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ON LIGHTLIKE HYPERSURFACES OF COSYMPLECTIC SPACE FORM

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Abstract. The main purpose of this paper is to study the lightlike hypersurface $(M, \bar{g})$ of cosymplectic space form $\bar{M}(c)$. In this paper, we computed the Gauss and Codazzi formulae of $(M, \bar{g})$ of cosymplectic manifold $(\bar{M}, g)$. We showed that we can't obtain screen semi-invariant lightlike hypersurface (SCI-LH) of $\bar{M}(c)$ with parallel second fundamental form h, parallel screen distribution and $c \neq 0$. We showed that if second fundamental form h and local second fundamental form B are parallel, then $(M, \bar{g})$ is totally geodesic. Finally we showed that if $(M, \bar{g})$ is umbilical, then cosymplectic manifold $(\bar{M}, g)$ is flat.

1. Introduction

In differential geometry, the concept of lightlike hypersurface theory of psedo-Riemannian manifolds is a very important and interesting topic for researchers to study. There are different types of sub-manifolds of psedo-Riemannian manifolds like lightlike, timelike and spacelike submanifolds, that depends on the structure of an induced metric on $(T_P\bar{M})$.

In lightlike hypersurfaces as an induced metric is degenerate, the study becomes quite contrasting and more complicated from non-degenerate concept of semi-Riemannian manifolds. One of the main difference or contrast between lightlike hypersurfaces and non-degenerate is that, in case of lightlike hypersurface, the normal vector and tangent vector bundles has non-trivial intersections.

Moreover in case of degenerate metric on hypersurfaces the tangent vector bundle contains normal vector bundle. In mathematical physics and general relativity the extensive theory of lightlike submanifolds has been studied from time to time and had found its applications in black horizons, the Kruskal and Kerr black holes.

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In (1996), Duggal and Bejancu [7] introduced the concept of degenerate particularly lightlike-geometry of semi-Riemannian manifolds, and they gave extrinsic approach in the field of differential geometry. This created widespread interest among researchers to study lightlike geometry of semi-Riemannian manifolds. With the passage of time several authors studied lightlike hypersurfaces of psedo-Riemannian manifolds. See also [3,8,9] and many more references there in.

N. Aktan [2] investigated lightlike hypersurfaces of indefinite Sasakian space form and [1] indefinite Kenmotsu space form, and proved non-existence of lightlike hypersurfaces of these manifolds. Recent years have witnessed a surge in interest in the fields of contact and almost contact geometry, as well as related topics such as super gravity and M. theory. An odd dimensional cosymplectic manifold has been proved to be a very valuable kind of almost contact manifold. In (1958), [10] Libermann proposed the concept of cosymplectic manifold, which, according to [13], means smooth odd-dimensional manifolds endowed with closed 1-form η and 2-form ω.

The odd dimensional equivalent of a Kaehlar manifold is a cosymplectic manifold and is locally the product of a line or a circle with a Kaehlar manifold. In fact, a cosymplectic structure on an odd dimensional smooth manifold is a normal almost contact metric structure such that the 1-form and fundamental 2-form are closed [13] and many more references therein.

2. Preliminaries

This section is devoted to give brief introduction of cosymplectic manifolds of constant ϕ-sectional curvature c and lightlike hypersurfaces of semi-Riemannian manifolds. A complete discussion on the contents used in this section can be found in [4, 5, 8, 11, 13].

