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CONFORMAL HEMI-SLANT SUBMERSIONS FROM COSYMPLECTIC MANIFOLDS

Vinay Kumar, Rajendra Prasad, and Sandeep Kumar Verma

Abstract. The main goal of the paper is the introduction of the notion of conformal hemi-slant submersions from almost contact metric manifolds onto Riemannian manifolds. It is a generalization of conformal anti-invariant submersions, conformal semi-invariant submersions and conformal slant submersions. Our main focus is conformal hemi-slant submersion from cosymplectic manifolds. We tend also study the integrability of the distributions involved in the definition of the submersions and the geometry of their leaves. Moreover, we get necessary and sufficient conditions for these submersions to be totally geodesic, and provide some representative examples of conformal hemi-slant submersions.

1. Introduction

The theory of smooth maps between Riemannian manifolds has been widely studied in Riemannian geometry. These maps are useful for comparing geometric structures between manifolds. The study of Riemannian submersions between Riemannian manifolds was initiated by O'Neil [16] and Grey [8]. Riemannian submersions are very important and interesting maps comparing with conformal submersions. Riemannian submersions between Riemannian manifold equipped with differentiable structures was studied by B. Watson [23]. Watson also showed that the base manifold and each fiber have the same kind of structure as the total space. D. Chinea [6,7] extend the notion of almost hermitian submersions to different subclasses of almost contact metric manifold. Gudmundsson and Wood [9, 10] introduced conformal holomorphic submersions.

M. A. Akyol [1,3–5] introduced the concept of conformal semi-slant submersions from almost hermitian manifolds onto Riemannian manifolds. There are several kinds of Riemannian submersions like, slant submersion, anti-invariant submersion, contact complex submersion, quaternionic submersions, H-slant

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submersion, semi-invariant submersions, para contact semi Riemannian submersion, etc. B. Sahin [19–21] define the notion of anti-invariant Riemannian submersions from almost hermitian manifolds. Later such submersions were considered between manifolds with differentiable structures. K. S. Park and R. Prasad [17] introduced semi slant submersions from almost hermitian manifolds onto Riemannian manifolds. C. Sayar, M. A. Akyol and R. Prasad [22] studied Bi-slant submersion in complex manifolds. Y. Gunduzalp [2, 11–13] studied semi slant submersions from almost product Riemannian manifolds. They showed that such submersions have rich geometric properties and they are useful for investigating the geometry of total space. We know that a semi slant submersion is the generalized version of a slant submersions. R. Prasad and S. Kumar [18] define and studied the conformal semi-invariant submersions from almost contact metric manifolds onto Riemannian manifolds. Conformal hemi-slant and conformal semi slant submersion in different manifolds were studied in [14, 15]. The conformal maps do not preserve the distance between points contrary to isometrics but they preserves angle between vector fields. This property allows one to transfer specific properties of manifolds to another manifolds by deforming such properties.

In the present paper, we study conformal hemi-slant submersions from cosymplectic manifolds onto Riemannian manifolds. This paper is divided into the following sections: In the second section, we provide main notions and formulae for other sections. In the third section, we give the definition of conformal hemi-slant submersions and some useful results. We tend also study the integrability of the distributions involved in the definition of the submersions and the geometry of their leaves. Finally, we provide some representative examples of conformal hemi-slant submersions.

2. Preliminaries

In this section, we recall main definitions and properties of cosymplectic manifolds. We also give definition of conformal hemi-slant submersions which we have used throughout the paper.

We consider M_1 is a $(2n+1)$-dimensional almost contact manifold which carries a tensor field ϕ of the tangent space, 1-form η and characteristic vector field ξ satisfying

$$\begin{aligned} &\phi^2 = -I + \eta \otimes \xi,\ \eta(\xi) = 1, \\ &\phi\xi = 0,\ \eta \circ \phi = 0, \end{aligned} \tag{1}$$

where $I : TM_1 \longrightarrow TM_1$ is the identity map.

Since any almost contact manifold (M_1, ϕ, ξ, η) admits a Riemannian metric g such that

$$g(\phi X_1, \phi X_2) = g(X_1, X_2) - \eta(X_1)\eta(X_2) \tag{2}$$

for any vector fields $X_1, X_2 \in \Gamma(TM_1)$, where $\Gamma(TM_1)$ represents the Lie algebra of vector fields on M_1. The manifold M_1 together with the structure (ϕ, ξ, η, g) is called an almost contact metric manifold.

The immediate consequence of (2), we have

$$\eta(X_1) = g(X_1, \xi) \quad \text{and} \quad g(\phi X_1, X_2) + g(X_1, \phi X_2) = 0$$

for all vector fields $X_1, X_2 \in \Gamma(TM_1)$.

