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HYPERBOLIC STRUCTURE OF POINTWISE INVERSE PSEUDO-ORBIT TRACING PROPERTY FOR C^1 DIFFEOMORPHISMS

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Abstract. We deal with a type of inverse pseudo-orbit tracing property with respect to the class of continuous methods, as suggested and studied by Pilyugin [54]. In this paper, we consider a continuous method induced through the diffeomorphism of a compact smooth manifold, and using the concept, we proved the following: (i) If a diffeomorphism f of a compact smooth manifold M has the robustly pointwise inverse pseudo-orbit tracing property, f is structurally stable. (ii) For a C^1 generic diffeomorphism f of a compact smooth manifold M, if f has the pointwise inverse pseudo-orbit tracing property, f is structurally stable. (iii) If a diffeomorphism f has the robustly pointwise inverse pseudo-orbit tracing property around a transitive set Λ, then Λ is hyperbolic for f. Finally, (iv) for C^1 generically, if a diffeomorphism f has the pointwise inverse pseudo-orbit tracing property around a locally maximal transitive set Λ, then Λ is hyperbolic for f. In addition, we investigate cases of volume preserving diffeomorphisms.

1. Introduction

Pseudo-orbit tracing theories deal with the structure of an orbit. Such notions are extremely useful for investigating the stability theory or hyperbolic structures, among other factors. If a diffeomorphism f of a compact smooth manifold M has a hyperbolic system, then f has the pseudo-orbit tracing property ([58]), and the inverse pseudo-orbit tracing property ([19]). Moreover, using the C^1 perturbation property (robust), if a diffeomorphism f of a compact smooth manifold M has the robust pseudo-orbit tracing property, it has a hyperbolic system ([59]), and if f has the robustly inverse pseudo-orbit tracing property, it then has a hyperbolic system ([19, 53]). According to the results,

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many different types of tracing properties were studied in [13, 19, 26, 28, 32–34, 36–38, 49, 53–55, 60, 61, 63]. In addition, a closed invariant set is hyperbolic if the robust and generic viewpoints of various types of pseudo-orbit tracing properties (see [3, 5, 15, 18, 20–25, 29–31, 39–43, 45–48, 50–52, 62, 64, 66]). Based on the results of these papers, we also studied cases of robust and generic diffeomorphisms of M.

Throughout this article, we assume that M is a compact smooth manifold with a Riemannian metric d induced from the tangent bundle TM. Denote by $\mathrm{Diff}(M) = \{f : M \to M : f \text{ is a diffeomorphism on } M\}$. For any $\delta > 0$, a bi-sequence $\{x_i : i \in \mathbb{Z}\}$ is the δ-pseudo-orbit of f if

$$d(f(x_i), x_{i+1}) < \delta, \ \forall i \in \mathbb{Z}.$$

For a closed f-invariant set $\Lambda \subset M$, a diffeomorphism f of M has the *pseudo orbit tracing property* ($POTP$) *around* Λ if for every positive ϵ, there is a positive constant δ such that for any δ-pseudo-orbit $\{x_i : i \in \mathbb{Z}\} \subset \Lambda$, there is a point $z \in M$ (where z is said to be a *pseudo-orbit tracing point*) such that

$$d(f^i(z), x_i) < \epsilon, \ \forall i \in \mathbb{Z}.$$

Based on this concept, we introduce the inverse pseudo-orbit tracing property, which was studied by Corless and Pilyugin [14], Lee [19], Lee and Lee [43], and Lee [42]. In general, the pseudo-orbit tracing property means that for any pseudo-orbit of f, a real orbit must exist, and thus the distance between the real orbit and a pseudo-orbit remains small at all times. In addition, the inverse pseudo-orbit tracing property means that any real orbit must exist in a pseudo-orbit, and both orbits must remain at a small distance at all times. Note that the inverse pseudo-orbit tracing property is a dual notion of the pseudo-orbit tracing property.

Now, we introduce the inverse pseudo-orbit tracing property with respect to the class of the continuous method.