2.1. Cosymplectic manifolds

Let $(\bar{M}, g)$ be an n-dimensional almost contact metric manifold endowed with $(\phi, \xi, \eta, \bar{g})$ as an almost contact metric structure, where ϕ, ξ, η and g represent a $(1,1)$-tensor field, an associated vector field, a 1-form and the Riemannian metric, respectively, satisfying

$$\phi^2 = -I + \eta \otimes \xi, \ \eta(\xi) = 1, \ \phi(\xi) = 0, \ \eta \circ \phi = 0, \tag{1}$$

$$g(\phi F, \phi J) = g(F, J) + \eta(F)\eta(J); \ \eta(F) = g(F, \xi), \tag{2}$$

where I represents identity tensor field and $F,\ J \in \chi(M)$. A normal contact metric manifold $(\bar{M}, g)$ is said to be a cosymplectic manifold if it satisfies [13].

$$(\bar{D}_F \phi)(J) = 0, \tag{3}$$

and

$$\bar{D}_F \xi = 0.$$

For any F, J on M, $\bar{D}$ represents Levi-Civita connection on M.

It is a well known fact that $(\phi, \xi, \eta, \bar{g})$ is cosymplectic if and only if $\bar{D}\eta = 0$ and $\bar{D}\Phi$ vanishes, where $\bar{D}$ represents covariant derivative with respect to Riemannian metric g. The fundamental 2-form is given by

$$g(\phi F, J) = g(F, \phi J) = \Phi(F, J).$$

The plane section $\bar{\sigma}$ in tangent space $T_P M$ is called a ϕ-section if spanned by F and ϕF, where F is a unit vector field and is orthogonal to ξ. ϕ-sectional curvature is the sectional curvature of a ϕ-section $\bar{\sigma}$. If a cosymplectic manifold $(\bar{M}, g)$ has constant ϕ-sectional curvature c at a point, then $(\bar{M}, g)$ is said to be a cosymplectic space form and is denoted as $\bar{M}(c)$. The curvature tensor R^1 of cosymplectic space form is given [12, 13]:

$$\begin{aligned} R^1(F,J)L = \frac{c}{4}\{&g(\phi J, \phi L)F - g(\phi F, \phi L)J \\ &+ \eta(J)g(F,L)\xi - \eta(F)g(J,L)\xi + g(\phi J, L)\phi F \\ &- g(\phi F, L)\phi J + 2g(F, \phi J)\phi L\} \end{aligned} \tag{4}$$

for any $F, J, L \in \Gamma(TM)$.

2.2. Light-like hypersurfaces

Let $(\bar{M}, g)$ be a semi-Riemannian manifold with index $(g) = q \geq 1$ and let $(M, \bar{g})$ be the hypersurface of $(\bar{M}, g)$ such that $g = \bar{g}|_M$. If an induced metric $\bar{g}$ on hypersurface $(M, \bar{g})$ is degenerate, then the hypersurface $(M, \bar{g})$ is said to be a lightlike hypersurface (see [6–8, 11]). On lightlike hypersurfaces there exists null vector field $\xi \neq 0$ such that

$$\bar{g}(\xi, F) = 0, \ \forall F \ \in \Gamma(T\bar{M}).$$

In the field of degenerate geometry of manifolds $RadT_xM$ is a radical space or null space of a tangent space T_xM and is a subspace at each point $F \in (M, \bar{g})$ and is defined by [7]

$$RadT_FM = \{\xi \in T_FM \,|\, g_F(\xi, F) = 0, \ \forall F \in \Gamma(TM)\}. \tag{5}$$

The nullity degree of an induced metric $\bar{g}$ is the dimension of $RadT_FM$ and it is well known that for light-like hypersurface $(M, \bar{g})$ the nullity degree of $\bar{g}$ is 1. $RadTM$ is called radical distribution and is spanned by null vector field ξ.