An almost contact structure (ϕ, ξ, η) is said to be normal if the almost complex structure ϕ on the product manifold $M_1 \times R$ is given by

$$\phi(U,\ f\frac{d}{dt}) = (\phi U - f\xi,\ \eta(U)\frac{d}{dt}),$$

where $\phi^2 = -I$ and f is the differentiable function on $M_1 \times R$ has no torsion i.e., ϕ is integrable. The condition for normality in terms of ϕ ξ and η is $[\phi, \phi] + 2d\eta \otimes \xi = 0$ on M_1, where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ. Now, the fundamental 2-form is defined by $\Phi(X_1, X_2) = g(X_1, \phi X_2)$.

An almost contact metric manifold is said to be a cosymplectic manifold if it is normal and both Φ and η are closed. The structure equation of a cosymplectic manifold is given by

$$(\nabla_{X_1}\phi)X_2 = 0$$

for all vector fields $X_1, X_2 \in \Gamma(TM_1)$, where ∇ represents the Levi-Civita connection of (M_1, g). Moreover, for a cosymplectic manifold, we have

$$\nabla_{X_1}\xi = 0$$

for every vector field $X_1 \in \Gamma(TM_1)$.

Example 2.1. We consider R^{2k+1} with Cartesian coordinates (x_i, y_i, z), where $(i = 1, \dots, k)$ and its usual contact form $\eta = dz$.

The characteristic vector field ξ is given by $\frac{\partial}{\partial z}$, and its Riemannian metric g and tensor field ϕ are given by

$$g = \sum_{i=1}^{k}((dx_i)^2 + (dy_i)^2) + (dz)^2,\ \phi = \begin{bmatrix} 0 & \delta_{ij} & 0 \\ -\delta_{ij} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ i = 1, \dots, k.$$

This gives a cosymplectic structure on R^{2k+1}. The vector fields $E_i = \frac{\partial}{\partial y_i}$, $E_{k+i} = \frac{\partial}{\partial x_i}$, $\xi = \frac{\partial}{\partial z}$ form a ϕ-basis for the cosymplectic structure. On the other hand, it can be shown that $(R^{2k+1}, \phi, \xi, \eta, g)$ is a cosymplectic manifold.

Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds of dimension m and n, respectively, where g_1 and g_2 are the Riemannian metrics on M_1 and M_2. Let $f : (M_1, g_1) \to (M_2, g_2)$ be a differentiable map. We call the map f a differentiable submersion if f is surjective and the differential $(f_*)_p$ has a maximal rank for any $p \in M_1$. The map f is said to be a Riemannian submersion if f is a differentiable submersion and $(f_*)_p : ((\ker(f_*)_p)^\perp, (g_1)_p) \to (T_{f(p)}M_1, (g_2)_{f(p)})$ is a linear isometry for each $p \in M_1$, where $(\ker(f_*)_p)^\perp$ is the orthogonal complement of the space $\ker(f_*)_p$ in the tangent space $T_P M_1$ of M_1 at p.

Now, we recall following definitions for later use:

Definition 1. Let π be a Riemannian submersion from an almost contact metric manifold $(M, \phi, \xi, \eta, g_m)$ onto a Riemannian manifold (N, g_n). Then we say that π is an invariant Riemannian submersion if the vertical distribution is invariant with respect to the complex structure π, i.e.,

$$\phi(\ker \pi_*) = \ker \pi_*.$$

Definition 2. Let $(M, \phi, \xi, \eta, g_m)$ be an almost contact metric manifold and (N, g_n) a Riemannian manifold. Let $\pi : M \to N$ be a smooth submersion. Then π is called a horizontally conformal submersion if there is a positive function λ such that

$$g_m(X, Y) = \frac{1}{\lambda^2} g_n(\pi_* X, \pi_* Y)$$

for every $X, Y \in \Gamma(\ker \pi_*)^\perp$. It is obvious that every Riemannian submersion is a particular horizontally conformal submersion with $\lambda = I$.

Definition 3. Let $(M, \phi, \xi, \eta, g_m)$ be an almost contact metric manifold and (N, g_n) a Riemannian manifold. A horizontal conformal submersion

$$\pi : (M, \phi, \xi, \eta, g_m) \to (N, g_n)$$

is called a conformal hemi-slant submersion if the vertical distribution $\ker \pi_*$ of π admits three orthogonal complementary distributions $\mathcal{D}^\theta$, $\mathcal{D}^\perp$ and ξ such that $\mathcal{D}^\theta$ is slant with angle θ and $\mathcal{D}^\perp$ is anti-invariant, i.e.,

$$\ker \pi_* = \mathcal{D}^\theta \oplus \mathcal{D}^\perp \oplus \langle \xi \rangle. \tag{3}$$

In this case, the angle θ is called the hemi-slant angle of the submersion.

It is known that the distribution $\ker \pi_*$ is integrable. Hence above definition implies that the integral manifold (fiber) $\pi^{-1}(q)$, $q \in N$ of $\ker \pi_*$ is a hemi-slant submanifold.

