Commun. Korean Math. Soc. **38** (2023), No. 1, pp. 257–266
https://doi.org/10.4134/CKMS.c220028
pISSN: 1225-1763 / eISSN: 2234-3024

EVALUATION SUBGROUPS OF THE PLÜCKER EMBEDDING OF SOME QUATERNION GRASSMANNIANS

Oteng Maphane

Abstract. Let $G_{k,n}(\mathbb{H})$ for $2 \leq k < n$ denote the Grassmann manifold of k-dimensional vector subspaces of $\mathbb{H}^n$. In this paper, we determine the evaluation subgroups of the Plüker embedding $G_{2,n}(\mathbb{H}) \hookrightarrow \mathbb{H}P^{N-1}$, where $N = \binom{n}{2}$.

1. Introduction

Let us remind the notion of a Gottlieb group (see, for example, [5]). Given a based CW-complex X, an element $\alpha \in \pi_n(X)$ is a Gottlieb element of X if $(\alpha, id_X) : X \vee S^n \to X$ extends to $\tilde{\alpha} : X \times S^n \to X$. The set $G_n(X)$ of all Gottlieb elements $\alpha \in \pi_n(X)$ is called the n-th Gottlieb group of X or the n-th evaluation subgroup of $\pi_n(X)$ [5].

Gottlieb groups play a profound role in topology, covering spaces, fixed point theory, homotopy theory of fibrations, and other fields. For instance, the triviality of Gottlieb groups is related to the cross section problem of fibrations.

Further, let $f : X \to Y$ be a based map of simply connected finite CW-complexes. As it was shown in [7], the evaluation at the basepoint of X gives the *evaluation map* $\omega : \mathrm{Map}(X, Y; f) \to Y$, where $\mathrm{Map}(X, Y; f)$ is the component of f in the space of mappings from X to Y. The image of the homomorphism induced in homotopy groups

$$\omega_\sharp : \pi_* \mathrm{Map}(X, Y; f) \to \pi_*(Y)$$

is called the n-th *evaluation subgroup of* f and it is denoted by $G_n(Y, X; f)$. Note that if $f = id_X$, the space $\mathrm{Map}(X, Y; f)$ is the monoid $\mathrm{aut}_1(X)$ of self-equivalences of X homotopic to the identity of X, then $ev : \mathrm{aut}_1(X) \to X$ is the evaluation map, and the image of the induced homomorphism

$$ev_\sharp : \pi_*(\mathrm{aut}_1(X)) \to \pi_*(X)$$

is $G_n(X)$, i.e., the n-th Gottlieb group.

Received February 4, 2022; Accepted June 2, 2022.

2020 *Mathematics Subject Classification.* Primary 55P62; Secondary 54C35.

Key words and phrases. Evaluation subgroups, Gottlieb group, Grassmann manifolds.

This work was completed with the support of the Botswana International University of Science and Tecnnology (BIUST).

In [12], Woo and Lee studied the relative evaluation subgroups $G_n^{rel}(X, Y; f)$ and proved that they fit in a sequence

$$\cdots \to G_{n+1}^{rel}(X, Y; f) \to G_n(X) \to G_n(X, Y; f) \to \cdots$$

called the G-sequence of f. This sequence is exact in some cases, for instance, if f is a homotopy monomorphism.

Recently, Smith and Lupton in [7] identified the homomorphism induced on rational homotopy groups by the evaluation map $\omega : \mathrm{Map}(X, Y; f) \to Y$, in terms of a map of complexes of derivations constructed directly from the Sullivan minimal model of f. In [4], relative Gottlieb groups of the Plücker embedding of some complex Grassmannians $G_{2,n}(\mathbb{C}) \hookrightarrow \mathbb{C}P^{N-1}$, where $N = n(n-1)/2$, were studied. More specifically, it was shown that; $G_*^{rel}(\mathbb{C}P^{N-1}, G_{2,n}(\mathbb{C})_{\mathbb{Q}}, h \circ i)$ splits as the direct sum of the suspension of $G_*(G_{2,n}(\mathbb{C})_{\mathbb{Q}})$ and $G_{2N-1}(\mathbb{C}P^{N-1}, G_{2,n}(\mathbb{C})_{\mathbb{Q}}, h \circ i)$, where $h : \mathbb{C}P^{N-1} \to \mathbb{C}P_{\mathbb{Q}}^{N-1}$ is the rationalization.