$S(TM)$ is a complementary vector bundle of $RadTM$ in tangent bundle TM called screen bundle of lightlike hypersurface (M). We note any $S(TM)$ is non degenerate and $S(TM)^{\perp}$ is the complementary vector bundle of $S(TM)$ in TM of rank 2 is called screen transversal bundle. Since radical distribution $RadTM$ is null bundle of screen bundle $S(TM)^{\perp}$ there exists X as a unique local section of $S(TM)^{\perp}$ [8] such that

$$g(X, W) = \ g(X, X) = 0, \ g(X, Z) = 1.$$

The pair (X, Z) is a local frame field of $S(TM)^{\perp}$ and note that X is transversal to hypersurface M. Then there exists a line bundle $ltrTM$ of tangent bundle of a manifold called as lightlike transversal bundle locally spanned by X. Hence we have

$$T\bar{M} = TM \oplus ltr(TM) = S(TM) \perp RadTM \oplus ltr(TM),$$

where $\oplus$ represents direct sum but not orthogonal [7, 8]. In view of (6) the Gauss and Weingarten formulas are given respectively by

$$\bar{D}_F J = D_F J + h(F, J),$$

$$\bar{D}_F X = -A_X F + D^t_F X.$$

If we set

$$B(F, J) = g(h(F, J), Z), \tag{6}$$

$$\tau(F) = g(D^t_F X, Z),$$

then the above equations become

$$\bar{D}_F J = D_F J + B(F, J)X, \tag{7}$$

$$\bar{D}_F X = -A_X F + \tau(F)X \tag{8}$$

for any $F, J \in \Gamma(TM)$. Here $X \in \Gamma ltr(TM)$, $D_F J, -A_X F \in \Gamma(TM)$ and $h(F, J), D^t_F X \in \Gamma(ltr(TM))$. Here A_X, B, D and D^t represent shape operator, second fundamental form, torsion free linear connection on lightlike hypersurface M and linear connection on $ltr(TM)$, respectively [7]. From (5), we can state that an induced connection on lightlike hypersurface M satisfies the following condition

$$(D_F g)(J, L) = B(F, J)\theta(L) + B(F, L)\theta(J).$$

For any F, J, $L \in \Gamma(TM)$ and θ is a differential 1-form on $\bar{M}$ locally defined by $\theta(F) = g(X, F)$ and D is an induced non-metric connection on smooth manifold M. It may be noted that B of $\bar{M}$ is independent of the choice of $S(T\bar{M})$ and may be represented as $B(-, Z) = 0$.

With respect to projection P local Gauss and Weingarten formulas is as

$$\nabla_F PJ = \nabla^*_F PJ + C(F, PJ)Z,$$

$$D_F Z = -A^*_Z F + \tau(F)Z,$$

where $\nabla_F PJ$ and $A^*_\zeta F \in S(TM)$, and A^*, C and ∇^* represent local shape operator, local second fundamental form and induced connection on screen bundle $S(TM)$, respectively. On lightlike hypersurface shape operators B and C are related as

$$\bar{g}(A_X F, PJ) = C(F, PJ),\ \bar{g}(A_X F, X) = 0,$$

$$\bar{g}(A^*_\xi F, PJ) = B(F, PJ),\ \bar{g}(A^*_\xi F, \xi) = 0$$

for all $F, J \in \Gamma(TM)$. With respect to Livi-Civita connection $\bar{D}$ let R^1 be a curvature tensor. Then we have (see [7])

$$R^1(F,J)L = R(F,J)L + A_{h(F,L)}J - A_{h(J,L)}F \\ + (D_F h)(J,L) - (D_J h)(F,L).$$

Let R, R^1 and R^* represent curvature tensors with respect $D, \bar{D}$ and ∇, respectively. For lightlike hypersurface $\bar{M}$ and screen bundle $S(TM)$ using Gauss formula and Weingarten formulae we have [8]

$$g(R^1(F,J)L, PO) = \bar{g}(R(F,J)L, PO) + B(F,L)C(J,PO) \\ - B(J,L)C(F,PO),$$

$$g(R^1(F,J)L, Z) = (\nabla_F B)(J,L) - (\nabla_J B)(F,L) \\ + \tau(F)B(J,L) - \tau(J)B(F,L), \tag{9}$$

$$\bar{g}(R(F,J)PL, X) = (\nabla_F C)(J,PL) - (\nabla_J C)(F,PL) \\ + \tau(J)C(F,PL) - \tau(F)B(J,PL),$$

$$g(R^1(F,J)L, X) = \bar{g}(R(F,J)L, X), \tag{10}$$

$$R^1(F,J)L = R(F,J)L + B(F,L)A_X J - B(J,L)A_X F \\ + \{(\nabla_F B)(J,L) - (\nabla_J B)(F,L) \\ + \tau(F)B(J,L) - \tau(J)B(F,L)\}X$$

for any $F, J, L \in \Gamma(T\bar{M})$.