Let $M^{\mathbb{Z}}$ be the product space of all bi-infinite sequences with the product topology. For any $\delta > 0$, we will denote $\Phi_f(\delta)$ as the set of all δ-pseudo orbits of f. We define a δ-method as a map $\varphi : M \to \Phi_f(\delta) (\subset M^{\mathbb{Z}})$ in which

$$\varphi(x)_0 = x_0 \ \text{ and } \varphi(x)_k = x_k, \ \forall k \in \mathbb{Z},$$

where $\varphi(x)_k$ indicates the k-th component of the sequence $\varphi(x)$. Then, φ is a *δ-pseudo orbit* of f through x.

Let $\mathcal{T}_d(f, \delta)$ be the set of δ-methods φ, which takes a diffeomorphism $g : M \to M$ such that (i) $d(f(x), g(x)) < \delta (x \in M)$ and (ii) $\varphi(x)_k = g^k(x)$ for all $k \in \mathbb{Z}$, where $d(\ ,\ )$ is the C^1 metric.

Definition 1.1. A diffeomorphism $f : M \to M$ has the *inverse pseudo-orbit tracing property with respect to* $\mathcal{T}_d(f, \delta)$ if given $\epsilon > 0$, there is $\delta > 0$ such that for any $\varphi \in \mathcal{T}_d(f, \delta)$, there is $y \in M$ in which $\varphi(y)$ is ϵ-pseudo traced by $x \in M$, that is,

$$d(f^i(x), \varphi(y)_i) < \epsilon, \ \forall i \in \mathbb{Z}.$$

Definition 1.2. A diffeomorphism $f : M \to M$ has the *pointwise inverse pseudo-orbit tracing property* (*PIPTP*) *with respect to* $\mathcal{T}_d(f, \delta)$ if f has the inverse pseudo-orbit tracing property with respect to $\mathcal{T}_d(f, \delta)$ at $z \in M$, that is, if given $\epsilon > 0$, there is $\delta > 0$ such that for any $\varphi \in \mathcal{T}_d(f, \delta)$, there is $y \in M$ in which $\varphi(y)$ is ϵ-pseudo traced by $z \in M$.

Remark 1.3. Throughout this paper, a diffeomorphism f has the inverse pseudo-orbit tracing property (or pointwise inverse pseudo-orbit tracing property), which means that f has the inverse pseudo-orbit tracing property (or, pointwise inverse pseudo-orbit tracing property) with respect to $\mathcal{T}_d(f, \delta)$.

Definition 1.4. A diffeomorphism $f : M \to M$ has the robustly PIPTP if there is a C^1 neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, g has the PIPTP.

A point $p \in M$ is *periodic* if $f^k(p) = p$ for some $k > 0$, and p is *hyperbolic* if $D_p f^{\pi(p)} : T_pM \to T_pM(f^{\pi(p)}(p) = p)$ has no eigenvalues of the norm. For a closed f-invariant set Λ, Λ is *hyperbolic* if $T_\Lambda M$ has the Df-invariant splitting $T_\Lambda M = E^s \oplus E^u$ and there is an $L \in \mathbb{N}$ such that

$$\|Df^n|_{E^s(x)}\| \leq \frac{1}{2} \text{ and } \|Df^{-n}|_{E^u(x)}\| \leq \frac{1}{2}$$

for all $x \in \Lambda$ and $n \geq L$. If f is *Anosov* if $\Lambda = M$ is hyperbolic.

We state that a diffeomorphism f satisfies *Axiom A* if $\Omega(f) = \overline{Per(f)}$ and is hyperbolic, where $\Omega(f)$ is the set of all non-wandering points of f and $Per(f)$ is the set of all periodic points of f. In addition, f is a *strong transversality condition* if for any hyperbolic periodic points $p, q \in \Omega(f)$, the stable manifold of $W^s(p)$ and the unstable manifold of $W^u(q)$ intersect transversally, where $W^s(p) = \{x \in M : d(f^i(p), f^i(x)) \to 0 \text{ as } i \to \infty\}$ and $W^u(q) = \{x \in M : d(f^i(q), f^i(x)) \to 0 \text{ as } i \to -\infty\}$.

A diffeomorphism $f : M \to M$ is *structurally stable* if there is a C^1 neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, there is a homeomorphism $h : M \to M$ such that $f \circ h = g \circ h$.