In this paper, we use a map of complexes of derivations of minimal Sullivan models of mapping spaces to compute rational relative Gottlieb groups of the Plücker embedding of some quaternion Grassmannians $G_{2,n}(\mathbb{H}) \hookrightarrow \mathbb{H}P^{N-1}$, where $N = \binom{n}{k}$. More precisely, we show that; $G_*^{rel}(\mathbb{H}P^{N-1}, G_{2,n}(\mathbb{H})_{\mathbb{Q}}, f \circ i)$ splits as the direct sum of the suspension of $G_*(G_{2,n}(\mathbb{H})_{\mathbb{Q}})$ and $G_{4N-1}(\mathbb{H}P^{N-1}, G_{2,n}(\mathbb{H})_{\mathbb{Q}}, f \circ i)$, where $f : \mathbb{H}P^{N-1} \to \mathbb{H}P_{\mathbb{Q}}^{N-1}$ is the rationalization.

2. Preliminaries

Through this paper, we rely on the theory of minimal Sullivan models in rational homotopy theory for which [2] is our standard reference. All vector spaces and algebras are taken over a field $\mathbb{Q}$ of rational numbers. We start with recalling some definitions.

Definition 2.1. A commutative graded differential algebra (cdga) is a graded algebra (A, d) such that $xy = (-1)^{|x||y|}yx$ and $d(xy) = (dx)y + (-1)^{|pq|}x(dy)$ for all $x \in A^p, y \in A^q$. It is said to be connected if $H^0(A) \cong \mathbb{Q}$. If $V = \oplus_{i\geq 1} V^i$ with $V^{\text{even}} := \oplus_{i\geq 1} V^{2i}$ and $V^{\text{odd}} := \oplus_{i\geq 1} V^{2i-1}$, then $\wedge V$ denotes the free commutative graded algebra defined by the tensor product

$$\wedge V = S(V^{\text{even}}) \otimes E(V^{\text{odd}}),$$

where $S(V^{\text{even}})$ is the symmetric algebra on V^{even} and $E(V^{\text{odd}})$ is the exterior algebra on V^{odd}.

Definition 2.2. A Sullivan algebra is a commutative differential graded algebra $(\wedge V, d)$, where $V = \cup_{k\geq 0} V(k)$ and $V(0) \subset V(1) \subset \cdots$ such that $dV(0) = 0$ and $dV(k) \subset \wedge V(k-1)$. It is called minimal if $dV \subset \wedge^{\geq 2} V$.

If (A, d) is a cdga of which the cohomology is connected and finite dimensional in each degree, then there always exists a quasi-isomorphism from a

Sullivan algebra $(\wedge V, d)$ to (A, d) [2]. To each simply connected space, Sullivan associates a cdga $A_{PL}(X)$ of rational polynomial differential forms on X that uniquely determines the rational homotopy type of X [11]. A minimal Sullivan model of X is a minimal Sullivan model of $A_{PL}(X)$. More precisely, $H^*(\wedge V, d) \cong H^*(X; \mathbb{Q})$ as graded algebras and $V \cong \pi_*(X) \otimes \mathbb{Q}$ as graded vector spaces.

Let (A, d) be a cdga. A derivation θ of degree k is a linear mapping $\theta : A^n \to A^{n-k}$ such that $\theta(ab) = \theta(a)b + (-1)^{k|a|}a\theta(b)$. Denote by $\operatorname{Der}_k A$ the vector space of all derivations of degree k, and $\operatorname{Der} A = \oplus_k \operatorname{Der}_k A$. The commutator bracket induces a graded Lie algebra structure on $\operatorname{Der} A$. Moreover, $(\operatorname{Der} A, \delta)$ is a differential graded Lie algebra (see, for example, [11]), with the differential δ defined in the usual way by