3. Lightlike hypersurfaces of cosymplectic space form

Tangent to the structure vector field ξ, i.e., $\xi \in \Gamma(TM)$, let $(M,\ \bar{g})$ be a hypersurface of a cosymplectic space form $\bar{M}(c)$. Then $(M,\ \bar{g})$ is called a lightlike hypersurface, if the induced metric $\bar{g}$ is degenerate. The manifold $\bar{M}$ having $ind(g) = 1$ and $ind(g) > 1$ is said to be a spacelike and timelike cosymplectic manifold. If Z is the local section of radical distribution, i.e., $Z \in Rad(TM)$, then ϕZ is tangent to M and $g(\phi Z, Z) = 0$. Therefore $\phi(Rad(TM))$ is 1-dimensional distribution on a lightlike hypersurface M.

Definition. Let $(\bar{M}, \phi, \xi, \eta, g)$ be a $(2n+1)$ dimensional cosyplectic manifold and $(M, \bar{g})$ be the lightlike hypersurface of $(\bar{M}, \phi, \xi, \eta, g)$. Then the hypersurface $(M, \bar{g})$ is a screen semi invariant lightlike hypersurface, if the following conditions are satisfied

$$\phi(Rad(TM)) \subset S(TM) \ \text{ and } \ \phi(ltr(TM)) \subset S(TM).$$

From equations (1) and (2), we have

$$g(\phi X, Z) = -g(X, \phi Z) = 0,\ g(\phi X, X) = 0,$$

$$g(\phi X, \phi Z) = 1. \tag{11}$$

Therefore $\phi(Rad(TM)) \oplus \phi(ltr(TM))$ is non degenerate vector sub-bundle of $S(TM)$ having rank 2. If a lightlike hypersurface M is tangent to ξ, then $\xi \in S(TM)$. Since $g(\phi Z, \xi) = g(\phi X, \xi) = 0$, there exists non degenerate distribution D_0 on a lightlike hypersurface M of rank $2n-4$ such that

$$S(TM) = \{\phi(TM^{\perp}) \oplus \phi(X(TM))\} \perp D_0 \perp \langle\xi\rangle, \tag{12}$$

where $\langle\xi\rangle = span\xi$.

The second fundamental form B is independent on screen distribution so we have the following equation.

$$B(\cdot, Z) = 0.$$

From the 1st equation of (2), we see that ϕZ and ϕX are degenerate vector fields and

$$\phi^2 Z = -Z \text{ and } \phi^2 X = -X.$$

Let us assume the local degenerate vector fields $Y = \phi X \in \phi(ltr(TM))$ and $V = \phi Z \in D$. Suppose P is the projection of screen bundle $S(TM)$ on a lightlike hypersurface M. Any $F \in \Gamma(TM)$ can be written as

$$F = SF + QF, \ QF = u(F)Y.$$

Applying ϕ to the above equation we get

$$\phi F = \phi(SF) + u(F)\phi Y,$$

where Q and S are the projection morphisms of tangent bundle into distributions G and G^1, respectively, and $u(F) = g(V, F)$ is the differential 1-form locally defined on a lightlike hypersurface. By putting $\bar{\phi}F = \phi(Sx)$ for any $F \in \Gamma(TM)$ in the last equation we get

