, there exists $b_n \in q^{-1}(c_n)$, for each n, such that the sequence $b_n \to_{\mathcal{M}^{\mathcal{N}}} b \in q^{-1}(c)$. Again, by the $(\mathcal{I}^{\mathcal{K}}, \mathcal{M}^{\mathcal{N}})$-sequence covering property of p, there exists $a_n \in p^{-1}(b_n) \in (p \circ q)^{-1}(c_n)$ such that $a_n \to_{\mathcal{I}^{\mathcal{K}}} a \in q^{-1}(b) \in (p \circ q)^{-1}(c)$.

(b) Let $q_i : X_i \to Y_i$ be an $(\mathcal{I}^{\mathcal{K}}, \mathcal{J}^{\mathcal{L}})$-sequence covering map for each $i = 1, 2, \ldots, N_0$. Consider the product map $\prod_{i=1}^{N_0} q_i : \prod_{i=1}^{N_0} X_i \to \prod_{i=1}^{N_0} Y_i$. Now suppose that $\{(b_{i,n})\}_{n \in \omega}$ is a sequence in $\prod_{i=1}^{N_0} Y_i$, $\mathcal{J}^{\mathcal{L}}$-converging to (b_i)

in $\prod_{i=1}^{N_0} Y_i$. Now, for each $i \in \{1, 2, 3, \ldots, N_0\}$, $b_{i,n} \to_{\mathcal{J}^{\mathcal{L}}} b_i \in Y_i$. By our assumption, there exists $a_{i,n} \in q^{-1}(y_{i,n})$ for each $n \in \omega$ such that the sequence $a_{i,n} \to_{\mathcal{I}^{\mathcal{K}}} a_i \in X_i$. Hence, the sequence $\{(a_{i,n})\}_{n\in\omega} \in \prod_{i=1}^{N_0} X_i$ is $\mathcal{I}^{\mathcal{K}}$-convergent to $(a_i) \in \prod_{i=1}^{N_0} X_i$.

(c) Consider an $(\mathcal{I}^{\mathcal{K}}, \mathcal{J}^{\mathcal{L}})$-sequence covering map q from X to Y. Suppose $q\big|_{q^{-1}(H)}$ is a restriction of q, where $H \subset Y$. Consider a sequence $\{q_n\}$ such that $b_n \to_{\mathcal{J}} b \in H \implies b_n \to_{\mathcal{J}} b$ in Y. By our assumption, there exists a sequence $\{a_n\}$, $a_n \in q^{-1}(b_n) \subset q^{-1}(H)$, for each n such that $a_n \to_{\mathcal{I}^{\mathcal{J}}} a \in q^{-1}(b) \subset q^{-1}(H)$, i.e., $q\big|_{q^{-1}(H)}$ is an $(\mathcal{I}^{\mathcal{K}}, \mathcal{J}^{\mathcal{L}})$-sequence covering map. □

Theorem 3.3 ([14]). *Let $\{(X_\alpha, \tau_\alpha) : \alpha \in \Lambda\}$ be an indexed family of topological spaces. Let $X = \prod_{\alpha\in\Lambda} X_\alpha$ be the product space and $\{x_\alpha(s)\}$ be a function in X_α for each $\alpha \in \Lambda$. Then $\{x_\alpha(s)\}$ is $\mathcal{I}^{\mathcal{K}}$-convergent to p_α, for all $\alpha \in \Lambda$ if and only if $\{(x_\alpha(s))_{\alpha\in\Lambda}\}$ is $\mathcal{I}^{\mathcal{K}}$-convergent to $(p_\alpha)_{\alpha\in\Lambda}$.*

Proposition 3.4. *Arbitrary product of $(\mathcal{I}^{\mathcal{K}}, \mathcal{J}^{\mathcal{L}})$-sequence covering maps is an $(\mathcal{I}^{\mathcal{K}}, \mathcal{J}^{\mathcal{L}})$-covering map.*

Proof. This proof immediately follows from Theorem 3.3. □

Theorem 3.5. *For X, Y be two topological spaces and $\mathcal{I}$, $\mathcal{K}$, $\mathcal{J}$, $\mathcal{L}$ be proper ideals of ω. If q is an $(\mathcal{I}^{\mathcal{K}}, \mathcal{J}^{\mathcal{L}})$-covering compact function from X to Y and $\exists$ a disjoint sequence of infinite subsets of ω, $\{Q_n\}$ where $Q_n \notin \mathcal{I}, \mathcal{K}$ for each n. Then for $b \in Y$, $\exists a \in q^{-1}(b)$ such that O is an open neighborhood of $a \implies q(O)$ is a sequential neighborhood of b, provided $\mathcal{I} \cup \mathcal{K}$ is an ideal.*

