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RICCI ρ-SOLITON IN A PERFECT FLUID SPACETIME WITH A GRADIENT VECTOR FIELD

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Abstract. In this paper, we studied several geometrical aspects of a perfect fluid spacetime admitting a Ricci ρ-soliton and an η-Ricci ρ-soliton. Beside this, we consider the velocity vector of the perfect fluid space time as a gradient vector and obtain some Poisson equations satisfied by the potential function of the gradient solitons.

1. Introduction

The spacetime of general relativity can be modelled as a 4-dimensional Lorentzian manifold of signature $(1, 3)$ or equivalently, $(3, 1)$. Relativistic fluid models are of great interest in different branches of astrophysics, nuclear physics etc. Perfect fluids are used in general relativity to model idealized distribution of matter, such as interior of a star or an isotropic universe. Einstein's gravitational equation can describe the behaviour of a perfect fluid inside a spherical object. In general relativity, the source for the gravitational field is the energy momentum tensor. A perfect fluid can be completely characterized by its rest frame mass density and isotropic pressure. It has no shear stresses, viscosity, nor heat conduction.

The general form of the energy momentum tensor T for a perfect fluid spacetime is [10]

$$T(X, Y) = pg(X, Y) + (\sigma + p)\eta(X)\eta(Y) \tag{1.1}$$

for any smooth vector fields X and Y, where p is the isotropic pressure, σ is the energy density, g is the metric tensor of Minkowski's spacetime, η is the dual 1-form of the velocity vector ξ of the fluid and $g(\xi, \xi) = -1$. If $\sigma = -p$, that is, $T = -\sigma g$, then the energy momentum tensor is Lorentz invariant and the medium is vacuum. If $\sigma = 3p$, then the medium is a radiation fluid.

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The field equation governing the perfect fluid motion is Einstein's gravitational equation [10] given by

$$kT(X,Y) = S(X,Y) + (\lambda - \frac{r}{2})g(X,Y) \tag{1.2}$$

for any smooth vector fields X and Y, where k is the gravitational constant, λ is the cosmological constant, S is the Ricci tensor and r is the scalar curvature of g.

From (1.1) and (1.2), we obtain

$$S(X,Y) = -(\lambda - \frac{r}{2} - kp)g(X,Y) + k(\sigma + p)\eta(X)\eta(Y). \tag{1.3}$$

If the Ricci tensor S of a manifold is a functional combination of g and $\eta \otimes \eta$, then the manifold is called quasi-Einstein [6]. Quasi-Einstein manifolds arose during the study of exact solutions of Einstein's field equations. Robertson-Walker spacetime are the examples of quasi-Einstein manifolds. In general relativity, quasi-Einstein manifolds can be taken as a model of the perfect fluid spacetime [7].

The Ricci-Bourguinon flow [5] is a generalization of the Ricci flow which is defined as follows:

$$\frac{\partial g}{\partial t} = -2(S - \rho r g), \quad g(0) = g_0,$$

where S is the Ricci tensor, r is the scalar curvature of g and ρ is a real non-zero constant. A Ricci ρ-soliton is a self similar solution to the Ricci-Bourguinon flow. In fact, the data (g, V, μ) on a pseudo-Riemannian manifold of dimension $n \geq 3$ is said to be a Ricci ρ-soliton if

$$(\mathcal{L}_V g)(X,Y) + 2S(X,Y) + 2(\mu + \rho r)g(X,Y) = 0, \tag{1.4}$$

where $\mathcal{L}_V$ denotes the Lie derivative operator along V and μ is an arbitrary constant. A Ricci ρ-soliton is called expanding if $\mu > 0$, steady if $\mu = 0$ and shrinking if $\mu < 0$.

Perfect fluid spacetime are extensively studied in many purpose of view. Here we would like to mention some recent studies. In 2019, Venkatesha and Kumara [15] studied Ricci solitons in a perfect fluid spacetime with a torse-forming vector field. In the same year, Siddiqi [11] studied Ricci ρ-soliton in a dust fluid and viscous fluid spacetime. In [12], the notion of conformal Ricci solitons in a perfect fluid spacetime were extensively studied. In 2020, Blaga [3, 4] studied Miao-Tam equation and η-Ricci soliton in a perfect fluid spacetime. For more studies, we refer the reader to go through the references [1, 2, 7–9, 13] and references therein.