$$\phi F = \bar{\phi}F - u(F)X, \tag{13}$$

where ϕ and $\bar{\phi}$ are 1-1 tensor fields on a cosymplectic manifold $\bar{M}$ and a lightlike hypersurface M, respectively. We note that $u(Y) = -1$ for all $J \in \Gamma(TM)$ $u(J) = 0$. Applying ϕ again we get

$$\phi^2 F = \phi\bar{\phi}F - u(F)\phi X,$$

$$-F + \eta(F)\xi = \phi\bar{\phi}F - u(F)Y.$$

In the above equations we note that if for any $F \in \Gamma(TM)$, $SF \in D$, $\phi F = \bar{\phi}(SF) \in D$, then $S(\bar{\phi}F) = \bar{\phi}F$. Which gives $\phi\bar{\phi}F = \bar{\phi}^2 F$ and we can write

$$\phi^2 F = -F + \eta(F)\xi + u(F)Y.$$

By using (7), (8) and (11) we get

$$(D_F\bar{\phi})J = -u(J)A_N F - B(F, J)Y,$$

and

$$(\bar{D}_F u)J = -B(F, \bar{\phi}J) - u(J)\tau(F).$$

Also for a lightlike hypersurface of a cosymplectic manifold we have

$$v(F) = g(X, F).$$

Theorem 3.1. *Let $(M, \bar{g})$ be an invariant lightlike hypersurface of a cosymplectic manifold $(\bar{M}, \phi, \xi, \eta, g)$ tangent to the structural vector field ξ. Then $(M, \bar{g}, \xi, \eta, \bar{\phi})$ defines an almost para-contact metric manifold.*

Proof. Assuming $(M, \bar{g})$ is an invariant lightlike hypersurface of a cosymplectic manifold $(\bar{M}, g)$. Then from equation (13) for any $F, J \in \Gamma(TM)$ we get

$$\bar{\phi}F = \phi F. \tag{14}$$

Using the 1st equation of (2) and (14), we get

$$\bar{\phi}^2 F = -F + \eta(F)\xi. \tag{15}$$

From equation (14), we get

$$\bar{\phi}\xi = 0. \tag{16}$$

From equations (15) and (17), we get

$$\eta \circ \bar{\phi} = 0, \ \eta(\xi) = 1.$$

From above equation and (14), (15), (16) and (2), we get

$$\bar{g}(\bar{\phi}F, \bar{\phi}J) = \bar{g}(F, J) - \eta(F)\eta(J). \tag{17}$$

As a result of equations (14), (15), (16) and (17), $(M, \bar{g}, \xi, \eta, \bar{\phi})$ defines an almost para-contact metric manifold. □

Theorem 3.2. *Let $(M, \bar{g})$ be an invariant lightlike hypersurface of a cosymplectic manifold $(\bar{M}, \phi, \xi, \eta, g)$ tangent to the structural vector field ξ. Then for all F, J and $L \in \Gamma(TM)$ we obtain*

$$(D_F\bar{\phi})J = 0, \ B(F, \phi J)X = B(F, \phi J)\phi X, \ \bar{\phi}(A_X F) = A_{\phi X}F. \tag{18}$$

Proof. We know that

$$(\bar{D}_F\phi)J = \bar{D}_F\phi J - \phi(\bar{D}_F J)(D_F\phi)J + B(F, \bar{\phi J})X - B(F, J)\phi X. \tag{19}$$

using equation (3) in (19) we get

$$(D_F\phi)J + B(F, \bar{\phi J})X - B(F, J)\phi X = 0. \tag{20}$$

By equating tangential and transversal components of above equation we get the 1st two equations of (18).

$$(\bar{D}_F\phi)X = \bar{D}_F\phi X - \phi(\bar{D}_F X). \tag{21}$$

By virtue of equations (3) and (8), we obtain the 3rd equation of (18). □

Proposition 3.3. *Let $(M, \bar{g})$ be a lightlike hypersurface of a cosymplectic manifold $(\bar{M}, \phi, \xi, \eta, g)$ tangent to the structural vector field ξ. Then*

$$\bar{\phi}^2 F = -F + \eta(F)\xi + u(\bar{\phi}F)X + u(F)\phi X, \tag{22}$$

$$B(F, \xi) = -u(F), \ C(F, \xi) = -v(F) \tag{23}$$

for any $F \in \Gamma(TM)$.