Let π be a conformal hemi-slant submersion from an almost contact metric manifold $(M, \phi, \xi, \eta, g_m)$ onto a Riemannian manifold (N, g_n). We observe that the notion of conformal hemi-slant submersion is a natural generalization of both the notions of anti-invariant, semi-invariant submersion [17] and conformal slant submersion [13]. More precisely if we denote the dimension of $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$ by m_1 and m_2, respectively, then we have the following:

(a) If $m_2 = 0$, then M is an conformal anti-invariant submersion.
(b) If $m_1 = 0$ and $\theta = 0$, then M is an conformal invariant submersion.
(c) If $m_1 = 0$ and $\theta \neq \frac{\pi}{2}$, then M is a proper conformal slant submersion with slant angle θ.
(d) If $\theta = \frac{\pi}{2}$, then M is a conformal anti-invariant submersion.

We say that the conformal hemi-slant submersion $\pi : (M, \phi, \xi, \eta, g_m) \to (N, g_n)$ is proper if $\mathcal{D}^\perp \neq 0$ and $\theta \neq 0, \frac{\pi}{2}$.

Lemma 2.2. *Let $(M, \phi, \xi, \eta, g_m)$ be an m-dimensional almost contact metric manifold and (N, g_n) be an n-dimensional Riemannian manifold. Let $\pi : M \to N$ be a differentiable map between them and $p \in M$. Then π is called horizontally weakly conformal or semi-conformal at p if either $(\pi_*)_p = 0$, or $(\pi_*)_p$ maps the horizontal space $\mathcal{H} = ((\ker \pi_*)_p)^\perp$ conformally onto $T_{f(p)}$.*

The second condition in the above definition exactly is the same as $(\pi_*)_p$ is symmetric and there exists a number $\chi(p) \neq 0$ such that

$$g_n(\pi_* X, \pi_* Y) = \chi(p) g_m(X, Y) \quad \text{for all } X, Y \in ((\ker \pi_*)_p)^\perp. \tag{4}$$

Here $\chi(p)$ is called the square dilation of π at p and its square root $\lambda(p) = \sqrt{\chi(p)}$ is called the dilation of π at p. The map π is called horizontally weakly conformal or semi-conformal on M if it is horizontally weakly conformal at every point on M. If π has no critical point, then it is said to be a (horizontally) conformal submersion.

We should mention that a horizontally conformally submersion $\pi : M \to N$ is called horizontally homothetic if the gradient of its dilation λ is vertical, i.e.,

$$\mathcal{H}(grad\lambda) = 0 \tag{5}$$

at $p \in M$, where $\mathcal{H}$ is the complement orthogonal distribution to $\mathcal{V} = \ker \pi_*$ in $\Gamma(T_p M)$.

Again, we recall the following definition.

Let $\pi : M \to N$ be a conformal submersion. A vector field E on M is called projectiable if there exists a vector field $\widehat{E}$ on N such that $\pi_*(E_p) = \widehat{E}_{f(p)}$ for any $p \in M$. In this case E and $\widehat{E}$ are called π-related. A horizontal vector field Y on M is called basic if it is projectiable. It is a well known fact that if $\widehat{Z}$ is a vector field on N, then there exists a unique basic vector field Z which is called the horizontal lift of $\widehat{Z}$.

The fundamental tensors $\mathcal{T}$ and $\mathcal{A}$ defined by O'Neill [16] for vector fields E and F on M such that

$$\mathcal{A}_E F = \mathcal{H}\nabla^M_{\mathcal{H}E} \mathcal{V}F + \mathcal{V}\nabla^M_{\mathcal{H}E} \mathcal{H}F, \tag{6}$$

$$\mathcal{T}_E F = \mathcal{H}\nabla^M_{\mathcal{V}E} \mathcal{V}F + \mathcal{V}\nabla^M_{\mathcal{V}E} \mathcal{H}F, \tag{7}$$

where $\mathcal{V}$ and $\mathcal{H}$ are the vertical and horizontal projections. On the other hand, from equations (3) and (4), we have

$$\begin{aligned} \nabla_U V &= \mathcal{T}_U V + \widehat{\nabla}_U V, \\ \nabla_U X &= \mathcal{H}\nabla_U X + \mathcal{T}_U X, \\ \nabla_X U &= \mathcal{A}_X U + \mathcal{V}\nabla_X U, \\ \nabla_X Y &= \mathcal{H}\nabla_X Y + \mathcal{A}_X Y \end{aligned} \tag{8}$$

for all $U, V \in \Gamma(\ker \pi_*)$ and $X, Y \in \Gamma(\ker \pi_*)^\perp$, where $\mathcal{V}\nabla_U V = \widehat{\nabla}_U V$. If X is basic, then $\mathcal{A}_X V = \mathcal{H}\nabla_X V$.