Proof. Contrapositively, let there be an element b in Y for which every $a \in q^{-1}(b)$ possesses neighborhood O_x of a such that $q(O_x)$ is not a sequential neighborhood of b. By our assumption, $q^{-1}(b)$ is compact in X and also $q^{-1}(b) \subset \cup_{a\in q^{-1}(b)} O_a$. So, we have $a_1, a_2, \ldots, a_{n_0} \in q^{-1}(b)$ for which $q^{-1}(b) \subset \cup_{i=1}^{n_0} O_{a_i}$. Again, the set $q(O_{a_m})$ is not a sequential neighborhood of b. Consider the sequence $\{b_{m,n}\}_{n=1}^{\infty}$ in Y where $b_{m,n} \notin q(O_{a_m})$ for all $m \in \{1, 2, 3, \ldots, n_0\}$, $n \in \omega$ such that $b_{m,n} \to b$. We define a sequence $\{b_n\}$ as follows: Let $b_k = b_{m,k}$ if $k \in M_m$, $m \in \{1, 2, 3, \ldots, n_0\}$ and $b_k = b$, otherwise. It implies that b_k converges to b which further implies that $b_k \to_{\mathcal{J}^{\mathcal{L}}} b$. By our assumption on q, there exists $a_k \in q^{-1}(b_k)$ for each k such that the sequence $a_k \to_{\mathcal{I}^{\mathcal{K}}} a \in q^{-1}(b)$. So, $a \in q^{-1}(b) \subset \cup_{i=1}^{n_0} O_{a_i}$. That way $\exists m_0$ for which $a \in O_{a_{m_0}}$ and the set $\{k : a_k \notin O_{a_{m_0}}\} \in \mathcal{I} \cup \mathcal{K}$ which implies that the set $\{k : q(a_k) \notin q(O_{a_{m_0}})\} \in \mathcal{I} \cup \mathcal{K}$. But, $M_{m_0} \subset \{k : b_k \notin q(O_{a_{m_0}})\} \in \mathcal{I} \cup \mathcal{K}$. Since $M_{m_0} \notin \mathcal{I} \cup \mathcal{K}$, that is a contradiction. Then $q(O)$ is a sequential neighborhood of b. □

Theorem 3.6. *Let X be a strongly Frechet space with property ωD. If $q : X \to Y$ is a closed and $(\mathcal{I}^{\mathcal{K}}, \mathcal{J}^{\mathcal{L}})$-covering map, then Y is strongly Frechet, provided $\mathcal{I} \cup \mathcal{K}$ is an ideal.*

Proof. It follows directly from Theorem 3.9 in [10] and Proposition 2.4. □

4. $\mathcal{I}^{\mathcal{K}}$-continuity and $(\mathcal{I}^{\mathcal{K}}, \mathcal{J}^{\mathcal{L}})$-quotient map

In this section, we study $\mathcal{I}^{\mathcal{K}}$-continuous maps defined on a topological space. The $\mathcal{I}^{\mathcal{K}}$-continuous maps are the generalization of $\mathcal{I}$-continuous maps [19] defined by Zhou et al.

Definition 4.1. Consider a mapping $q : X \to Y$, where X and Y are two topological spaces. Then for two given ideals $\mathcal{I}$ and $\mathcal{K}$ on ω, we have the following definitions.

(1) q is said to be possesses the property of preserving $\mathcal{I}^{\mathcal{K}}$-convergence if for a given sequence $\{a_n\}$ in X with $\mathcal{I}^{\mathcal{K}} - \lim a_n = a \in X$, the sequence $\{q(a_n)\}$ is $\mathcal{I}^{\mathcal{K}}$-convergent to $q(a)$.
(2) q is an $\mathcal{I}^{\mathcal{K}}$-continuous map if O is an $\mathcal{I}^{\mathcal{K}}$-open set in Y, then $q^{-1}(O)$ is $\mathcal{I}^{\mathcal{K}}$-open in X.

Theorem 4.2. *For a given mapping $q : X \to Y$, where X, Y be topological spaces. Then*

(1) *The property of continuity of q implies that of property of preserving $\mathcal{I}^{\mathcal{K}}$-convergence* [13].
(2) *q preserves $\mathcal{I}^{\mathcal{K}}$-convergence implies that q is an $\mathcal{I}^{\mathcal{K}}$-continuous map.*

Proof. Consider the mapping $q : X \to Y$. Then

(1) Reader may refer to [13, Theorem 3.11] for the proof.