According to Robinson [57], we can see that a diffeomorphism f is structurally stable if and only if f satisfies Axiom A and the strong transversality condition, and it is known that a diffeomorphism f is Anosov; thus, f is structurally stable. In addition, f has the pseudo-orbit tracing property, and f has the inverse pseudo-orbit tracing property with respect to the class of continuous methods (see [19, 53]). Pilyugin [53] proved that if a diffeomorphism f has the robust inverse pseudo-orbit tracing property with respect to continuous methods induced through continuous maps or a sequence of continuous maps, it is structurally stable. In addition, Lee [19] proved that if a diffeomorphism f has the robustly inverse pseudo-orbit tracing property with respect to $\mathcal{T}_d(f, \delta)$, and thus is structurally stable.

From these results, the main research topic is a study on the qualitative theory of an orbit structure. In this research, we also study the quality of an orbit structure for using the pointwise inverse pseudo-orbit tracing property, which is the main theorem.

A subset $\mathcal{R}$ of Diff(M) is *residual* if it contains the intersection of the countable family of open and dense sets of Diff(M). It is known that $\mathcal{R}$ is dense in Diff(M).

According the concepts, Abdenur and Diáz [2] suggested a problem related to the pseudo-orbit tracing property, that is, if a diffeomorphism f in a residual set $\mathcal{R}$ of Diff(M) has the pseudo-orbit tracing property, then it has a hyperbolic structure. Many results have been published in [3,5,13,18,20,35,39,42,45,46]. Regarding this problem, we consider the following:

Problem. For a C^1 generic diffeomorphism $f \in$ Diff(M), if f has the inverse pseudo-orbit tracing property, then does it have a hyperbolic structure (e.g., structural stability, Axiom A)?

Regarding the above problem and C^1 perturbation property, we prove the following main theorem applied in this study:

Theorem A. *Let $f : M \to M$ be a diffeomorphism. We therefore have the following*:

(a) *If f has the robustly PIPTP, then f is structurally stable.*
(b) *For a C^1 generic f, if f has the PIPTP, then f is structurally stable.*

1.1. Proof of Theorem A

For a hyperbolic periodic point p of f, the dimension $W^s(p)$ is said to be the *index* of p and is denoted as index(p). It should be noted that $T_pW^s(p) = E^s(p)$ and $\dim W^s(p) = \dim E^s(p)$.

Recall that if a periodic point p with a periodic $\pi(p)$ is hyperbolic, then there is a C^1 neighborhood $\mathcal{U}(f)$ of f and a neighborhood U of p such that for any $g \in \mathcal{U}(f)$, there exists a unique $p_g = \bigcap_{n\in\mathbb{Z}} g^n(U)$ such that $g^{\pi(p)}(p) = p$ and p_g is hyperbolic for g.

Lemma 1.5. *If a periodic point p of f is non-hyperbolic, then there is a diffeomorphism g C^1 close to f such that g has two distinct hyperbolic periodic points q, r with* index$(q) \neq$ index(r).

Proof. See [39, Lemma 2.6]. □

Lemma 1.6 ([54, Theorem 1])**.** *The following statements are equivalent*:

(a) *A diffeomorphism f has the PIPTP*;
(b) *$W^u(p)$ and $W^s(q)$ are transverse at x for hyperbolic points $p, q \in P(f)$.*

A diffeomorphism $f : M \to M$ is a *star* if there is a C^1 neighborhood $\mathcal{U}(f)$ of f such that every periodic point of $Per(g)$ is hyperbolic for any diffeomorphism $g \in \mathcal{U}(f)$, where $Per(g)$ is the set of periodic points of g.