It is easily seen that for $p \in M$, $V \in \mathcal{V}_p$ and $X \in \mathcal{H}_p$ the linear operators

$$\mathcal{T}_V, \mathcal{A}_X : T_pM \to T_pM$$

are skew-symmetric, that is

$$g(\mathcal{A}_X E, F) = -g(E, \mathcal{A}_X F) \text{ and } g(\mathcal{T}_V E, F) = -g(E, \mathcal{T}_V F)$$

for all $E, F \in T_pM$. We also see that the restriction of $\mathcal{T}$ to the vertical distribution $\mathcal{T}$ is the second fundamental form of the fibres of π. Since $\mathcal{T}_{\mathcal{V}}$ is skew-symmetric, we get π has totally geodesic fibres if and only if $\mathcal{T} = 0$.

Let $(M, \phi, \xi, \eta, g_m)$ be an almost contact metric manifold and (N, g_n) be a Riemannian manifold. Let $\pi : M \to N$ be a smooth map. Then the second fundamental form of π is given by

$$(\nabla \pi_*)(X, Y) = \nabla^{\pi}_X \pi_* Y - \pi_*(\nabla_X Y) \text{ for all } X, Y \in \Gamma(T_pM),$$

where we denote conveniently by ∇ the Levi-Civita connections of the metrics g_m and g_n and ∇^f is the pullback connection. We also know that π is said to be totally geodesic map if $(\nabla \pi_*)(X, Y) = 0$ for all $X, Y \in \Gamma(TM)$.

Lemma 2.3. *Let $\pi : M \to N$ be a horizontal conformal submersion. Then, for any horizontal vector fields X, Y and vertical vector fields U, V, we have*
(i) $(\nabla \pi_*)(X, Y) = X(\ln \lambda)\pi_*(Y) + Y(\ln \lambda)\pi_*(X) - g_m(X, Y)\pi_*(grad \ln \lambda)$,
(ii) $(\nabla \pi_*)(U, V) = -\pi_*(\mathcal{T}_U V)$,
(iii) $(\nabla \pi_*)(X, U) = -\pi_*(\nabla^M_X U) = -\pi_*(\mathcal{A}_X U)$.

3. Conformal hemi-slant submersions

In this section, we define and study conformal hemi-slant submersions from almost contact metric manifolds.

Let π be a conformal hemi-slant from an almost contact metric manifold $(M, \phi, \xi, \eta, g_m)$ onto a Riemannian manifold (N, g_n). Then, we have

$$TM = (\ker \pi_*) \oplus (\ker \pi_*)^{\perp}.$$

For any $X \in \Gamma(\ker \pi_*)$, we put

$$X = PX + QX + \eta(X)\xi, \tag{9}$$

where $PX \in \Gamma(\mathcal{D}^{\theta})$ and $QX \in \Gamma(\mathcal{D}^{\perp})$.

For all $X \in \Gamma(\ker \pi_*)$, we have

$$\phi X = \psi X + \omega X, \tag{10}$$

where ψX and ωX are vertical and horizontal components of ϕX, respectively.

Also for $V \in \Gamma(\ker \pi_*)^{\perp}$, we have

$$\phi V = BV + CV, \tag{11}$$

where $BV \in \Gamma(\ker \pi_*)$ and $CV \in \Gamma(\mu)$. Then, the horizontal distribution $\Gamma(\ker \pi_*)^{\perp}$ decomposed as

$$\Gamma(\ker \pi_*)^{\perp} = \omega \mathcal{D}^{\theta} \oplus \phi \mathcal{D}^{\perp} \oplus \mu,$$

where μ is the orthogonal complement of $\omega\mathcal{D}^\theta \oplus \phi\mathcal{D}^\perp$ in $\Gamma(\ker\pi_*)^\perp$ and it is invariant with respect to ϕ.

Lemma 3.1. *Let f be a conformal hemi-slant submersion from an almost contact metric manifold $(M, \phi, \xi, \eta, g_m)$ onto a Riemannian manifold (N, g_n). Then we have*

$$\text{(a)}\ \ \psi\mathcal{D}^\theta = \mathcal{D}^\theta, \qquad \text{(b)}\ \ \psi\mathcal{D}^\perp = \{0\},$$
$$\text{(c)}\ \ B\omega\mathcal{D}^\theta = \mathcal{D}^\theta, \qquad \text{(d)}\ \ B\phi\mathcal{D}^\perp = \mathcal{D}^\perp.$$

Lemma 3.2. *Let $(M, \phi, \xi, \eta, g_m)$ be an almost contact metric manifold and (N, g_n) be a Riemannian manifold. If $\pi : (M, \phi, \xi, \eta, g_m) \to (N, g_n)$ is a conformal hemi-slant submersion, then*

$$\psi^2 X + B\omega X = -X + \eta(X) \otimes \xi, \qquad \omega\psi X + C\omega X = 0,$$
$$\psi BZ + BCZ = 0, \qquad \omega BZ + C^2 Z = -Z$$

for all $V \in \Gamma(\ker\pi_*)$ and $Z \in \Gamma(\ker\pi_*)^\perp$.