Proof. From the 1st equation of (1) and (13), we get (18), and by using (3), (7), (13) and then comparing tangential and transversal parts we get (19). □

Definition. A lightlike hypersurface $(M, \bar{g}, \xi, \eta, \bar{\phi})$ of a cosymplectic manifold is D^0 totally geodesic if $B(F, J) = 0$ for any $F, J \in D^0$.

Theorem 3.4. *Let $(M, \bar{g}, \xi, \eta, \bar{\phi})$ be a lightlike hypersurface of $(\bar{M}, \phi, \xi, \eta, g)$ tangent to the structural vector field ξ. Then for any $F, J \in \Gamma(TM)$, a lightlike hypersurface $(M, \bar{g}, \xi, \eta, \bar{\phi})$ is totally geodesic if h is parallel, where h is the second fundamental form.*

Proof. Let's pretend h is parallel.

$$(\bar{D}_F h)(J, \xi) = D_F h(J, \xi) - h(D_F J, \xi) - h(J, D_F \xi).$$

Using equation (3) in (20), we get our desired result. □

Theorem 3.5. *Let $(M, \bar{g}, \xi, \eta, \bar{\phi})$ be a screen semi-invariant lightlike hypersurface of a cosymplectic space form $\bar{M}(c)$ that is tangent to the structural vector field ξ and the local second fundamental form B is parallel. If $\tau(Z) \neq 0$, then c=0 if and only if it is totally geodesic.*

Proof. By putting $J = Z$ in (9) and (4) we obtain

$$\begin{aligned} g(R^1(F, Z)L, Z) = (\nabla_F B)(Z, L) - (\nabla_Z B)(F, L) \\ + \tau(F)B(Z, L) - \tau(Z)B(F, L), \end{aligned}$$

and

$$\begin{aligned} g(R^1(F, Z)L, Z)) = \frac{c}{4}\{&g(\phi Z, \phi L)g(F, Z) - g(\phi F, \phi L)g(Z, Z) \\ &+ \eta(Z)g(F, L)g(\xi, Z) - \eta(F)g(Z, L)g(\xi, Z) \\ &+ g(\phi Z, L)g(\phi F, Z) - g(\phi F, L)g(\phi Z, Z) \\ &+ 2g(F, \phi Z)g(\phi L, Z)\}, \end{aligned} \tag{24}$$

respectively. Since B is parallel, then from equations (21) and (22) we obtain

$$\frac{c}{4}u(F)u(L) = \tau(Z)B(F, L).$$

By replacing F and L by Y in (23) we get

$$\frac{c}{4} = \tau(Z)B(Y, Y).$$

If $\tau(Z) \neq 0$ we get our required result. □

Definition. If the lightlike hypersurface $(M, \bar{g}, \xi, \eta, \bar{\phi})$ of a cosymplectic manifold satisfies

$$B(F, J) = \lambda\bar{g}(F, J),$$

then $(M, \bar{g}, \xi, \eta, \bar{\phi})$ is called a totally umbilical lightlike hypersurface, where λ represents a smooth function.