Lemma 1.7. *If a diffeomorphism f has the robustly PIPTP, then f is a star.*

Proof. By contradiction, suppose that f is not a star. Then, there is a diffeomorphism g C^1 close to f and a periodic point p of g such that p is not hyperbolic for g. By Lemma 1.5, there is a diffeomorphism g_1 C^1 close to g (in addition, C^1 is close to f such that g_1 has two distinct hyperbolic periodic points q and r with $\text{index}(q) \neq \text{index}(r)$. Because f has the robustly PIPTP, by Lemma 1.6, the manifolds $W^s(q)$ and $W^u(r)$ intersect transversally, and $W^u(q)$ and $W^s(r)$ intersect transversally. This means that $\text{index}(q) = \text{index}(r)$. This is a contradiction. Thus, if a diffeomorphism f has the robustly PIPTP, then f is a star. □

Proof of Theorem A(a). Based on Aoki [4] and Hyashi [16], if a diffeomorphism f is a star, then f is Axiom A and has no cycles. Thus, according to Lemma 1.7, if a diffeomorphism f has the robustly PIPTP, then f is Axiom A and has no cycles. To prove Theorem A, it is sufficient to show a strong transversality condition. Because f has the PIPTP, based on Lemma 1.6, we see that for any two hyperbolic periodic points p and q in the non-wandering set $\Omega(f)$, the manifolds $W^s(p)$ and $W^u(q)$ intersect transversally. Thus, if a diffeomorphism f has the robustly PIPTP, then it is structurally stable. □

To prove Theorem A(b), we need the following lemma.

Lemma 1.8. *There is a residual set $\mathcal{R}$ of* Diff(M) *such that for any $f \in \mathcal{R}$, we have the following*:

(a) *f is Kupka-Smale, that is, every periodic point of f is hyperbolic and the manifolds $W^s(p)$ and $W^u(q)$ are a transverse intersection for any two distinct $p, q \in Per(f)$ (see* [17]*).*
(b) *if there is a diffeomorphism g C^1 close to f such that g has two hyperbolic periodic points p and q with* $\text{index}(p) \neq \text{index}(q)$*, then f has two hyperbolic periodic points p_f and q_f with* $\text{index}(p_f) \neq \text{index}(q_f)$ *(see* [45]*).*

Proof of Theorem A(b). Let $f \in \mathcal{R}$ be the PIPTP. By Lemma 1.6, to prove the structural stability, it is sufficient to show that f satisfies Axiom A. If a diffeomorphism f is a star, then f satisfies Axiom A by Aoki [4] and Hayashi [16]. Thus, we show that f is a star. By contradiction, suppose that f is not a star. Then, there is a diffeomorphism g C^1 close to f and a periodic point p of g such that p is not hyperbolic for g. By Lemma 1.5, there is a diffeomorphism h C^1 close to g (in addition, C^1 is close to f) such that h has two distinct hyperbolic periodic points p and q with $\text{index}(p) \neq \text{index}(q)$. Because $f \in \mathcal{R}$, f has two distinct hyperbolic periodic points p_f and q_f with $\text{index}(p_f) \neq \text{index}(q_f)$. Because f has the PIPTP, by Lemma 1.6, the manifolds $W^s(p_f)$ and $W^u(q_f)$ intersect transversally. This is a contradiction. Thus, if $f \in \mathcal{R}$ has the PIPTP, then f is structurally stable. □

2. Pointwise inverse pseudo-orbit tracing property around a closed invariant set Λ

For a closed f-invariant set Λ of subset M, Λ is *locally maximal* if there is a neighborhood U of Λ such that $\Lambda = U_f = \bigcap_{n\in\mathbb{Z}} f^n(U)$.

Definition 2.1. We state that a diffeomorphism $f : M \to M$ has the *robustly PIPTP* around Λ if (i) Λ is locally maximal, and (ii) there is a C^1 neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, g has the PIPTP around U_g, where $U_g = \bigcap_{n\in\mathbb{Z}} g^n(U)$ is the continuation of Λ.

For a closed f-invariant set Λ, we introduce closed f invariant sets, which are a chain class $C(p)$, a homoclinic class $H(p)$, and a transitive set Λ.

For any $\delta > 0$ and any points $x, y \in M$, we state that x has *a chain relation with* y (which is written as $x >^C y$) if $x_0 = x$, $x_n = y$, and $d(f(x_i), x_{i+1}) < \delta$ for all $0 \leq i \leq n$.

For a hyperbolic periodic point p of f, we denote $C(p) = \{x \in M : x >^C p$ and $p >^C x\}$, which is said to be a *chain class*.