$$\delta\theta = d \circ \theta + (-1)^{k+1}\theta \circ d.$$

Let $(\wedge V, d)$ be a Sullivan algebra where V is spanned by $\{v_1, \dots, v_k\}$. Then, $\operatorname{Der} \wedge V$ is spanned by $\theta_1, \dots, \theta_k$, where θ_i is the unique derivation of $\wedge V$ defined by $\theta_i(v_j) = \delta_{ij}$. The derivation θ_i will be denoted by $(v_i, 1)$. It is known (see [2]), that an element $v \in V \cong \pi_*(X) \otimes \mathbb{Q}$ is a Gottlieb element of $\pi_*(X) \otimes \mathbb{Q}$ if and only if there is a derivation θ of $\wedge V$ satisfying $\theta(v) = 1$ and such that $\delta\theta = 0$. Let $\phi : (A, d) \to (B, d)$ be a morphism of cdga's. A ϕ-derivation of degree k is a linear mapping $\theta : A^n \to B^{n-k}$ for which $\theta(ab) = \theta(a)\phi(b) + (-1)^{k|a|}\phi(a)\theta(b)$. We consider only derivations of positive degree. Denote by $\operatorname{Der}_n(A, B; \phi)$ the vector space of ϕ-derivations of degree n for $n > 0$, and by $\operatorname{Der}(A, B; \phi) = \oplus_n \operatorname{Der}_n(A, B; \phi)$ the graded vector space of all ϕ-derivations. The differential graded vector space of ϕ-derivations is denoted by $(\operatorname{Der}(A, B; \phi), \partial)$, where the differential ∂ is defined by $\partial\theta = d_B \circ \theta + (-1)^{k+1}\theta \circ d_A$. In the case $A = B$ and $\phi = 1_B$, the vector space $(\operatorname{Der}(B, B; 1), \partial)$ is just a usual differential graded Lie algebra of derivations on the cdga B (see [7]). We note that, there is an isomorphism of graded vector spaces

$$\operatorname{Der}(A, B; \phi) \cong \operatorname{Hom}(V, B).$$

If $\{v_i\}$ is a basis of V, then the vector space $\operatorname{Der}(A, B; \phi)$ is spanned by the unique ϕ-derivation θ denoted by (v_i, b_i) and $(v_i, 1)$ such that

$$\begin{cases} \theta_i(v_i) = b_i, \\ \theta_i(v_j) = 0, \quad i \neq j,\ b_i \in B. \end{cases}$$

It was shown in [7] that a pre-composition with ϕ gives a chain complex map $\phi^* : \operatorname{Der}(B, B; 1) \to \operatorname{Der}(A, B; \phi)$, and a post-composition with the augmentation $\varepsilon : B \to \mathbb{Q}$ gives a chain complex map $\varepsilon_* : \operatorname{Der}(A, B; \phi) \to \operatorname{Der}(A, \mathbb{Q}; \varepsilon)$. The evaluation subgroup of ϕ is defined as follows:

$$G_n(A, B; \phi) = \operatorname{Im}\{H(\varepsilon_*) : H_n(\operatorname{Der}(A, B; \phi)) \to H_n(\operatorname{Der}(A, \mathbb{Q}; \varepsilon))\}.$$

In the case $A = B$ and $\phi = 1_B$, we get the Gottlieb group of (B, d) defined as

$$G_n(B) = \operatorname{Im}\{H(\varepsilon_*) : H_n(\operatorname{Der}(B, B; 1)) \to H_n(\operatorname{Der}(B, \mathbb{Q}; \varepsilon))\}.$$

In particular, $G_n(B) \cong G_n(X_{\mathbb{Q}})$, if B is the minimal Sullivan model of a simply connected space X [2, Proposition 29.8].

Definition 2.3. A simply connected space X is called formal (see [3]) if there is a quasi-isomorphism $(\wedge V, d) \to H^*(\wedge V, d)$, where $(\wedge V, d)$ is the minimal Sullivan model of X.

Examples of formal spaces include spheres, projective complex spaces, homogeneous spaces G/H where G and H have equal rank, and compact Kähler manifolds.

3. Evaluation subgroups of a map

The quaternion Grassmannian $G_{k,n}(\mathbb{H})$ is a homogeneous space as $G_{k,n}(\mathbb{H}) \cong Sp(n)/(Sp(k) \times Sp(n-k))$ for $1 \le k < n$, of equal rank as well as a Kähler manifold, hence it is formal (see, for example, [11, §12] and [1]). It is a Kähler manifold of dimension $2m$, where $m = 2k(n-k)$. The method to compute a Sullivan model of the homogeneous space $G_{k,n}(\mathbb{H})$ is given in details in [6, 9]. Thus, a Sullivan model of $G_{k,n}(\mathbb{H})$ for $1 \le k < n$ is given by (see [9])

$$(\wedge(b_4, b_8, \ldots, b_{4k}, x_4, x_8, \ldots, x_{4(n-k)}, y_3, y_7, \ldots, y_{4n-1}), d)$$

with

$$db_i = 0 = dx_j,\ dy_{4p-1} = \sum_{p_1+p_2=p} b_{4p_1} \cdot x_{4p_2},\ \ 1 \le p \le n. \tag{3.1}$$

Lemma 3.1. *If $2 \le k < n$, then the minimal Sullivan model of $G_{k,n}(\mathbb{H})$ is given by*

$$(\wedge(b_4, \ldots, b_{4k}, y_{4(n-k)+3}, \ldots, y_{4n-1}), d),$$

where $db_i = 0$ and $dy_{4(n-k)+3} \in \wedge(b_4, \ldots, b_{4k})$.