2. Perfect fluid spacetime

Let (M^4, g) be a general relativistic perfect fluid spacetime satisfying (1.3). Consider $\{e_i\}_{1\leq i\leq 4}$ an orthonormal frame field; that is, $g(e_i, e_i) = \epsilon_{ij}\delta_{ij}$ for

$i,j \in \{1,2,3,4\}$ with $\epsilon_{11} = -1$, $\epsilon_{ii} = 1$ for $i \in \{2,3,4\}$ and $\epsilon_{ij} = 0$ for $i \neq j$; $i,j \in \{1,2,3,4\}$. Let $\xi = \sum_{i=1}^{4} \xi^i e_i$. Then

$$-1 = g(\xi,\xi) = \sum_{i=1}^{4} \epsilon_{ii}(\xi^i)^2$$

and

$$\eta(e_i) = g(e_i,\xi) = \sum_{j=1}^{4} \xi^i g(e_i,e_j) = \epsilon_{ii}\xi^i.$$

Now, contracting (1.3) and considering $g(\xi,\xi) = -1$, we obtain

$$r = 4\lambda + k(\sigma - 3p). \tag{2.1}$$

Substituting (2.1) in (1.3), we obtain

$$S(X,Y) = (\lambda + \frac{1}{2}k(\sigma - p))g(X,Y) + k(\sigma + p)\eta(X)\eta(Y). \tag{2.2}$$

Since $g(\xi,\xi) = -1$, then $\nabla g = 0$ implies

$$g(\nabla_X \xi, \xi) = 0. \tag{2.3}$$

Now, if $\xi = Df$ for some smooth function f on M^4, where D is the gradient operator, then from (2.3), we have $g(\nabla_\xi \xi, X) = 0$ for any smooth vector field X and this implies $\nabla_\xi \xi = 0$. Therefore, we can state the following:

Proposition 2.1. *Let (M^4, g) be a general relativistic perfect fluid spacetime. If the velocity vector ξ of the fluid is a gradient vector field, then the integral curves of ξ are geodesics.*

Theorem 2.2 (Blaga, Prop. 2.2, [3]). *Let (M^4, g) be a general relativistic perfect fluid spacetime satisfying* (2.2) *with p and σ constant*:

(1) *If S is Ricci symmetric, then $\sigma = -p$ or $\nabla\xi = 0$.*
(2) *If S is* (*weakly*) *pseudo-Ricci symmetric, then $p = \frac{2\lambda}{3k} - \frac{\sigma}{3}$. In this case, ξ is a torse-forming* (*in particular, irrotational and geodesic*) *vector field and η is a closed* (*and Codazzi*) 1*-form.*

It is known that locally symmetricness ($\nabla R = 0$) implies Ricci symmetricness ($\nabla S = 0$) but the converse is not true in general. However the converse is true in dimension three. Also weakly symmetricness implies (weakly) pseudo-Ricci symmetricness ([14]). Therefore we can state the following:

Corollary 2.3. *Let (M^4, g) be a general relativistic perfect fluid spacetime satisfying* (2.2) *with p and σ constant*:

(1) *If S is locally symmetric, then $\sigma = -p$ or $\nabla\xi = 0$.*
(2) *If S is weakly symmetric, then $p = \frac{2\lambda}{3k} - \frac{\sigma}{3}$. In this case, ξ is a torse-forming* (*in particular, irrotational and geodesic*) *vector field and η is a closed* (*and Codazzi*) 1*-form.*

3. Ricci ρ-soliton

In this section, we study the data (g, ξ, μ) as a Ricci ρ-soliton in a perfect fluid spacetime (M^4, g) whose velocity vector ξ acts as a potential vector of the soliton.

Considering $V = \xi$ in (1.4), we obtain

$$(\mathcal{L}_\xi g)(X,Y) + 2S(X,Y) + 2(\mu + \rho r)g(X,Y) = 0. \tag{3.1}$$

Now,

$$(\mathcal{L}_\xi g)(X,Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi). \tag{3.2}$$

Substituting (3.2) in (3.1), we have

$$S(X,Y) = -(\mu + \rho r)g(X,Y) - \frac{1}{2}[g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi)]. \tag{3.3}$$

Contracting (3.3), we get

$$r = -4(\mu + \rho r) - div(\xi), \tag{3.4}$$

where '*div*' stands for divergence. Now, equating (2.2) and (3.3), we have

$$\begin{aligned}(\lambda + \frac{1}{2}k(\sigma - p))g(X,Y) + k(\sigma + p)\eta(X)\eta(Y) \\ = -(\mu + \rho r)g(X,Y) - \frac{1}{2}[g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi)].\end{aligned} \tag{3.5}$$