(2) Let us assume that q possesses the property of preserving $\mathcal{I}^{\mathcal{K}}$-convergence and let O be $\mathcal{I}^{\mathcal{K}}$-closed in Y. Consider $q^{-1}(O)$ to be non-$\mathcal{I}^{\mathcal{K}}$-closed in X. Therefore, there exists a sequence $a_n \to_{\mathcal{I}^{\mathcal{K}}} a \notin q^{-1}(O)$. Then $\mathcal{I}^{\mathcal{K}} - \lim q(a_n) = q(a)$ but $q(a) \notin O$. This is a contradiction. Hence the result follows. □

For a maximal ideal $\mathcal{J}$ on ω, the converse of Theorem 4.2(2) is also true. Suppose that q is $\mathcal{I}^{\mathcal{J}}$-continuous and the sequence $a_n \to_{\mathcal{I}^{\mathcal{J}}} a$ in X. Let O be an open neighborhood of $q(a)$. Then, $q^{-1}(O)$ is $\mathcal{I}^{\mathcal{J}}$-open in X containing a. But, by Remark 2.12, there exists $M \in \mathcal{I}^*$ such that $\{n \in M : a_n \notin q^{-1}(O)\} \in \mathcal{J}$. Hence, $\{n \in M : q(a_n) \notin O\} \in \mathcal{J}$.

Theorem 4.3. *Consider a mapping $q : X \to Y$, where X and Y are two topological spaces. Then the following are equivalent.*

(1) *q is a continuous map.*
(2) *q possesses the property of preserving $\mathcal{I}^{\mathcal{K}}$-convergence.*
(3) *q is $\mathcal{I}^{\mathcal{K}}$-continuous.*

Proof. (1) $\implies$ (2) Reader may refer to Theorem 4.2(1).

(2) $\implies$ (1) Let q possess the property of preserving $\mathcal{I}^{\mathcal{K}}$-convergence. If possible, let q be not continuous. Then, there is an open set $O \subset Y$ such that $q^{-1}(O)$ is not open in X. So, $q^{-1}(O)$ is not $\mathcal{I}^{\mathcal{K}}$-open in X. That means there exists a sequence $\{a_n\}$ in $X \setminus q^{-1}(O)$ which $\mathcal{I}^{\mathcal{K}}$-converges to $\xi \in f^{-1}(O)$. Thus,

$\{f(a_n)\}$ is a sequence in the closed set $Y \setminus O$ which does not $\mathcal{I}^{\mathcal{K}}$-converges to $q(\xi) \in O$, as O is an $\mathcal{I}^{\mathcal{K}}$-open subset of Y. Hence, q does not possess the property of preserving $\mathcal{I}^{\mathcal{K}}$-convergence.

(3) $\Longrightarrow$ (1) Let q be an $\mathcal{I}^{\mathcal{K}}$-continuous map and let H be an arbitrary closed set in Y. Then H is $\mathcal{I}^{\mathcal{K}}$-closed in Y. Since q is an $\mathcal{I}^{\mathcal{K}}$-continuous map, $q^{-1}(H)$ is $\mathcal{I}^{\mathcal{K}}$-closed in X. As the space X is $\mathcal{I}^{\mathcal{K}}$-sequential, so, $q^{-1}(H)$ is closed in X. Thus, q is a continuous map. □

Corollary 4.4. *Let X be a sequential space. Let $q : X \to Y$ be a mapping from a sequential space X to a topological space Y. Then the following are equivalent.*

(1) *q is a continuous map.*
(2) *q possesses the property of preserving $\mathcal{I}^{\mathcal{K}}$-convergence.*
(3) *q is an $\mathcal{I}^{\mathcal{K}}$-continuous map.*
(4) *q is a sequentially continuous map.*

Proof. The proof follows immediately, as each sequential space is an $\mathcal{I}$-sequential space and the notions of continuity and sequentially continuity coincide in a sequential space. □

In view of the above, we now have the following question.

Question 4.5. In Theorem 4.3, whether the condition of X to be an $\mathcal{I}^{\mathcal{K}}$-sequential space is necessary for the properties of continuity and preserving $\mathcal{I}^{\mathcal{K}}$-convergence to coincide?

Answer to Question 4.5 is in the affirmative whenever $\mathcal{K}$ is a maximal ideal: Let us assume that X is not a $\mathcal{I}^{\mathcal{K}}$-sequential space. Then, consider the map $i : (X, \tau) \to (X, \tau_{\mathcal{I}^{\mathcal{K}}})$, where $\tau_{\mathcal{I}^{\mathcal{K}}}$ is as mentioned in Remark 2.10. Since X is not $\mathcal{I}^{\mathcal{K}}$-sequential, the topology $\tau_{\mathcal{I}^{\mathcal{K}}}$ is finer than τ. Hence the identity map $i : \tau \to \tau_{\mathcal{I}^{\mathcal{K}}}$ is not continuous. Suppose $\{x_n\}$ is $\mathcal{I}^{\mathcal{K}}$-convergent to y in (X, τ). Then every open neighborhood of P of y in $(X, \tau_{\mathcal{I}^{\mathcal{K}}})$ is $\mathcal{I}^{\mathcal{K}}$-open in (X, τ). So, P contains a $\mathcal{K}$-tail of $\{x_n\}$ by Proposition 2.8, that is, $\{n \in \omega : x_n \in P\} \in \mathcal{K}^*$, so $\{n \in \omega : x_n \notin P\} \in \mathcal{K}$. Since A is arbitrary, so, $x_n \to_{\mathcal{I}^{\mathcal{K}}} y$ in $(X, \tau_{\mathcal{I}^{\mathcal{K}}})$. Hence, the properties of continuity and preserving $\mathcal{I}^{\mathcal{K}}$-convergence do not coincide.