Proof. Consider the Sullivan model from equation (3.1)

$$(\wedge(b_4, b_8, \ldots, b_{4k}, x_4, x_8, \ldots, x_{4(n-k)}, y_3, y_7, \ldots, y_{4n-1}), d)$$

of $G_{k,n}(\mathbb{H})$ for $2 \le k < n$,

$$\begin{aligned} dy_3 &= b_4 + x_4, \\ dy_7 &= b_8 + x_8 + b_4 x_4, \\ &\vdots \\ dy_{2n-1} &= b_{4k} x_{4(n-k)}. \end{aligned}$$

The model is not minimal as the linear part is not zero. To find its minimal Sullivan model, we make a change of variable $t_4 = b_4 + x_4$ and replace x_4 by $t_4 - b_4$ wherever it appears in the differential. This gives an isomorphic Sullivan algebra

$$(\wedge(b_4, t_4, b_8, \ldots, b_{4k}, x_4, \ldots, x_{4(n-k)}, y_3, y_7, \ldots, y_{4n-1}), d),$$

where

$$\begin{aligned} dy_3 &= t_4, \\ dy_7 &= b_8 + x_8 + b_4(t_4 - b_4), \\ &\vdots \\ dy_{4n-1} &= b_{4k}x_{4(n-k)}. \end{aligned}$$

As the ideal generated by y_3 and t_4 is acyclic, the above Sullivan algebra is quasi-isomorphic to

$$(\wedge(b_4, b_8, \ldots, b_{4k}, x_8, \ldots, x_{4(n-k)}, y_7, \ldots, y_{4n-1}), d),$$

where

$$\begin{aligned} dy_7 &= b_8 + x_8 - b_4^2, \\ &\vdots \\ dy_{4n-1} &= b_{4k}x_{4(n-k)}. \end{aligned}$$

One continues in this fashion and make another change of variable, $t_8 = b_8 + x_8 - b_4^2$ and replace x_8 by $t_8 - b_8 + b_4^2$ wherever it appears in the differential and do so until they reach a change of variable of the form

$$\begin{aligned} t_{4(n-k)} &= b_{4(n-k)} + x_{4(n-k)} + \alpha \ \text{ for } \ n = 4k, \ \text{ or} \\ t_{4(n-k)} &= x_{4(n-k)} + \beta \ \text{ for } \ n > 4k, \end{aligned}$$

where $\alpha \in \wedge(b_4, \ldots, b_{4(k-1)})$, $\beta \in \wedge(b_4, \ldots, b_{4k})$ and replace

$$x_{4(n-k)} = \begin{cases} t_{4(n-k)} - b_{4k} + \alpha & \text{for } \ n = 4k, \\ t_{4(n-k)} + \beta & \text{for } \ n > 4k, \end{cases}$$

wherever it appears in the differential. This gives an isomorphic Sullivan algebra

$$(\wedge(b_4, \ldots, b_{4k}, y_{4(n-k)-1}, y_{4(n-k)+3}, \ldots, y_{4n-1}), d),$$

where

$$\begin{aligned} dy_{4(n-k)-1} &= t_{4(n-k)}, \\ &\vdots \\ dy_{4n-1} &= b_{4k}x_{4(n-k)}. \end{aligned}$$

As the ideal generated by $t_{4(n-k)}$ and $y_{4(n-k)-1}$ is acyclic, we get the minimal Sullivan model:

$$(\wedge(b_4, \ldots, b_{4k}, y_{4(n-k)+3}, \ldots, y_{4n-1}), d)$$

with $db_i = 0$ and $dy_{4(n-k)+3} \in \wedge(b_4, \ldots, b_{4k})$. $\square$

On the other hand, the minimal Sullivan model of the quaternion projective space $\mathbb{H}P^n$ is given by $(\wedge(x_4, x_{4n+3}), d)$ with $dx_4 = 0$, $dx_{4n+3} = x_4^{n+1}$ (see [8]).