We define the covariant derivatives of ψ and ω as follows:

$$(\nabla_V \psi)W = \widehat{\nabla}_V \psi W - \psi\widehat{\nabla}_V W,$$
$$(\nabla_V \omega)W = \mathcal{H}\nabla_V \omega W - \omega\widehat{\nabla}_V W$$

for all $V, W \in \Gamma(\ker\pi_*)$, where $\widehat{\nabla}_V W = \mathcal{V}\widehat{\nabla}_V W$.

Lemma 3.3. *Let $(M, \phi, \xi, \eta, g_m)$ be a cosymplectic manifold and (N, g_n) be a Riemannian manifold. If $\pi : (M, \phi, \xi, \eta, g_m) \to (N, g_n)$ is a conformal hemi-slant submersion, then*

(i)
$$\mathcal{V}\nabla_X \psi Y + \mathcal{T}_X \omega Y = B\mathcal{T}_X Y + \psi\mathcal{V}\nabla_X Y,$$
$$\mathcal{T}_X \psi Y + \mathcal{H}\nabla_X \omega Y = C\mathcal{T}_X Y + \omega\mathcal{V}\nabla_X Y$$

for all $X, Y \in \Gamma(\ker\pi_*)$.

(ii)
$$\mathcal{T}_X BV + \mathcal{H}\nabla_X CV = C\mathcal{H}\nabla_X V + \omega\mathcal{T}_X V,$$
$$\mathcal{V}\nabla_X BV + \mathcal{T}_X CV = B\mathcal{H}\nabla_X V + \psi\mathcal{T}_X V$$

for all $X \in \Gamma(\ker\pi_*)$ and $V \in \Gamma(\ker\pi_*)^\perp$.

(iii)
$$\mathcal{V}\nabla_V \psi X + \mathcal{A}_V \omega X = B\mathcal{A}_V X + \psi\mathcal{V}\nabla_V X,$$
$$\mathcal{A}_V \psi X + \mathcal{H}\nabla_V \omega X = C\mathcal{A}_V X + \omega\mathcal{V}\nabla_V X$$

for all $X \in \Gamma(\ker\pi_*)$ and $V \in \Gamma(\ker\pi_*)^\perp$.

(iv)
$$\mathcal{A}_U BV + \mathcal{H}\nabla_U CV = C\mathcal{H}\nabla_U V + \omega\mathcal{A}_U V,$$
$$\mathcal{V}\nabla_U BV + \mathcal{A}_U CV = B\mathcal{H}\nabla_U V + \psi\mathcal{A}_U V$$

for all $U, V \in \Gamma(\ker\pi_*)^\perp$.

Now, we define

$$(\nabla_X\psi)Y = \mathcal{V}\nabla_X\psi Y - \psi\mathcal{V}\nabla_X Y,$$
$$(\nabla_X\omega)Y = \mathcal{H}\nabla_X\omega Y - \omega\mathcal{V}\nabla_X Y,$$
$$(\nabla_V B)W = \mathcal{V}\nabla_V BW - B\mathcal{H}\nabla_V W,$$
$$(\nabla_V C)W = \mathcal{H}\nabla_V BW - C\mathcal{H}\nabla_V W$$

for all $X, Y \in \Gamma(\ker\pi_*)$ and $V, W \in \Gamma(\ker\pi_*)^\perp$.

Lemma 3.4. *Let $(M, \phi, \xi, \eta, g_m)$ be a cosymplectic manifold and (N, g_n) be a Riemannian manifold. If $\pi : (M, \phi, \xi, \eta, g_m) \to (N, g_n)$ is a conformal hemi-slant submersion, then*

$$(\nabla_X\psi)Y = B\mathcal{T}_X Y - \mathcal{T}_X\omega Y,$$
$$(\nabla_X\omega)Y = C\mathcal{T}_X Y - \mathcal{T}_X\psi Y,$$
$$(\nabla_V B)W = \omega\mathcal{A}_V W - \mathcal{A}_V BW,$$
$$(\nabla_V C)W = \psi\mathcal{A}_V W - \mathcal{A}_V CW$$

for all $X, Y \in \Gamma(\ker\pi_*)$ and $V, W \in \Gamma(\ker\pi_*)^\perp$.

If the tensors ψ and ω are parallel with respect to the linear connection ∇ on M, respectively, then

$$B\mathcal{T}_X Y = \mathcal{T}_X\omega Y,$$
$$C\mathcal{T}_X Y = \mathcal{T}_X\psi Y$$

for all $X, Y \in \Gamma(\ker\pi_*)$.