Consider $\{e_i\}_{1 \le i \le 4}$ an orthonormal frame field and let $\xi = \sum_{i=1}^4 \xi^i e_i$. We have seen that $\sum_{i=1}^4 \epsilon_{ii}(\xi^i)^2 = -1$ and $\eta(e_i) = \epsilon_{ii}\xi^i$. Multiplying (3.5) by ϵ_{ii} and summing over i for $X = Y = e_i$, we obtain

$$4\mu + 4\rho r = -4\lambda - k(\sigma - 3p) - div(\xi). \tag{3.6}$$

Now, substituting $X = Y = \xi$ in (3.5), we get

$$\mu + \rho r = -\lambda + \frac{1}{2}k(\sigma + 3p). \tag{3.7}$$

The equations (3.6) and (3.7) together implies

$$div(\xi) = -3k(\sigma + p). \tag{3.8}$$

Substituting the value of r from (2.1) in (3.7), we obtain

$$\mu = -\lambda(1 + 4\rho) + \frac{1}{2}k[(\sigma + 3p) - 2\rho(\sigma - 3p)]. \tag{3.9}$$

In view of (3.8), we can state the following:

Theorem 3.1. *Let (M^4, g) be a general relativistic perfect fluid spacetime. If (g, ξ, μ) represents a Ricci ρ-soliton with $\xi = Df$ for some smooth function f on M^4, then the Poisson equation satisfied by f is*

$$\Delta f = -3k(\sigma + p).$$

From (3.9), we can state the following:

Theorem 3.2. *Let (M^4, g) be a general relativistic perfect fluid spacetime. If (g, ξ, μ) represents a Ricci ρ-soliton in (M^4, g), then the soliton is*

(1) *expanding if* $\frac{1}{2}k[(\sigma + 3p) - 2\rho(\sigma - 3p)] > \lambda(1 + 4\rho)$.
(2) *steady if* $\frac{1}{2}k[(\sigma + 3p) - 2\rho(\sigma - 3p)] = \lambda(1 + 4\rho)$.
(3) *shrinking if* $\frac{1}{2}k[(\sigma + 3p) - 2\rho(\sigma - 3p)] < \lambda(1 + 4\rho)$.

If $\sigma = -p$, then the medium is vacuum. Therefore, from (3.8), we write the following:

Theorem 3.3. *If (g, ξ, μ) represents a Ricci ρ-soliton in the vacuum, then the velocity vector ξ is harmonic.*

Remark 3.4. If $\rho = 0$, then (3.1) reduces to

$$(\mathcal{L}_\xi g)(X, Y) + 2S(X, Y) + 2\mu g(X, Y) = 0.$$

This is the Ricci soliton equation. Now, from (3.9), we have

$$\mu = -\lambda + \frac{1}{2}k(\sigma + 3p).$$

Therefore, the Ricci soliton is steady if $p = \frac{2\lambda}{3k} - \frac{\sigma}{3}$; expanding if $p > \frac{2\lambda}{3k} - \frac{\sigma}{3}$ and shrinking if $p < \frac{2\lambda}{3k} - \frac{\sigma}{3}$ (see also [3]).

Remark 3.5. If we take radiation fluid, that is, $\sigma = 3p$; then from (3.9),

$$\mu = -\lambda(1 + 4\rho) + k\sigma.$$

So, the Ricci ρ-soliton in a radiation fluid is steady if $\lambda(1+4\rho) = k\sigma$; expanding if $\lambda(1 + 4\rho) < k\sigma$ and shrinking if $\lambda(1 + 4\rho) > k\sigma$.

Remark 3.6. In a general relativistic perfect fluid spacetime, if the vector field ξ is torse-forming with

$$\nabla_X \xi = s[X + \eta(X)\xi]$$

for any smooth vector field X and s a non-zero real number, then

$$div(\xi) = 3s. \tag{3.10}$$

Therefore, from (3.8) and (3.10), we get

$$s = -k(\sigma + p).$$

So, in this case, the existence of a Ricci ρ-soliton from Plebański energy conditions $\sigma \geq 0$ and $-\sigma \leq p \leq \sigma$ for perfect fluids implies $-2k\sigma \leq s < 0$ (precisely, $s = -k(\sigma + p)$).

Remark 3.7. If the vector field ξ is conformal Killing, that is, $\mathcal{L}_\xi g = sg$ with a non-zero real number s, then the existence of a Ricci ρ-soliton implies the vacuum case. Indeed, equation (3.5) can be written as

$$\begin{aligned}&(\lambda + \frac{1}{2}k(\sigma - p))g(X, Y) + k(\sigma + p)\eta(X)\eta(Y)\\ =\ & -(\mu + \rho r)g(X, Y) - \frac{1}{2}(\mathcal{L}_\xi g)(X, Y),\end{aligned}$$

which implies

$$(3.11)\qquad (\lambda+\frac{1}{2}k(\sigma-p))g(X,Y)+k(\sigma+p)\eta(X)\eta(Y)$$
$$=-(\mu+\rho r)g(X,Y)-\frac{1}{2}sg(X,Y).$$