We state that the Λ subset of M is *transitive* if the closure of the orbit of x is Λ, for some $x \in \Lambda$. Equivalently, the omega limit set of x (written as $\omega(x)$) is Λ. For this notion, if $\Lambda = M$, then we state that f is transitive.

For a hyperbolic periodic point q, of f, we write $q \sim p$ if $W^s(p)$ and $W^u(q)$, and $W^u(p)$ and $W^s(q)$, intersect transversally, that is, $W^s(p) \pitchfork W^u(q) \neq \emptyset$ and $W^u(p) \pitchfork W^s(q) \neq \emptyset$. Denote the homoclinic class $H(p)$ as the closure of $\{q \in Per(f) : q \sim p\}$, which is a closed, f-invariant, and transitive set. Moreover, it is known that $H(p) \subset C(p)$.

For a robust property of a diffeomorphism, Lee et al. [48] proved that if a diffeomorphism f has the robust pseudo-orbit tracing property around $C(p)$, then $C(p)$ is hyperbolic. A closed f-invariant set $\Lambda \subset M$, Λ, has a *dominated splitting* if the tangent bundle over Λ splits $T_\Lambda M = E \oplus F$, which is Df-invariant such that $C > 0$ and $\lambda \in (0, 1)$ satisfying

$$\|Df^n_{E(x)}\|\|Df^{-n}|_{F(f^n(x))}\| \leq C\lambda^n$$

for all $x \in \Lambda$ and $n \geq 0$. It is clear that if Λ is hyperbolic, then Λ has a dominated splitting.

Lee and Lee [43] proved that if a diffeomorphism f has the robust inverse pseudo-orbit tracing property around a transitive set Λ, then Λ has dominated splitting. Recently, Lee [42] proved that if a diffeomorphism f has the robustly inverse pseudo-orbit tracing property around Λ, then it is hyperbolic. This result is a generalization of the results in [43].

For a C^1 generic diffeomorphism $f : M \to M$, Lee and Lee [46] proved that if f has the pseudo-orbit tracing property around $C(p)$, then $C(p)$ is hyperbolic, Abdenur and Diáz [2] proved that if f has the pseudo-orbit tracing property around a locally maximal transitive set Λ, then Λ is hyperbolic for f. In addition, Lee [20] proved that if a C^1 generic diffeomorphism f has the robustly inverse pseudo-orbit tracing property around $C(p)$, then $C(p)$ is hyperbolic. However, we do not know whether a C^1 generic diffeomorphism f having the inverse pseudo-orbit tracing property around a closed f-invariant set is hyperbolic?

The following is the result of a C^1 generic diffeomorphism f, which has the pointwise inverse pseudo-orbit tracing property around a closed f-invariant set.

Theorem B. *Let $\Lambda \subset M$ be a transitive set of f. We thus have the following*:

(a) *If a diffeomorphism f has the robustly PIPTP around Λ, then Λ is hyperbolic for f.*

(b) *For a C^1 generic diffeomorphism f, if f has the PIPTP around a locally maximal Λ, then Λ is hyperbolic for f.*

2.1. Proof of Theorem B

Lemma 2.2. *Let $\Lambda \subset M$ be a closed f-invariant set and $\mathcal{U}(f)$ and U be as defined in 2.1. If a periodic point p of f in Λ is non-hyperbolic, then there is a diffeomorphism $g \in \mathcal{U}(f)$ such that g has two distinct hyperbolic periodic points $q, r \in U_g$ with* $\text{index}(q) \neq \text{index}(r)$.

Proof. See [63, Lemma 2.4]. □

From the results of [41, 42], for a transitive set Λ, if every periodic point in Λ is hyperbolic and Λ is locally maximal, then Λ is hyperbolic.

To prove Theorem B(a), it is sufficient to show that every periodic point in Λ is hyperbolic.