Lemma 3.5. *Let $\pi : (M, \phi, \xi, \eta, g_m) \to (N, g_n)$ be a conformal hemi-slant submersion from an almost contact metric manifold $(M, \phi, \xi, \eta, g_m)$ onto a Riemannian manifold (N, g_n). Then π is a proper conformal hemi-slant submersion if and only if there exists a constant $\lambda \in [0, 1]$ such that*

$$\psi^2 X = -(\cos^2\theta)X \textit{ for all } X \in \Gamma(\mathcal{D}^\theta).$$

Proof. For any non-zero vector field $X \in \Gamma(\mathcal{D}^\theta)$, we have

$$\cos\theta = \frac{\| \psi X \|}{\| \phi X \|},$$

and

$$\cos\theta = \frac{g_m(\phi X, \psi X)}{\| \psi X \| \| \phi X \|}, \tag{12}$$

where $\theta(X)$ is the hemi-slant angle.

Using equations (2) and (10), we get

$$\cos\theta = -\frac{g_m(X, \psi^2 X)}{\| \psi X \| \| \phi X \|} \tag{13}$$

From equations (12) and (13), we have

$$\psi^2 X = -(\cos^2\theta)X. \qquad \square$$

From Lemma 3.6, equations (1) and (2), we easily proof.

Lemma 3.6. *Let $\pi : (M,\phi,\xi,\eta,g_m) \to (N,g_n)$ be a conformal hemi-slant submersion from an almost contact metric manifold (M,ϕ,ξ,η,g_m) onto a Riemannian manifold (N,g_n). Then*

$$g_m(\psi X,\psi Y) = \cos^2\theta g_m(X,Y),$$
$$g_m(\omega X,\omega Y) = \sin^2\theta g_m(X,Y)$$

for all $X, Y \in \Gamma(\mathcal{D}^\theta)$.

Lemma 3.7. *Let $\pi : (M,\phi,\xi,\eta,g_m) \to (N,g_n)$ be a conformal hemi-slant submersion from a cosymplectic manifold (M,ϕ,ξ,η,g_m) onto a Riemannian manifold (N,g_n) with the slant angle $\theta \in [0,\frac{\pi}{2}]$. If ω is parallel with respect to ∇ on $\mathcal{D}^\theta$, then we have*

$$\mathcal{T}_{\psi X}\psi X = -\cos^2\theta\cdot\mathcal{T}_X X \quad \textit{for } X \in \Gamma(\mathcal{D}^\theta).$$

Proof. If ω is parallel ($(\nabla_X\omega)Y = \mathcal{H}\nabla_X\omega Y - \omega\mathcal{V}\nabla_X Y = 0$), then from Lemma 3.6, we have

$$\begin{aligned} &\mathcal{H}\nabla_X\omega Y - \omega\mathcal{V}\nabla_X Y = C\mathcal{T}_X Y - \mathcal{T}_X\psi Y,\\ &C\mathcal{T}_X Y = \mathcal{T}_X\psi Y \quad \text{for all } X,Y \in \Gamma(\mathcal{D}^\theta). \end{aligned} \tag{14}$$

Interchanging the role of X and Y in equation (14), we have

$$C\mathcal{T}_Y X = \mathcal{T}_Y\psi X \quad \text{for all } X,Y \in \Gamma(\mathcal{D}^\theta). \tag{15}$$

Since $\mathcal{T}$ is symmetric, from equations (14) and (15), we get

$$\mathcal{T}_{\psi X}\psi X = \cos^2\theta\cdot\mathcal{T}_X X \quad \text{for all } X \in \Gamma(\mathcal{D}^\theta). \qquad \square$$

Lemma 3.8. *Let $\pi : (M,\phi,\xi,\eta,g_m) \to (N,g_n)$ be a conformal hemi-slant submersion from a cosymplectic manifold (M,ϕ,ξ,η,g_m) onto a Riemannian manifold (N,g_n). Then, we have*

(i) $g_m(\nabla_X Y,\xi) = 0$,
(ii) $g_m([X,Y],\xi) = 0$,

where $X, Y \in \Gamma(\mathcal{D}^\theta \oplus \mathcal{D}^\perp)$.

Theorem 3.9. *Let π be a conformal hemi-slant submersion from a cosymplectic manifold (M,ϕ,ξ,η,g_m) onto a Riemannian manifold (N,g_n). Then the slant distribution $\mathcal{D}^\theta$ is integrable if and only if*

$$\begin{aligned} &g_m(\mathcal{T}_X\omega\psi Y,V) - g_m(\mathcal{T}_Y\omega\psi X,V)\\ &= \frac{1}{\lambda^2}g_n((\nabla\pi_*)(Y,\omega X) - (\nabla\pi_*)(X,\omega Y),\pi_*(\phi V))\\ &\quad + \frac{1}{\lambda^2}g_n(\nabla^\pi_X\pi_*(\omega Y) - \nabla^\pi_Y\pi_*(\omega X),\pi_*(\phi V)) \end{aligned}$$

for all $X, Y \in \Gamma(\mathcal{D}^\theta)$ and $V \in \Gamma(\mathcal{D}^\perp)$.