Theorem 3.6. *Let* $(M, \bar{g}, \xi, \eta, \bar{\phi})$ *be a screen semi-invariant lightlike hypersurface (SCI-LH) of a cosymplectic space form* $\bar{M}(c)$ *tangent to the structural vector field* ξ*. Then the lightlike hypersurface* $(M, \bar{g}, \xi, \eta, \bar{\phi})$ *is flat if* M *is totally umbilical.*

Proof. From equation (4) we get

$$\begin{aligned} g(R^1(F,J)L,Z) = \frac{c}{4}\{&g(\phi J, \phi L)g(F,Z) - g(\phi F, \phi L)g(J,Z) \\ &+ \eta(J)g(F,L)g(\xi,Z) - \eta(F)g(J,L)g(\xi,Z) \\ &+ g(\phi J, L)g(\phi F, Z) - g(\phi F, L)g(\phi J, Z) \\ &+ 2g(F, \phi J)g(\phi L, Z)\}. \end{aligned}$$

By putting $\phi Z = V$, $\phi X = Y$ and $g(F,Z) = 0$ in last equation, we obtain

$$\begin{aligned} g(R^1(F,J)L,Z) = \frac{c}{4}\{&g(\phi F, L)g(J,V) - g(\phi J, L)g(F,V) \\ &- 2g(\phi F, J)g(L,V)\} \end{aligned} \tag{25}$$

for all $F, J, L \in \Gamma(TM)$.

By replacing F, J and L by PF, Z and PL, respectively, in (25) and (4) we get

$$\begin{aligned} g(R^1(PF,Z)PL,Z) = \frac{c}{4}\{&g(\phi Z, PL)g(Z,V) - g(\phi Z, PL)g(PF,V) \\ &- 2g(\phi PF, Z)g(PL,V)\}, \end{aligned}$$

$$g(R^1(PF,Z)PL,Z) = \frac{-3c}{4}\{u(PF)u(PL)\}, \tag{26}$$

and

$$\begin{aligned} g(R^1(PF,Z)PL,Z) = &-B(D_{PF}Z, PL) - ZB(PF,PL) \\ &+ B(D_Z PF, PL) + B(PF, D_Z PL) \\ &- \tau(Z)B(PF,PL). \end{aligned} \tag{27}$$

From equations (26) and (27), and then using equation (18), the Gauss and Weingarten formulas in the resulting equation we have

$$\frac{-3c}{4}\{u(PF)u(PL) = \{\lambda^2 - Z(\lambda) - \lambda(\tau Z)\}\bar{g}(\phi F, PL). \tag{28}$$

By setting $F = L = Y \in S(TM)$, we get $PF = PL = Y, u(Y) = 1$ and $g(Y,Y) = 0$.

From (28) we obtain

$$c = 0.$$

So $(M, \bar{g}, \xi, \eta, \bar{\phi})$ is flat. ∎

Lemma 3.7. *Let $(M, \bar{g}, \xi, \eta, \bar{\phi})$ be a SCI-LH of cosymplectic space form $\bar{M}(c)$ tangent to the structural vector field ξ. Then*

$$\begin{aligned} R(F,J)L = \frac{c}{4}\{&g(\phi J,\phi L)F - g(\phi F,\phi L)J \\ &+ \eta(J)g(F,L)\xi - \eta(F)g(J,L)\xi + g(\phi J,L)\bar{\phi}F \\ &- g(\phi F,L)\bar{\phi}J + 2g(F,\phi J)\bar{\phi}L\} \\ &- B(F,L)A_X J + B(J,L)A_X F, \end{aligned} \tag{29}$$

and

$$\begin{aligned} (D_F h)(J,L) - (D_J h)(F,L) = \frac{c}{4}\{&g(\phi F,L)u(J) - g(\phi J,L)u(F) \\ &- 2g(F,\phi J)u(L)\}X \end{aligned} \tag{30}$$

are the Gauss and Codazzi formulas, respectively, for any $F, J, L \in \Gamma(TM)$.