It follows by an induction argument that

$$N = \binom{n}{k} > k(n-k).$$

Therefore, $H^{\geq 4N}(G_{k,n}(\mathbb{H}), \mathbb{Q}) = 0$. Hence, b_4^N is coboundary in

$$(\wedge(b_4, \ldots, b_{4k}, y_{4(n-k)+3}, \ldots, y_{4n-1}), d).$$

Note that the minimal Sullivan model of $\mathbb{H}P^{N-1}$ is given by

$$(\wedge(x_4, x_{4N-1}), d),$$

where $dx_4 = 0$ and $dx_{4N-1} = x_4^N$.

A Sullivan model of the inclusion $i : G_{2,n}(\mathbb{H}) \hookrightarrow \mathbb{H}P^{N-1}$ is given by

$$\phi : (\wedge(x_4, x_{4N-1}), d) \to (\wedge(b_4, b_8, y_{4n-5}, y_{4n-1}), d),$$

where

$$\phi(x_4) = b_4,\ \phi(x_{4N-1}) = y,\ dy = y_4^N.$$

Theorem 3.2. *Let $B = (\wedge(b_4, b_8, y_{4n-5}, y_{4n-1}), d)$. Then*

$$G_n(B) = \langle [y^*_{4n-5}], [y^*_{4n-1}] \rangle.$$

Proof. Let $\alpha_{4n-1} = (y_{4n-1}, 1)$ and $\alpha_{4n-5} = (y_{4n-5}, 1)$. Then $\delta\alpha_{4n-1} = \delta_{4n-5} = 0$. Moreover, α_{4n-1} and α_{4n-5} can not be boundaries for degree reason. Therefore, $[\alpha_{4n-1}]$ and $[\alpha_{4n-5}]$ are non zero homology classes in $H_*(\mathrm{Der}(B, B; 1))$. Further, $\varepsilon_*(\alpha_{4n-2}) = y^*_{4n-1}$ and $\varepsilon_*(\alpha_{4n-5}) = y^*_{4n-5}$. As $G_{2,n}(\mathbb{H})$ is a finite CW-complex, then $G_{even}(B) = 0$ (see [2, Page 379]). Hence, $G_n(B) = \langle [y^*_{4n-5}], [y^*_{4n-1}] \rangle$. □

It is easy to see by an induction argument that $\binom{n}{2} = n(n-2)/2$.

Lemma 3.3. *Let $N = n(n-2)/2$. Then $4(N-1) > 8n - 16$ for $n \geq 4$.*

Proof. The inequality $4(N-1) > 8n-16$ simplifies to the inequality $2(N-1) > 4n - 8$ for $n \geq 4$, which is the inequality given in [4, Lemma 6]. □

Lemma 3.4. *Given $\phi : A = (\wedge(x_4, x_{4N-1}), d) \to (\wedge(b_4, b_8, y_{4n-5}, y_{4n-1}), d) = B$, where $\phi(x_4) = b_4$, $\phi(x_{4N-1}) = y$, $dy = y_4^N$. There exists a ϕ-derivation θ_4 such that $\theta_4(x_4) = 1$ and it is a cycle.*

Proof. As $\theta_4(x_4) = 1$, then $\partial(\theta_4)(x_4) = 0$. Now it only remains to define θ_4 on x_{4N-1} such that

$$d\theta_4(x_{4N-1}) - \theta_4(dx_{4N-1}) = 0.$$

Hence,

$$\begin{aligned} d\theta_4(x_{4N-1}) - \theta_4(dx_{4N-1}) &= d\theta_4(x_{4N-1}) - \theta_4(x_4^N) \\ &= d\theta_4(x_{4N-1}) - Nx_4^{N-1}. \end{aligned}$$

As the dimension of $G_{2,n}(\mathbb{H})$ is less than $4(N-1)$ by Lemma 3.3, then Nx_4^{N-1} is boundary, that is, $Nx_4^{N-1} = dr$. Define $\theta_4(x_{4N-1}) = r$. Moreover, $\partial\theta_4 = 0$. Therefore, θ_4 cannot be a boundary, hence it is a non-zero homology class in $H_*(\mathrm{Der}(A, B; \phi), \partial)$. □