Proof of Theorem B(a). Let $\mathcal{U}(f)$ and U be as defined in Definition 2.1. By contrast, suppose that there is a periodic point p of f in Λ such that p is not hyperbolic. From Lemma 2.2, we have a diffeomorphism $g \in \mathcal{U}(f)$ such that g has two distinct hyperbolic periodic points $q, r \in U_g$ with $\text{index}(q) \neq \text{index}(r)$. Because g has the PIPTP around U_g, by Lemma 1.6, we see that $\text{index}(q) = \text{index}(r)$. This is a contradiction. Thus, if f has the robustly PIPTP around Λ, then every periodic point in Λ is hyperbolic. Because Λ is locally maximal in U, by [50, Lemma 2.3], we see that $\Lambda = \overline{\Lambda \cap Per(f)}$. Then, as in the previous arguments, according to the results of Lee [41, 42], Λ is hyperbolic. □

Note that if a transitive set Λ is locally maximal, it guarantees the existence of a periodic point in Λ by Pugh's closing lemma.

Lemma 2.3. *There is a residual subset $\mathcal{G}_1$ of* $\text{Diff}(M)$ *such that for any $f \in \mathcal{G}_1$, if a closed f-invariant set Λ is transitive and it is locally maximal, then $\Lambda = \overline{\Lambda \cap Per(f)}$.*

Proof. Let $\mathcal{G}_1$ be as in [1], and let $f \in \mathcal{G}_1$. Because Λ is locally maximal, Λ is a homoclinic class $H(p)$. Because $\overline{H(p) \cap Per(f)} = H(p)$, we see that $\Lambda = \overline{\Lambda \cap Per(f)}$. □

The following is a version of the diffeomorphisms of [7, Lemma 5.1].

Lemma 2.4. *There is a residual subset $\mathcal{G}_2$ of* $\mathrm{Diff}(M)$ *such that, for any $f \in \mathcal{G}_2$, if for any $\epsilon > 0$, there exists a diffeomorphism g such that g has two distinct hyperbolic periodic points p and q in U with* $\mathrm{index}(p) \neq \mathrm{index}(q)$*, then f has two distinct hyperbolic periodic points p_f and q_f in U with* $\mathrm{index}(p_f) \neq \mathrm{index}(q_f)$.

Lemma 2.5. *There is a residual subset $\mathcal{G}_3$ of* $\mathrm{Diff}(M)$ *such that for any $f \in \mathcal{G}_3$, if f has the PIPTP around Λ, then every periodic point in Λ is hyperbolic.*

Proof. Let $\mathcal{G}_0$ be the same as in [35, Lemma 2.4], and let $f \in \mathcal{G}_3 = \mathcal{R} \cap \mathcal{G}_0 \cap \mathcal{G}_1 \cap \mathcal{G}_2$ have the PIPTP. By contrast, suppose that there is a periodic point p in Λ such that p is non-hyperbolic. By Lemmas 2.2 and 2.4, f has two distinct hyperbolic periodic points q and r in Λ with $\mathrm{index}(q) \neq \mathrm{index}(r)$. Because f has the PIPTP around Λ, by Lemma 1.6, we can see that $\mathrm{index}(q) = \mathrm{index}(r)$. This is a contradiction. Thus, if $f \in \mathcal{G}_4$ has the PIPTP, then every periodic point in Λ is hyperbolic. □

Note that every periodic point in Λ is hyperbolic, and if Λ is locally maximal, there is a C^1 neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, every periodic point is in $U_g = \bigcap_{n\in\mathbb{Z}} g^n(U)(\subset U)$ and is hyperbolic. Thus, we can see that f satisfies the local star condition on Λ, that is, there is a C^1 neighborhood $\mathcal{U}(f)$ of f and a neighborhood U of Λ such that for any $g \in \mathcal{U}(f)$, every periodic point in $U_g = \bigcap_{n\in\mathbb{Z}} g^n(U)$ is hyperbolic.

Proof of Theorem B(b). Let $\mathcal{G}_4$ be as in Lemma [35, Lemma 2.8] and let $f \in \mathcal{G} = \mathcal{G}_3 \cap \mathcal{G}_4$ have the PIPTP around Λ. By Lemma 2.5, every periodic point in Λ is hyperbolic; thus, f satisfies the local star condition on Λ. Then, as in the proof of [35, Theorem 1.1], Λ is hyperbolic. □

From now, we consider the closed f-invariant set, which is the chain class $C(p)$. To prove this, we need the results of a C^1 generic diffeomorphism $f : M \to M$.