Proof. Since $(M, \bar{g}, \xi, \eta, \bar{\phi})$ is a SCI-LH of $\bar{M}(c)$, using (4) and (24), we derive

$$\begin{aligned} R(F,J)L = \frac{c}{4}\{&g(\phi J,\phi L)F - g(\phi F,\phi L)J \\ &+ \eta(J)g(F,L)\xi - \eta(F)g(J,L)\xi + g(\phi J,L)\phi F \\ &- g(\phi F,L)\phi J + 2g(F,\phi J)\phi L\} - A_{h(F,L)}J + A_{h(J,L)}F \\ &- (D_F h)(J,L) + (D_J h)(F,L). \end{aligned} \tag{31}$$

Using equation (13) in (31) we obtain

$$\begin{aligned} R(F,J)L = \frac{c}{4}\{&g(\phi J,\phi L)F - g(\phi F,\phi L)J + \eta(J)g(F,L)\xi \\ &- \eta(F)g(J,L)\xi + g(\phi J,L)\bar{\phi}F - g(\phi J,L)u(F)X \\ &- g(\phi F,L)\bar{\phi}J + g(\phi F,L)u(J)X + 2g(F,\phi J)\bar{\phi}L \\ &- 2g(F,\phi J)u(L)X\} - A_{h(F,L)}J + A_{h(J,L)}F \\ &- (D_F h)(J,L) + (D_J h)(F,L). \end{aligned} \tag{32}$$

Hence we derive (29) and (30) correspondingly by comparing the tangential and transversal vector bundle components of equation (32). □

Lemma 3.8. *Let $(M, \bar{g}, \xi, \eta, \bar{\phi})$ be a SCI-LH of cosymplectic space form $\bar{M}(c)$, tangent to the structural vector field ξ. Then for any F,J,L $\in \Gamma(TM)$*

$$g(R^1(F,Z)L,X) = \frac{-c}{4}\{g(\phi F,\phi L) - u(L)\theta(\phi F) - 2u(F)\theta(\phi L)\}.$$

Proof. By putting Z for J in equation (12) and then taking inner product of resulting equation with X we obtain our result. □

Theorem 3.9. *There does not exist any SCI-LH $(M, \bar{g}, \xi, \eta, \bar{\phi})$ of cosymplectic space form $\bar{M}(c)$ if h is parallel and $c \neq 0$.*

Proof. Suppose there exists a SCI-LH $(M, \bar{g}, \xi, \eta, \bar{\phi})$ of cosymplectic space form $\bar{M}(c)$ with $c \neq 0$ and h to be parallel. By putting $J = Z$ and $L = \phi X$ in (30) we get

$$\frac{c}{4} u(F)X = 0.$$

By replacing F by ϕX we get

$$c = 0,$$

which is contradicting our supposition. □

Theorem 3.10. *In a SCI-LH $(M, \bar{g}, \xi, \eta, \bar{\phi})$ if $(c \neq 0)$ and screen distribution is parallel, then there does not exist any SCI-LH $(M, \bar{g}, \xi, \eta, \bar{\phi})$ of $\bar{M}(c)$.*

Proof. Assume $c \neq 0$ and screen distribution is parallel, then from equation (4), by replacing F, J, and L by $Z, \phi X$ and ϕZ we get

$$g(R^1(Z, \phi X)\phi Z, X) = \frac{c}{4}. \tag{33}$$

From equation (10) we get

$$\begin{aligned} g(R^1(Z, \phi X)\phi Z, X) = {} & (D_E C)(\phi X, P\phi Z) - (D_{\phi X} C)(\phi Z, P\phi Z) \\ & + \tau(\phi X)C(Z, P\phi Z) - \tau(Z)C(\phi X, P\phi Z). \end{aligned}$$

As assumed $S(TM)$ is parallel we get

$$g(R^1(Z, \phi X)\phi Z, X) = 0. \tag{34}$$

□

From equations (33) and (34), we obtain $c = 0$. Which is a contradiction, hence there does not any SCI-LH $(M, \bar{g}, \xi, \eta, \bar{\phi})$ of $\bar{M}(c)$ with $c \neq 0$ and parallel screen distribution.

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