Theorem 3.5. *Consider the inclusion $G_{2,n}(\mathbb{H}) \hookrightarrow \mathbb{H}P^{N-1}$, and let $\phi : A \to B$ be its Sullivan model. Then $G_*(A, B; \phi) = \langle [x_4^*], [x_{4N-1}^*] \rangle$.*

Proof. Define the derivation $\theta_{4N-1} = (x_{4N-1}, 1)$ in $\mathrm{Der}(A, B; \phi)$. Then $\partial\theta_{4N-1} = 0$. Moreover, $[\theta_{4N-1}]$ is a non-zero homology in $H_*(\mathrm{Der}(A, B; \phi), \partial)$, and $[\theta_4]$ is a non-zero homology class in $H_*(\mathrm{Der}(A, B; \phi), \partial)$ by Theorem 3.4. Further, $H(\varepsilon_*)([\theta_4]) = [x_4^*]$ and $H(\varepsilon_*)([\theta_{4N-1}]) = [x_{4N-1}^*]$. It then follows that $G_*(A, B; \phi) = \langle [x_4^*], [x_{4N-1}^*] \rangle$. □

Definition 3.6. Let $\phi : A \to B$ be a map of differential graded vector spaces. A differential graded vector space, $Rel_*(\phi)$, called the mapping cone of ϕ (see, for example, [7, 10]) is defined by $Rel_n(\phi) = A_{n-1} \oplus B_n$ for all $n > 1$, with the differential $\delta(a, b) = (-d_A(a), \phi(a) + d_B(b))$. There are inclusion and projection chain maps $J : B_n \to Rel_n(\phi)$ and $P : Rel_n(\phi) \to A_{n-1}$ defined by $J(w) = (0, w)$ and $P(a, b) = a$, respectively. These yields a short exact sequence of chain complexes

$$0 \to B_* \xrightarrow{J} Rel_*(\phi) \xrightarrow{P} A_{*-1} \to 0$$

and a long exact homology sequence of ϕ

$$\cdots \to H_{n+1}(Rel(\phi)) \xrightarrow{H(P)} H_n(A) \xrightarrow{H(\phi)} H_n(B) \xrightarrow{H(J)} H_n(Rel(\phi)) \to \cdots$$

whose connecting homomorphism is $H(\phi)$.

Following [7], we consider a commutative diagram of differential graded vector spaces;

$$\begin{array}{ccc} \mathrm{Der}(B, B; 1) & \xrightarrow{\phi^*} & \mathrm{Der}(A, B; \phi) \\ \downarrow{\scriptstyle \varepsilon_*} & & \downarrow{\scriptstyle \varepsilon_*} \\ \mathrm{Der}(B, \mathbb{Q}; \varepsilon) & \xrightarrow{\widehat{\phi^*}} & \mathrm{Der}(A, \mathbb{Q}; \varepsilon), \end{array}$$

where ε is the augmentation of either A or B. On passing to homology and using the naturality of the mapping cone construction, we obtain the following homology ladder for $n \geq 2$,

$$\begin{array}{ccccccc} \cdots \to H_{n+1}(Rel(\phi^*)) & \xrightarrow{H(P)} & H_n(\mathrm{Der}(B, B; 1)) & \xrightarrow{H(\phi^*)} & H_n(\mathrm{Der}(A, B; \phi)) \to \cdots \\ \downarrow{\scriptstyle H(\varepsilon_*, \varepsilon_*)} & & \downarrow{\scriptstyle H(\varepsilon_*)} & & \downarrow{\scriptstyle H(\varepsilon_*)} \\ \cdots \to H_{n+1}(Rel(\widehat{\phi^*})) & \xrightarrow{H(\widehat{P})} & H_n(\mathrm{Der}(B, \mathbb{Q}; \varepsilon)) & \xrightarrow{H(\widehat{\phi^*})} & H_n(\mathrm{Der}(A, \mathbb{Q}; \varepsilon)) \to \cdots . \end{array}$$

The n-th relative evaluation subgroup of ϕ is defined as follows:

$$G_n^{rel} = \mathrm{Im}\{H(\varepsilon_*, \varepsilon_*) : H_n(Rel(\phi^*)) \to H_n(Rel(\widehat{\phi^*}))\}.$$

The G-sequence of the map $\phi : A \to B$ is given by the sequence

$$\cdots \overset{H(\widehat{J})}{\to} G^{rel}_{n+1}(A, B; \phi) \overset{H(\widehat{P})}{\to} G_n(B) \overset{H(\widehat{\phi^*})}{\to} G_n(A, B; \phi) \overset{H(\widehat{J})}{\to} \cdots$$

which ends in $G_2(A, B; \phi)$. Moreover, as it was shown in [7, Theorem 3.5], this can be applied to the Sullivan model $\phi : A \to B$ of the map $f : X \to Y$.