Proof. We note that $\mathcal{D}^\theta$ is integrable if and only if

$$g_m([X,Y],V) = 0,\ g_m([X,Y],W) = 0 \text{ and } g_m([X,Y],\xi) = 0$$

for all $X, Y \in \Gamma(\mathcal{D}^\theta)$, $V \in \Gamma(\mathcal{D}^\perp)$ and $W \in (\ker \pi_*)^\perp$. Since $\ker \pi_*$ is integrable then, $g_m([X,Y],W) = 0$ and from Lemma 3.8, $g_m([X,Y],\xi) = 0$. Thus, $\mathcal{D}^\theta$ is integrable if and only if $g_m([X,Y],V) = 0$.

From equations (2), (10) and Lemma 3.6, we have

$$\begin{aligned}
&g_m([X,Y],V)\\
&= g_m(\nabla_X Y, V) - g_m(\nabla_Y X, V)\\
&= g_m(\nabla_X \psi Y, \phi V) + g_m(\nabla_X \omega Y, \phi V) - g_m(\nabla_Y \psi X, \phi V) - g_m(\nabla_Y \omega X, \pi V)\\
&= \cos^2\theta g_m([X,Y],V) - g_m(\nabla_X \omega\psi Y, V) + g_m(\nabla_Y \omega\psi X, V)\\
&\quad + g_m(\nabla_X \omega Y, \phi V) - g_m(\nabla_Y \omega X, \phi V).
\end{aligned}$$

Next, using equation (6), we have

$$\begin{aligned}
\sin^2\theta g_m([X,Y],V) = &- g_m(\mathcal{T}_X \omega\psi Y, V) + g_m(\mathcal{T}_Y \omega\psi X, V) + g_m(\mathcal{H}\nabla_X \omega Y, \phi V)\\
&- g_m(\mathcal{H}\nabla_Y \omega X, \phi V).
\end{aligned}$$

Since π is a conformal submersion, using Lemma 3.6 and (2), we have

$$\begin{aligned}
\sin^2\theta g_m([X,Y],V) = &- g_m(\mathcal{T}_X \omega\psi Y, V) + g_m(\mathcal{T}_Y \omega\psi X, V)\\
&+ \frac{1}{\lambda^2} g_n((\nabla\pi_*)(Y,\omega X) - (\nabla\pi_*)(X,\omega Y), \pi_*(\phi V))\\
&+ \frac{1}{\lambda^2} g_n(\nabla^\pi_X \pi_*(\omega Y) - \nabla^\pi_Y \pi_*(\omega X), \pi_*(\phi V)).
\end{aligned}$$

□

Theorem 3.10. *Let π be a conformal hemi-slant submersion from a cosymplectic manifold $(M, \phi, \xi, \eta, g_m)$ onto a Riemannian manifold (N, g_n). Then the anti-invariant distribution $\mathcal{D}^\perp$ is always integrable.*

Theorem 3.11. *Let π be a conformal hemi-slant submersion from a cosymplectic manifold $(M, \phi, \xi, \eta, g_m)$ onto a Riemannian manifold (N, g_n). Then the distribution $(\ker \pi_*)^\perp$ is integrable if and only if*

$$\begin{aligned}
&\frac{1}{\lambda^2} g_n(\nabla^\pi_Y \pi_*(CX) - \nabla^\pi_X \pi_*(CY), \pi_*(\omega Z))\\
&= g_m(\mathcal{A}_Y BX - \mathcal{A}_X BY - CY(\ln\lambda)X + CX(\ln\lambda)Y + 2g_m(X, CY) grad \ln\lambda, \omega Z)\\
&\quad + g_m(\mathcal{V}\nabla_Y BX - \mathcal{V}\nabla_X BY + \mathcal{A}_Y CX - \mathcal{A}_X CY, \psi Z)
\end{aligned}$$

for all $Z \in \Gamma(\ker \pi_*)$ and $X, Y \in \Gamma(\ker \pi_*)^\perp$.

Theorem 3.12. *Let π be a conformal hemi-slant submersion from a cosymplectic manifold $(M, \phi, \xi, \eta, g_m)$ onto a Riemannian manifold (N, g_n). Then any two conditions below imply the third*:

(i) $(\ker \pi_*)^\perp$ *is integrable,*

(ii) π *is a horizontally homothetic map,*

(iii) $\frac{1}{\lambda^2}g_n(\nabla^{\pi}_{Y}\pi_*(CX)-\nabla^{\pi}_{X}\pi_*(CY),\pi_*(\omega Z))$
$= g_m(\mathcal{A}_Y BX - \mathcal{A}_X BY, \omega Z)$
$+ g_m(\mathcal{V}\nabla_Y BX - \mathcal{V}\nabla_X BY + \mathcal{A}_Y CX - \mathcal{A}_X CY, \psi Z)$

for all $Z \in \Gamma(\ker\pi_*)$ *and* $X, Y \in \Gamma(\ker\pi_*)^{\perp}$.