We, therefore, have the following result.

Theorem 4.6. *Let $q : X \to Y$ be a function and let $\mathcal{K}$ be a maximal ideal. The following are equivalent.*

(i) *X is an $\mathcal{I}^{\mathcal{K}}$-sequential space.*
(ii) *q is a continuous map if and only if it possesses the property of preserving $\mathcal{I}^{\mathcal{K}}$-convergence.*

4.1. $(\mathcal{I}^{\mathcal{K}}, \mathcal{J}^{\mathcal{L}})$-quotient map

Definition 4.7. Let X, Y be topological spaces and $q : X \to Y$ be a mapping. Then

(i) q is said to be a quotient (resp. an $\mathcal{I}$-quotient) mapping [19, Definition 5.1] if for each $q^{-1}(O)$ is an open set (resp. $\mathcal{I}$-open) in X if and only if O is an open subset (resp. $\mathcal{I}$-open) in Y.
(ii) q is said to be an $(\mathcal{I}^{\mathcal{K}}, \mathcal{J}^{\mathcal{L}})$-quotient map if $q^{-1}(O)$ is $\mathcal{I}^{\mathcal{K}}$-open in X if and only if O is $\mathcal{J}^{\mathcal{L}}$-open in Y.

Theorem 4.8. *Every $(\mathcal{I}^{\mathcal{K}}, \mathcal{J}^{\mathcal{L}})$-covering mapping is $(\mathcal{I}^{\mathcal{K}}, \mathcal{J}^{\mathcal{L}})$-quotient.*

Proof. Let q be an $(\mathcal{I}^{\mathcal{K}}, \mathcal{J}^{\mathcal{L}})$-covering map from X onto Y. Suppose that G is a non-$\mathcal{J}^{\mathcal{L}}$-closed subset of Y. So, there exists a sequence $\{b_n\}$ in G such that $\mathcal{J}^{\mathcal{L}} - \lim b_n = b \notin G$. Since q is an $(\mathcal{I}^{\mathcal{K}}, \mathcal{J}^{\mathcal{L}})$-covering map, there exists a sequence $\{b_n\}$, where $a_n \in q^{-1}(b_n)$ for all ω and $a \in q^{-1}(b)$ such that $\mathcal{I}^{\mathcal{K}} - \lim a_n = a$. We observe that $\{a_n : n \in \omega\} \subset f^{-1}(G)$ and $a \notin q^{-1}(G)$. Therefore, $q^{-1}(G)$ is not $\mathcal{I}^{\mathcal{K}}$-closed in X. Thus, q is $(\mathcal{I}^{\mathcal{K}}, \mathcal{J}^{\mathcal{L}})$-quotient. □

Remark 4.9. It is immediate that every quotient (resp. an $\mathcal{I}^{\mathcal{K}}$-quotient) map is continuous (resp. an $\mathcal{I}^{\mathcal{K}}$-continuous).

In the sequel, we refer to $(\mathcal{I}^{\mathcal{K}}, \mathcal{I}^{\mathcal{K}})$-quotient map as $\mathcal{I}^{\mathcal{K}}$-quotient map.

Theorem 4.10. *Consider a mapping $q : X \to Y$, where X and Y are two topological spaces.*
(1) *X is $\mathcal{I}^{\mathcal{K}}$-sequential $\implies$ Y is $\mathcal{I}^{\mathcal{K}}$-sequential and q is an $\mathcal{I}^{\mathcal{K}}$-quotient map, provided q is a quotient map.*
(2) *q is a quotient map if q is an $\mathcal{I}^{\mathcal{K}}$-quotient map, provided Y is an $\mathcal{I}^{\mathcal{K}}$-sequential space.*

Proof. (1) We claim that the space Y is $\mathcal{I}^{\mathcal{K}}$-sequential. Consider an $\mathcal{I}^{\mathcal{K}}$-open set O in Y. By our assumption, it suffices to show that $q^{-1}(O)$ is an $\mathcal{I}^{\mathcal{K}}$-open subset of X. Consider a sequence $\{a_n\} \subset X$ with $\mathcal{I}^{\mathcal{K}} - \lim a_n = a \in q^{-1}(O)$ in X. By the continuity of q and Theorem 4.2, $\mathcal{I}^{\mathcal{K}} - \lim q(a_n) = a \in O$. On the other hand, O is an $\mathcal{I}^{\mathcal{K}}$-open subset of Y. By Lemma 2.5, $|\{n \in \omega : q(a_n) \in O\}| = \omega$ which further means $|\{n \in \omega : a_n \in q^{-1}(O)\}| = \omega$. By Lemma 2.5, we therefore conclude that $q^{-1}(O)$ is $\mathcal{I}^{\mathcal{K}}$-open in X.