Lemma 2.6. *There is a residual subset $\mathcal{S}$ of* $\mathrm{Diff}(M)$ *such that given $f \in \mathcal{S}$, the following hold*:

(a) *For any $\rho > 0$, if for a C^1 neighborhood $\mathcal{U}(f)$ of f, there is a diffeomorphism $g \in \mathcal{U}(f)$ such that g has a ρ-simply periodic curve $\mathcal{I}_{p_g}$ with the two endpoints of $\mathcal{I}_{p_g}$ being homo clinically related with p_g, then f has a 2ρ-simple periodic curve $\mathcal{J}_p$ such that the two endpoints of $\mathcal{J}_p$ are homo clinically related to p (see* [67]*),*

(b) *A homoclinic class $H(p)$ having the hyperbolic periodic point p is a chain class $C(p)$ having the hyperbolic periodic point p (see* [11]*).*

(c) *A homoclinic class $H(p)$ is not hyperbolic if and only if $H(p)$ has a weak hyperbolic periodic point (see* [65]*).*

Theorem 2.7. *For a C^1 generic diffeomorphism $f : M \to M$, if f has a PIPTP around a chain class $C(p)$, then $C(p)$ is hyperbolic.*

Proof. Let $\mathcal{C} = \mathcal{S} \cap \mathcal{G}$, and let $f \in \mathcal{C}$ have the PIPTP around $C(p)$. By contrast, suppose that $C(p)$ is not hyperbolic. Because $f \in \mathcal{S}$, $C(p) = H(p)$, and thus $H(p)$ has a weak hyperbolic periodic point q with $q \sim p$. Then, as in the proof of [67, Proposition A], f has a ρ-simple periodic curve $\mathcal{J}_p$ in $C(p)$ with the two endpoints x and y in $\mathcal{J}_p$ being homoclinically related to p. Then, we can see that the manifolds $W^s(x)$ and $W^u(y)$, or $W^u(x)$ and $W^s(y)$, do not intersect transversally. Because f has the PIPTP around $C(p)$, this is a contradiction. Thus, if $f \in \mathcal{C}$ has the PIPTP around $C(p)$, then $H(p)$ is hyperbolic. □

For a closed f-invariant Λ subset M, we state that Λ is *Lyapunov stable* if for any neighborhood U of Λ, there is a neighborhood V of Λ such that $f^n(\overline{V}) \subset U$ for all $n \geq 0$, and is *bi-Lyapunov stable* if it is Lyapunov stable for f and f^{-1}.

Potrie [56] proved that for a C^1 generic diffeomorphism $f : M \to M$, if homoclinic class $H(p)$ is bi-Lyapunov stable, then $H(p)$ has a dominated splitting.

Lemma 2.8. *There is a residual subset $\mathcal{B}$ of* Diff(M) *such that given $f \in \mathcal{B}$, if $H(p)$ is Lyapunov stable for f then there exists $\mathcal{U}(f)$ a C^1 neighborhood of f such that $H(p_g)$ is Lyapunov stable for every $g \in \mathcal{U}(f)$ (see* [56]*).*

For a C^1 generic diffeomorphism $f : M \to M$, if every hyperbolic periodic point in $H(p)$ has the same index, then $H(p)$ is hyperbolic, and thus, for a C^1 generic diffeomorphism $f : M \to M$, if a homoclinic class $H(p)$ is bi-Lyapunov stable and f has the pseudo-orbit tracing property around $H(p)$, then $H(p)$ is hyperbolic. From these results, we have the following.

Corollary 2.9. *For a C^1 generic diffeomorphism $f : M \to M$, if f has the PIPTP around a bi-Lyapunov stable homoclinic class $H(p)$, then $H(p)$ is hyperbolic.*

Proof. Let $f \in \mathcal{C} \cap \mathcal{B}$ have a PIPTP around $H(p)$, and let $H(p)$ be a bi-Lyapunov stable. Then, as in the proof of [5, Theorem1.2], a bi-Lyapunov stable $H(p)$ is hyperbolic if $H(p)$ is homogeneous. Thus, we show that if a homoclinic class $H(p)$ is bi-Lyapunov stable, then $H(p)$ is homogeneous. Because f has the PIPTP around $H(p)$ by Lemma 1.6, every periodic point in $H(p)$ has the same index, that is, $H(p)$ is homogeneous, and thus, $H(p)$ is hyperbolic. □

3. Volume preserving diffeomorphisms with pointwise inverse pseudo-orbit tracing property

We assume that M is a compact smooth Riemannian manifold with a volume form μ. Let $\mathcal{V}(M)$ denote the space of volume-preserving diffeomorphisms of M. Because f is a volume-preserving diffeomorphism, it is known that the non-wandering set $\Omega(f)$ is the whole space M.