Now, contracting X and Y in (3.11), we obtain

$$(3.12)\qquad 4[\lambda+\frac{1}{2}k(\sigma-p)]-k(\sigma+p)=-4(\mu+\rho r+\frac{1}{2}s).$$

Again, putting $X=Y=\xi$ in (3.11), we get

$$(3.13)\qquad -[\lambda+\frac{1}{2}k(\sigma-p)]+k(\sigma+p)=(\mu+\rho r+\frac{1}{2}s).$$

Solving (3.12) and (3.13), we obtain $k(\sigma+p)=0$. As $k\neq 0$, $\sigma=-p$ and this describes the vacuum case.

4. η-Ricci ρ-soliton

In this section, we consider a slightly more general notion of the Ricci ρ-soliton, called η-Ricci ρ-soliton, which can be obtained by adding certain multiple of the $(0,2)$-tensor field $\eta\otimes\eta$ in (1.4). Therefore, an η-Ricci ρ-soliton is given by

$$(4.1)\qquad (\mathcal{L}_V g)(X,Y)+2S(X,Y)+2(\mu+\rho r)g(X,Y)+2\omega\eta(X)\eta(Y)=0,$$

where μ and ω are real constants. An η-Ricci ρ-soliton is called expanding if $\mu>0$, steady if $\mu=0$ and shrinking if $\mu<0$.

In this section, we study the data an η-Ricci ρ-soliton in a perfect fluid spacetime (M^4,g) whose velocity vector ξ acts as a potential vector of the soliton. For this, taking $V=\xi$ in (4.1), we get

$$(4.2)\quad S(X,Y)=-(\mu+\rho r)g(X,Y)-\omega\eta(X)\eta(Y)-\frac{1}{2}[g(\nabla_X\xi,Y)+g(X,\nabla_Y\xi)].$$

The equations (2.2) and (4.2) together implies

$$(4.3)\qquad (\lambda+\frac{1}{2}k(\sigma-p))g(X,Y)+k(\sigma+p)\eta(X)\eta(Y)$$
$$=-(\mu+\rho r)g(X,Y)-\omega\eta(X)\eta(Y)-\frac{1}{2}[g(\nabla_X\xi,Y)+g(X,\nabla_Y\xi)].$$

Contracting X and Y in the foregoing equation, we obtain

$$(4.4)\qquad 4(\lambda+\frac{1}{2}k(\sigma-p))-k(\sigma+p)=-4(\mu+\rho r)+\omega-div(\xi).$$

Putting $X=Y=\xi$ in (4.3) yields

$$(4.5)\qquad -(\lambda+\frac{1}{2}k(\sigma-p))+k(\sigma+p)=(\mu+\rho r)-\omega.$$

The equations (4.4) and (4.5) together implies

$$\omega = -k(\sigma + p) - \frac{1}{3}div(\xi). \tag{4.6}$$

Substituting the value of ω from (4.6) in (4.5), we infer that

$$\mu + \rho r = -(\lambda + \frac{1}{2}k(\sigma - p)) - \frac{1}{3}div(\xi).$$

Now, putting the value of r from (2.1) in the foregoing equation yields

$$\mu = -(1 + 4\rho)\lambda - \frac{1}{2}k[(\sigma - p) + 2\rho(\sigma - 3p)] - \frac{1}{3}div(\xi). \tag{4.7}$$

Therefore, we are in a position to state the following:

Theorem 4.1. *Let (M^4, g) be a general relativistic perfect fluid spacetime. If (g, ξ, μ, ω) represents an η-Ricci ρ-soliton in (M^4, g), then*

$$\mu = -(1 + 4\rho)\lambda - \frac{1}{2}k[(\sigma - p) + 2\rho(\sigma - 3p)] - \frac{1}{3}div(\xi),$$
$$\omega = -k(\sigma + p) - \frac{1}{3}div(\xi).$$

If we consider ξ as a gradient vector field, then in view of (4.6), we have the following:

Theorem 4.2. *Let (M^4, g) be a general relativistic perfect fluid spacetime admitting an η-Ricci ρ-soliton (g, ξ, μ, ω), where $\xi = Df$ for some smooth function f on M^4. Then f satisfies the Poisson equation*

$$\Delta f = -3[\omega + k(\sigma + p)].$$

Example 4.3. An η-Ricci ρ-soliton (g, ξ, μ, ω) in a radiation fluid is given by

$$\mu = -(1 + 4\rho)\lambda - kp - \frac{1}{3}div(\xi),$$
$$\omega = -4kp - \frac{1}{3}div(\xi).$$

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