Let $q^{-1}(O)$ be an $\mathcal{I}^{\mathcal{K}}$-open subset of X, where $O \subset Y$. By our assumption, $q^{-1}(O)$ is an open subset of X and so, O is open in Y. By above proof, the set O is $\mathcal{I}^{\mathcal{K}}$-open in Y. This proves that q is an $\mathcal{I}^{\mathcal{K}}$-quotient map.

(2) Consider that $q^{-1}(O)$ is an open subset of X, where $O \subset Y$. By our assumption, $q^{-1}(O)$ is an $\mathcal{I}^{\mathcal{K}}$-open subset in X, so, O is an $\mathcal{I}^{\mathcal{K}}$-open subset of Y and hence, O is an open set of Y. □

By Theorem 4.3 and Theorem 4.10 we have the following corollary.

Corollary 4.11. *Let X be an $\mathcal{I}^{\mathcal{K}}$-sequential space and let Y be an arbitrary topological space. Let $q : X \to Y$ be a continuous mapping. Then q is quotient if and only if q is $\mathcal{I}^{\mathcal{K}}$-quotient and Y is an $\mathcal{I}^{\mathcal{K}}$-sequential space.*

Lemma 4.12. *The identity map id_X :$\mathcal{I}^{\mathcal{J}}$-$sX \to X$ is a continuous $\mathcal{I}^{\mathcal{J}}$-covering map.*

Proof. The continuity of the identity map $id_X : \mathcal{I}^{\mathcal{J}}\text{-}sX \to X$ is immediate (Since, whenever O is open in X, O is $\mathcal{I}^{\mathcal{J}}$-open in X, that is O is open in $\mathcal{I}^{\mathcal{J}}\text{-}sX$).

Suppose that $\{a_n\}$ is a sequence in X such that $\mathcal{I}^{\mathcal{J}}\text{-}\lim a_n = a \in X$. Consider an arbitrary open set O in $\mathcal{I}^{\mathcal{J}}\text{-}sX$ with $a \in O$, then O in $\mathcal{I}^{\mathcal{J}}$-open in X. By Corollary 2.5 and the fact that $\mathcal{J}$ is a maximal ideal, we have $\{n \in \omega : a_n \in O\} \in \mathcal{J}^*$ which implies that $\{n \in \omega : a_n \notin O\} \in \mathcal{J}$. Therefore, $\{a_n\}$ is $\mathcal{I}^{\mathcal{J}}$-convergent to $a \in \mathcal{I}^{\mathcal{J}}\text{-}sX$. Hence, id_X is an $\mathcal{I}^{\mathcal{J}}$-covering map. □

Theorem 4.13. *Let X be a topological space and let $\mathcal{I}$, $\mathcal{J}$ be two ideals on ω, where $\mathcal{J}$ is a maximal ideal. Then X is an $\mathcal{I}^{\mathcal{J}}$-sequential space if and only if every $\mathcal{I}^{\mathcal{J}}$-quotient mapping onto X is quotient.*

Proof. Direct implication is immediate by Theorem 4.10(ii). On the other hand, for converse part, suppose that every $\mathcal{I}^{\mathcal{J}}$-quotient mapping onto X is quotient. Then by Theorem 4.8 and Lemma 4.12, the map $id_X : \mathcal{I}^{\mathcal{J}}\text{-}sX \to X$ is a continuous $\mathcal{I}^{\mathcal{J}}$-quotient map. Again, from Remark 2.10(ii), $\mathcal{I}^{\mathcal{J}}\text{-}sX$ is an $\mathcal{I}^{\mathcal{J}}$-sequential space. Thus, it is clear from Theorem 4.10(1) that X is an $\mathcal{I}^{\mathcal{J}}$-sequential space. □

Proposition 4.14. *Let $\mathcal{I}$, $\mathcal{K}$ be two ideals on ω. Let $q : X \to Y$ be an $\mathcal{I}^{\mathcal{J}}$-quotient map. Then for each $\mathcal{I}^{\mathcal{J}}$-convergent sequence $\{b_n\}$ in Y with $b_n \to_{\mathcal{I}^{\mathcal{J}}} b$ and $b_n \neq b$ for each $n \in \omega$, there exists a sequence $\{a_k\}_{k\in\omega}$ such that $\{a_k : k \in \omega\} \subseteq q^{-1}(\{b_n : n \in \omega\})$ and $a_k \to_{\mathcal{I}^{\mathcal{J}}} a \notin q^{-1}(\{b_n : n \in \omega\})$.*