Given the inclusion $G_{2,n}(\mathbb{H}) \hookrightarrow \mathbb{H}P^{N-1}$, and $\phi : A = (\wedge(x_4, x_{4N-1}), d) \to (\wedge(b_4, b_8, y_{4n-5}, y_{4n-1}), d) = B$ its Sullivan model. Note that $Rel_*(\phi^*) = \operatorname{Ker}(\varepsilon_*, \varepsilon_*) \oplus M$, where M is isomorphic to $\operatorname{Im}(\varepsilon_*, \varepsilon_*)$. Then a non-zero element in $G^{rel}_*(A, B; \phi)$ is an element in M modulo the kernel of $(\varepsilon_*, \varepsilon_*)$ (see [4]). Therefore, the vector space M is spanned by

$$\{(0, \theta_4), (0, \theta_{4N-1}), (s\gamma_4, 0), (s\gamma_8, 0), (s\gamma_{4n-5}, 0), (s\gamma_{4n-1}, 0)\},$$

where γ_i is a derivation in $\operatorname{Der}(B, B; 1)$.

Lemma 3.7. $(0, \theta_4) = J(\theta_4)$ *is a non-zero homology class in* $H_4(Rel_*(\phi^*))$.

Proof. Since $(0, \theta_{4n})$ is a cycle, it is left to show that it is not a boundary. We claim there is $\gamma'_4 \in \operatorname{Der} B$ such that $D(s\gamma'_4, 0) = (-s\delta\gamma'_4, \phi^*(\gamma'_4)) = (0, \theta_4)$. Hence, $\delta\gamma'_4 = 0$, and $\phi^*(\gamma'_4) = \theta_4$. Therefore, $\phi^*(\gamma'_4)(x_4) = \theta_4(x_4) = 1$. It follows that $\gamma'_4(b_4) = 1$. But γ'_4 cannot be a cycle, because $H(\varepsilon_*)([\gamma'_4]) = [b^*_4] \neq 0$ would be a non-zero element in $\pi_4(X_{\mathbb{Q}})$, which contradicts [2, Proposition 28.8]. Thus, $[(0, \theta_4)]$ is a non-zero homology class in $H_4(Rel(\phi^*))$. □

We now give the following result.

Theorem 3.8. $G^{rel}_*(A, B; \phi) \cong sG_*(B) \oplus G_{4N-1}(A, B; \phi)$. *Moreover, the* G-*sequence*

$$0 = G_4(B) \to G_4(A, B; \phi) \overset{H(\hat{J})}{\to} G^{rel}_4(A, B; \phi) \to 0 \tag{3.2}$$

is not exact.

Proof. It is easy to see that $\partial\gamma_4 \neq 0$ and $\partial\gamma_8 \neq 0$ in $\operatorname{Der}(A, B; \phi)$. Further, $(s\gamma_{4n-5}, 0)$ and $(s\gamma_{4n-1}, 0)$ are non-zero homology classes in $Rel_*(\phi^*)$. Note that in $Rel_*(\phi^*)$, one gets $D(sb^*_4, 0) = (0, \phi^*(b^*_4)) = (0, x^*_4)$. Hence, $H(\varepsilon_*, \varepsilon_*)([(0, \theta_4)]) = [(0, x^*_4)]$ is zero in $H(Rel_*(\phi^*))$. We deduce that

$$G^{rel}_4(A, B; \phi) = 0.$$

Therefore,

$$\begin{aligned} G^{rel}_*(A, B; \phi) &= \operatorname{Im} H(\varepsilon_*, \varepsilon_*), \\ &= \langle [(sy^*_{4n-5}, 0)], [(sy^*_{4n-1}, 0)], [(0, x^*_{4N-1})] \rangle, \\ &= sG_*(B) \oplus G_{4N-1}(A, B; \phi). \end{aligned}$$

The G-sequence (3.2) is not exact as $H(\hat{J})([x^*_4]) = 0$. Thus, $H(\hat{J})$ is not injective. □