For a robust property of the volume-preserving diffeomorphism of M, Bessa [8] and Lee [27] proved that if $f \in \mathcal{V}(M)$ has the robustly pseudo-orbit tracing property or the inverse pseudo-orbit tracing property, then f is Anosov. For a general type of the pseudo-orbit tracing property, if $f \in \mathcal{V}(M)$ robustly has various types of pseudo-tracing properties, then f is Anosov (see [8, 9, 27, 30, 36, 37, 39, 44]).

For a C^1 generic volume-preserving diffeomorphism of M, Bonatti and Crovisier [11] showed that a C^1 generic volume-preserving diffeomorphism f of M is transitive. Moreover, it is a homoclinic class $H(p)$. In Bessa, Lee, and Wen [10], if a C^1 generic $f \in \mathcal{V}(M)$ has a pseudo-orbit tracing property, then f is Anosov. In addition, if a C^1 generic $f \in \mathcal{V}(M)$ has various types of pseudo-orbit tracing properties, then f is Anosov (see [10, 27, 29, 37]). From the previous results, we prove the following:

Theorem C. *Let $f \in \mathcal{V}(M)$. We therefore have the following*:

(a) *If f has the robustly PIPTP, then f is Anosov.*
(b) *For a C^1 generic f, if f has the PIPTP, then f is Anosov.*

3.1. Proof of Theorem C

By [6, Theorem 1.1], if a volume-preserving diffeomorphism $f : M \to M$ is a star, then f is Anosov. To prove that a volume-preserving diffeomorphism $f : M \to M$ is a star, we use the C^1 perturbation lemma, which is called the Franks' lemma of the version of volume-preserving diffeomorphisms (see [12]).

Lemma 3.1. *If a periodic point p of f is non-hyperbolic, then there is a volume preserving diffeomorphism g C^1 close to f such that g has two hyperbolic periodic points q and r with* $\text{index}(q) \neq \text{index}(r)$.

Proof. See [10, Lemma 2.5]. □

Proof of Theorem C(a). Since f has the robustly PIPTP, as [6, Theorem 1.1], to prove, it is sufficient to show that f is a star. By contrast, suppose that f is not a star. Then, there is a volume-preserving diffeomorphism g C^1 close to f and a periodic point p of g, such that p is non-hyperbolic for g. By Lemma 3.1, there is a volume-preserving diffeomorphism g_1 C^1 close to g (in addition, C^1 close to f) such that g_1 has two distinct hyperbolic periodic points q and r with $\text{index}(q) \neq \text{index}(r)$. Because f has the PIPTP, based on Lemma 1.6, we can see that $\text{index}(q) = \text{index}(r)$. This is a contradiction. □

Lemma 3.2. *There is a residual subset $\mathcal{N}$ of $\mathcal{V}(M)$ such that for any $f \in \mathcal{N}$, if a periodic point p of f is non-hyperbolic, then f has two distinct hyperbolic periodic points q and r with* $\text{index}(q) \neq \text{index}(r)$.

Proof. See [10, Proposition2.4]. □

Proof of Theorem C(b). Let $f \in \mathcal{N}$ have the PIPTP. Suppose that there is a periodic point p of f such that p is not hyperbolic. By Lemma 3.2, f has two distinct hyperbolic periodic points q and r with $\text{index}(q) \neq \text{index}(r)$. Because f has the PIPTP, this is a contradiction by Lemma 1.6. Thus, every periodic point f is hyperbolic if $f \in \mathcal{N}$ has the PIPTP. According to the hyperbolicity, we can see that f is a star, and thus f is Anosov. □

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