Proof. Suppose that q is an $\mathcal{I}^{\mathcal{J}}$-quotient map and $\{b_n\}$ is a sequence in Y with $b_n \to_{\mathcal{I}^{\mathcal{J}}} b$ and $b_n \neq b$, for each n. Then, the subset $Y \setminus \{b_n\}$ is not $\mathcal{I}^{\mathcal{J}}$-open in Y. Then, $q^{-1}(Y \setminus \{b_n : n \in \omega\}) = X \setminus q^{-1}(\{b_n : n \in \omega\})$ is not $\mathcal{I}^{\mathcal{J}}$-open in X. So, there exists a sequence $\{a_k\}_{k\in\omega}$ in $q^{-1}(\{b_n : n \in \omega\})$ such that $a_k \to_{\mathcal{I}^{\mathcal{J}}} a \notin q^{-1}(\{b_n : n \in \omega\})$. □

Proposition 4.15. *Let q be an $\mathcal{I}^{\mathcal{J}}$-continuous mapping from X to Y. If $\mathcal{J}$ is a maximal ideal, then the following statements are equivalent.*

(i) *If a sequence $b_j \to_{\mathcal{I}^{\mathcal{J}}} b$ in Y, then there exists a sequence $\{a_i\}_{i\in\omega}$ in X with $\{a_i : i \in \omega\} \subset q^{-1}(\{b_j : j \in \omega\})$ such that $a_i \to_{\mathcal{I}^{\mathcal{J}}} a \in q^{-1}(b)$.*
(ii) *If $P \subset Y$, then $q([q^{-1}(P)]_{\mathcal{I}^{\mathcal{J}}}) = [P]_{\mathcal{I}^{\mathcal{J}}}$.*
(iii) *If $b \in [P]_{\mathcal{I}^{\mathcal{J}}} \subset Y$, then $q^{-1}(b) \cap [q^{-1}(P)]_{\mathcal{I}^{\mathcal{J}}} \neq \phi$.*
(iv) *If $b \in [P]_{\mathcal{I}^{\mathcal{J}}} \subset Y$, then there exists an element $a \in q^{-1}(b)$ for which if O is an $\mathcal{I}^{\mathcal{J}}$-sequential neighborhood of a, $b \in [q(O) \cap P]_{\mathcal{I}^{\mathcal{J}}}$.*
(v) *If $b \in [P]_{\mathcal{I}^{\mathcal{J}}} \subset Y$, then there exists an element $a \in q^{-1}(b)$ such that whenever O is an $\mathcal{I}^{\mathcal{J}}$-sequential neighborhood of a, $q(O) \cap P \neq \phi$.*
(vi) *If $b \in Y$, if O is an $\mathcal{I}^{\mathcal{J}}$-sequential neighborhood of $q^{-1}(b)$, then $q(O)$ is an $\mathcal{I}^{\mathcal{J}}$-neighborhood of b.*

Further assuming one of the above condition would imply that q is an $\mathcal{I}^{\mathcal{J}}$-quotient map.

Proof. (i) $\implies$ (ii) Let us assume that $a \in [q^{-1}(P)]_{\mathcal{I}^{\mathcal{J}}}$. Then, $\exists$ a sequence $a_n \to_{\mathcal{I}^{\mathcal{J}}} a$, where $a_n \in q^{-1}(P)$ for each $n \in \omega$. It implies that $q(a_n) \in P$ for each $n \in \omega$. Since q is $\mathcal{I}^{\mathcal{J}}$-continuous, $q(a_n) \to_{\mathcal{I}^{\mathcal{J}}} q(a)$ or $q(a) \in [P]_{\mathcal{I}^{\mathcal{J}}}$. Therefore, $q([q^{-1}(P)]_{\mathcal{I}^{\mathcal{J}}}) \subset [P]_{\mathcal{I}^{\mathcal{J}}}$.

For the reverse inequality, let $b \in [P]_{\mathcal{I}^{\mathcal{J}}}$. Then there exists a sequence $b_j \to_{\mathcal{I}^{\mathcal{J}}} b$, where $b_j \in P$ for each j. So, by our assumption, there exists a sequence $a_i \to_{\mathcal{I}^{\mathcal{J}}} a \in q^{-1}(b)$ with $\{a_i : i \in \omega\} \subset q^{-1}(\{b_j : j \in \omega\}) \subset q^{-1}(P)$. Thus, $a \in [q^{-1}(P)]_{\mathcal{I}^{\mathcal{J}}}$, i.e., $q(a) \in q([q^{-1}(P)]_{\mathcal{I}^{\mathcal{J}}})$ and therefore, $[P]_{\mathcal{I}^{\mathcal{J}}} \subset q([q^{-1}(P)]_{\mathcal{I}^{\mathcal{J}}})$.

(ii) $\implies$ (iii) Let $b \in [P]_{\mathcal{I}^{\mathcal{J}}} \subset Y$, by (ii) we have $b \in q([q^{-1}(P)]_{\mathcal{I}^{\mathcal{J}}})$. Hence, $q^{-1}(b) \cap [q^{-1}(P)]_{\mathcal{I}^{\mathcal{J}}} \neq \phi$.