Example 1. Consider $G_{2,5}(\mathbb{H})$ for which the minimal Sullivan model is given by

$$(\wedge(b_4, b_8, y_{15}, y_{19}), d),$$

where $db_4 = db_8 = 0$, $dy_{15} = b_4^4 + 2b_8 b_4^2$, $dy_{19} = 2b_4^5 + 2b_8 b_4^3$. The inclusion $G_{2,5}(\mathbb{H}) \hookrightarrow \mathbb{H}P^9$ is modelled by

$$\phi : A = (\wedge(x_4, x_{39}), d) \to (\wedge(b_4, b_8, y_{15}, y_{19}), d) = B,$$

where $\phi(x_4) = b_4$. As $G_{2,5}(\mathbb{H})$ is a smooth manifold of dimension 24, then the cohomology class $[b_4^{10}]$ is zero. Thus, there is a $y \in B$ such that $dy = b_4^{10}$. A straightforward calculation shows that $b_4^9 = d(2b_4^4 y_{19} - b_4^5 y_{15})$, therefore $b_4^{10} = d(2b_4^5 y_{19} - b_4^6 y_{15})$. Hence, $\phi(x_{39}) = 2b_4^5 y_{19} - b_4^6 y_{15}$.

Note that $\theta_4 \in \mathrm{Der}(A, B; \phi)$ is a ϕ-derivation defined by $\theta_4(x_4) = 1$ and $\theta_4(x_{39}) = 10(2b_4^4 y_{19} - b_4^5 y_{15})$. Therefore, θ_4 is a cycle, and $H(\varepsilon_*)([\theta_4]) = [x_4^*] \in G_4(A, B; \phi)$ is in the kernel of $H(\hat{J})$.

References

[1] P. Deligne, P. Griffiths, J. Morgan, and D. Sullivan, *Real homotopy theory of Kähler manifolds*, Invent. Math. **29** (1975), no. 3, 245–274. https://doi.org/10.1007/BF01389853

[2] Y. Félix, S. Halperin, and J.-C. Thomas, *Rational Homotopy Theory*, Graduate Texts in Mathematics, 205, Springer-Verlag, New York, 2001. https://doi.org/10.1007/978-1-4613-0105-9

[3] Y. Félix, J. Oprea, and D. Tanré, *Algebraic Models in Geometry*, Oxford Graduate Texts in Mathematics, 17, Oxford University Press, Oxford, 2008.

[4] J. B. Gatsinzi, V. Onyango-Otieno, and P. A. Otieno, *Relative Gottlieb groups of the Plücker embedding of some complex Grassmanians*, Commun. Korean Math. Soc. **35** (2020), no. 1, 279–285. https://doi.org/10.4134/CKMS.c180453

[5] D. H. Gottlieb, *Evaluation subgroups of homotopy groups*, Amer. J. Math. **91** (1969), 729–756. https://doi.org/10.2307/2373349

[6] W. Greub, S. Halperin, and R. Vanstone, *Connections, curvature, and cohomology*, Pure and Applied Mathematics, Vol. 47-III, Academic Press, New York, 1976.

[7] G. Lupton and S. B. Smith, *Rationalized evaluation subgroups of a map. I. Sullivan models, derivations and G-sequences*, J. Pure Appl. Algebra **209** (2007), no. 1, 159–171. https://doi.org/10.1016/j.jpaa.2006.05.018

[8] L. Menichi, *Rational homotopy—Sullivan models*, in Free loop spaces in geometry and topology, 111–136, IRMA Lect. Math. Theor. Phys., 24, Eur. Math. Soc., Zürich, 2015.

[9] A. Murillo, *The top cohomology class of classical compact homogeneous spaces*, Algebras Groups Geom. **16** (1999), no. 4, 531–550.

[10] E. H. Spanier, *Algebraic Topology*, corrected reprint of the 1966 original, Springer-Verlag, New York, 1995.

[11] D. Sullivan, *Infinitesimal computations in topology*, Inst. Hautes Études Sci. Publ. Math. No. 47 (1977), 269–331 (1978).

[12] M. H. Woo and K. Y. Lee, *On the relative evaluation subgroups of a CW-pair*, J. Korean Math. Soc. **25** (1988), no. 1, 149–160.

Oteng Maphane
Department of Mathematics and Statistics
Falculty of Science
Botswana International University of Science and Technology
Palapye
Botswana
Email address: maphaneot@biust.ac.bw