(iii) $\implies$ (iv) Let $b \in [P]_{\mathcal{I}^{\mathcal{J}}}$ for each $P \subset Y$. Then by (iii), there exists an element $a \in f^{-1}(b) \cap [q^{-1}(P)]_{\mathcal{I}^{\mathcal{J}}}$. Since, $a \in [q^{-1}(A)]_{\mathcal{I}^{\mathcal{J}}}$, there exists a sequence $\{a_n\}_{n\in\omega}$ in $q^{-1}(P)$ such that $a_n \to_{\mathcal{I}^{\mathcal{J}}} a$. If O is an $\mathcal{I}^{\mathcal{J}}$-sequential neighborhood of a, then $\{a_n\}$ has an $\mathcal{I}$-tail which is $\mathcal{K}$-eventually in O. That means there is a set $M = \{n_1, n_2, \ldots, n_k, \ldots\} \in \mathcal{I}^*$ for which we must have $E \in \mathcal{K}$ such that $a_{n_k} \in O$ for all $n_k \notin E$. Therefore, $q(a_{n_k}) \in q(O) \cap P$ for all $n_k \notin E$. Now, take an element $x \in q(O) \cap P$. Consider the sequence $\{b_n\}$ such that $b_n = q(a_n)$, if $n \in M$, $n \notin E$ and x, otherwise. Then $\{b_n : n \in \omega\} \subset q(O) \cap P$ and also, $b_n \to_{\mathcal{I}^{\mathcal{J}}} q(a) = b$. Hence, $b \in [q(O) \cap P]_{\mathcal{I}^{\mathcal{J}}}$.

(iv) $\implies$ (v) It is obvious.

(v) $\implies$ (vi) Suppose that $b \in Y$ and O is an $\mathcal{I}^{\mathcal{J}}$-sequential neighborhood of $q^{-1}(b)$. If $q(O)$ is not an $\mathcal{I}^{\mathcal{J}}$-sequential neighborhood of b, then $b \in Y \setminus (q(O))_{\mathcal{I}^{\mathcal{J}}} = [Y \setminus q(O)]_{\mathcal{I}^{\mathcal{J}}}$. By (v), it is immediate that $q(O) \cap (Y \setminus q(O)) = \phi$. This contradicts our assumption.

(vi) $\implies$ (iii) Consider $b \in [P]_{\mathcal{I}^{\mathcal{J}}} \subset Y$ and $q^{-1}(b) \cap [q^{-1}(P)]_{\mathcal{I}^{\mathcal{J}}} = \phi$. Then $q^{-1}(b) \subset X \setminus [q^{-1}(P)]_{\mathcal{I}^{\mathcal{J}}} = (X \setminus q^{-1}(P))_{\mathcal{I}^{\mathcal{J}}}$. This implies that $X \setminus q^{-1}(P)$ is an $\mathcal{I}^{\mathcal{J}}$-sequential neighborhood of $q^{-1}(b)$. By (vi), we have $q(X \setminus q^{-1}(P))$ is an $\mathcal{I}^{\mathcal{J}}$-sequential neighborhood of b, i.e., $b \in (q(X \setminus q^{-1}(P)))_{\mathcal{I}^{\mathcal{J}}} = (Y \setminus P)_{\mathcal{I}^{\mathcal{J}}} = Y \setminus [P]_{\mathcal{I}^{\mathcal{J}}}$. This is a contradiction.

(iii) $\implies$ (i) Let us consider a sequence $b_n \to_{\mathcal{I}^{\mathcal{J}}} b$ in Y. Consider the range $R = \{b_n : n \in \omega\}$, then $b \in [P]_{\mathcal{I}^{\mathcal{J}}}$. By (iii), there exists an element $a \in q^{-1}(b) \cap [q^{-1}(R)]_{\mathcal{I}^{\mathcal{J}}}$. Then, we have there exists a sequence in $a_n \in q^{-1}(R)$ such that $a_n \to_{\mathcal{I}^{\mathcal{J}}} a \in q^{-1}(b)$.

Now, assume statement (i) and consider a non-$\mathcal{I}^{\mathcal{J}}$-closed subset O in Y. Then, there exists $\{b_n\}$ such that $b_j \to_{\mathcal{I}^{\mathcal{J}}} b \in Y \setminus O$, where $b_j \in O$ for each j. It is clear that $b \neq b_j$ for each $j \in \omega$. By (i), there exists a sequence $a_i \to_{\mathcal{I}^{\mathcal{J}}} a \in q^{-1}(b) \notin q^{-1}(O)$, where $a_i \in q^{-1}(\{b_j : j \in \omega\})$. Thus, $q^{-1}(O)$ is not an $\mathcal{I}^{\mathcal{J}}$-closed subset in X. $\square$

5. Conclusion and future scope

Convergence discussed in this paper is a non-axiomatised generalization of several convergence modes, viz., usual convergence, statistical convergence, $\mathcal{I}$-convergence and $\mathcal{I}^*$ convergence. We have presented a study on different mappings in the context of $\mathcal{I}^{\mathcal{K}}$ convergence in a topological space. With an assumption of ideality condition on the ideals $\mathcal{I}$ and $\mathcal{K}$, this work on $\mathcal{I}^{\mathcal{K}}$-convergence includes more class of ideals as compared to the assumption $\mathcal{I} \subset \mathcal{K}$. Work may be carried out further to investigate the operators in $\mathcal{I}^{\mathcal{K}}$-closure and $\mathcal{I}^{\mathcal{K}}$-interior of a topological space in case of non-maximal ideals. Further, it will be interesting to explore similar development for G-convergence [7] as well.

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References

[1] J. R. Boone and F. Siwiec, *Sequentially quotient mappings*, Czechoslovak Math. J. **26(101)** (1976), no. 2, 174–182.
[2] P. Das, S. Sengupta, and J. Šupina, *$\mathcal{I}^{\mathcal{K}}$-convergence of sequences of functions*, Math. Slovaca **69** (2019), no. 5, 1137–1148. https://doi.org/10.1515/ms-2017-0296
[3] P. Das, M. Sleziak, and V. Toma, *$\mathcal{I}^{\mathcal{K}}$-Cauchy functions*, Topology Appl. **173** (2014), 9–27. https://doi.org/10.1016/j.topol.2014.05.008
[4] K. P. Hart, J. Nagata, and J. E. Vaughan, *Encyclopedia of General Topology*, Elsevier Science Publishers, B.V., Amsterdam, 2004.
[5] P. Kostyrko, T. Šalát, and W. Wilczyński, *$\mathcal{I}$-convergence*, Real Anal. Exchange **26** (2000/01), no. 2, 669–685.
[6] B. K. Lahiri and P. Das, *I and I^*-convergence in topological spaces*, Math. Bohem. **130** (2005), no. 2, 153–160.
[7] S. Lin and L. Liu, *G-methods, G-sequential spaces and G-continuity in topological spaces*, Topology Appl. **212** (2016), 29–48. https://doi.org/10.1016/j.topol.2016.09.003
[8] S. Lin and P. Yan, *Sequence-covering maps of metric spaces*, Topology Appl. **109** (2001), no. 3, 301–314. https://doi.org/10.1016/S0166-8641(99)00163-7
[9] M. Mačaj and M. Sleziak, *$\mathcal{I}^{\mathcal{K}}$-convergence*, Real Anal. Exchange **36** (2010/11), no. 1, 177–193. http://projecteuclid.org/euclid.rae/1300108092
[10] S. K. Pal, N. Adhikary, and U. Samanta, *On ideal sequence covering maps*, Appl. Gen. Topol. **20** (2019), no. 2, 363–377. https://doi.org/10.4995/agt.2019.11238
[11] V. Renukadevi and B. Prakash, *On statistically sequentially covering maps*, Filomat **31** (2017), no. 6, 1681–1686. https://doi.org/10.2298/FIL1706681R
[12] V. Renukadevi and B. Prakash, *On statistically sequentially quotient maps*, Korean J. Math. **25** (2017), no. 1, 61–70. https://doi.org/10.11568/kjm.2017.25.1.61
[13] A. Sharmah and D. Hazarika, *Further aspects of $\mathcal{I}^{\mathcal{K}}$-convergence in topological spaces*, Appl. Gen. Topol. **22** (2021), no. 2, 355–366. https://doi.org/10.4995/agt.2021.14868
[14] A. Sharmah and D. Hazarika, *On $\mathcal{I}^{\mathcal{K}}$-convergence in a topological space via semi-open sets*, arXiv 2021.
[15] B. C. Tripathy and B. Hazarika, *Paranorm I-convergent sequence spaces*, Math. Slovaca **59** (2009), no. 4, 485–494. https://doi.org/10.2478/s12175-009-0141-4
[16] B. C. Tripathy and B. Hazarika, *$\mathcal{I}$-monotonic and $\mathcal{I}$-convergent sequences*, Kyungpook Math. J. **51** (2011), no. 2, 233–239. https://doi.org/10.5666/KMJ.2011.51.2.233

[17] X. Zhou, *On $\mathcal{I}$-quotient mappings and $\mathcal{I}$-cs$'$-networks under a maximal ideal*, Appl. Gen. Topol. **21** (2020), no. 2, 235–246. https://doi.org/10.4995/agt.2020.12967

[18] X. Zhou and L. Liu, *On $\mathcal{I}$-covering mappings and 1-$\mathcal{I}$-covering mappings*, J. Math. Res. Appl. **40** (2020), no. 1, 47–56. https://doi.org/10.3770/j.issn:2095-2651.2020.01.005

[19] X. Zhou, L. Liu, and S. Lin, *On topological spaces defined by $\mathcal{I}$-convergence*, Bull. Iranian Math. Soc. **46** (2020), no. 3, 675–692. https://doi.org/10.1007/s41980-019-00284-6

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