Theorem 3.13. *Let* π *be a conformal hemi-slant submersion from a cosymplectic manifold* $(M, \phi, \xi, \eta, g_m)$ *onto a Riemannian manifold* (N, g_n). *Then the distribution* $(\ker\pi_*)^{\perp}$ *is totally geodesic foliation on* M *if and only if for*

$$\begin{aligned}
&\frac{1}{\lambda^2}\{g_n(\nabla^{\pi}_{X}\pi_*(Y),\pi_*(\omega\psi PV)) - g_n(\nabla^{\pi}_{X}\pi_*(CY),\pi_*(\omega V))\}\\
=\ &\cos^2\theta g_m(\mathcal{A}_X Y, PV) + g_m(\mathcal{A}_X BY, \omega V)\\
&+ g_m(X, grad\ln\lambda)g_m(Y,\omega\psi PV) + g_m(Y, grad\ln\lambda)g_m(X,\omega\psi PV)\\
&- g_m(X,Y)g_m(grad\ln\lambda,\omega\psi PV) - g_m(CY, grad\ln\lambda)g_m(X,\omega V)\\
&+ g_m(X, CY)g_m(grad\ln\lambda,\omega V)
\end{aligned}$$

for all $X, Y \in \Gamma(\ker\pi_*)^{\perp}$ *and* $V \in \Gamma(\ker\pi_*)$.

Proof. For all $X, Y \in \Gamma(\ker\pi_*)^{\perp}$ and $V \in \Gamma(\ker\pi_*)$, using equations (1), (2), (7), (8), (9), (10) and Lemma 3.6, we have

$$\begin{aligned}
&g_m(\nabla_X Y, V)\\
&= g_m(\nabla_X Y, PV + QV)\\
&= g_m(\nabla_X \phi Y, \psi PV) + g_m(\nabla_X \phi Y, \omega PV) + g_m(\nabla_X \phi Y, \phi QV)\\
&= \cos^2\theta g_m(\nabla_X Y, PV) - g_m(\nabla_X Y, \omega\psi PV) + g_m(\nabla_X \phi Y, \omega PV)\\
&\quad + g_m(\nabla_X \phi Y, \phi QV)\\
&= \cos^2\theta g_m(\nabla_X Y, PV) - g_m(\mathcal{H}\nabla_X Y, \omega\psi PV) + g_m(\nabla_X BY, \omega PV + \phi QV)\\
&\quad + g_m(\nabla_X CY, \omega PV + \phi QV).
\end{aligned}$$

Since $\omega V = \omega PV + \phi QV$, we have

$$\begin{aligned}
g_m(\nabla_X Y, V) =\ &\cos^2\theta g_m(\nabla_X Y, PV) - g_m(\mathcal{H}\nabla_X Y, \omega\psi PV)\\
&+ g_m(\nabla_X BY, \omega V) + g_m(\nabla_X CY, \omega V)\\
=\ &\cos^2\theta g_m(\nabla_X Y, PV) - g_m(\mathcal{H}\nabla_X Y, \omega\psi PV)\\
&+ g_m(\mathcal{A}_X BY, \omega V) + g_m(\mathcal{H}\nabla_X CY, \omega V).
\end{aligned}$$

Since π is a conformal submersion using (2) and Lemma 3.6, we have

$$\begin{aligned}
&g_m(\nabla_X Y, V)\\
=\ &\cos^2\theta g_m(\mathcal{A}_X Y, PV) + g_m(\mathcal{A}_X BY, \omega V)\\
&- \frac{1}{\lambda^2}g_n(\nabla^{f}_{X}\pi_*(Y),\pi_*(\omega\psi PV)) + \frac{1}{\lambda^2}g_n(\nabla^{\pi}_{X}\pi_*(CY),\pi_*(\omega V))
\end{aligned}$$

$$
\begin{aligned}
&+ g_m(X, grad \ln \lambda) g_m(Y, \omega\psi PV) + g_m(Y, grad \ln \lambda) g_m(X, \omega\psi PV)\\
&- g_m(X, Y) g_m(grad \ln \lambda, \omega\psi PV) - g_m(CY, grad \ln \lambda) g_m(X, \omega V)\\
&+ g_m(X, CY) g_m(grad \ln \lambda, \omega V).
\end{aligned}
$$
□

Theorem 3.14. *Let π be a conformal hemi-slant submersion from a cosymplectic manifold $(M, \phi, \xi, \eta, g_m)$ onto a Riemannian manifold (N, g_n). Then the distribution $(\ker \pi_*)$ is a totally geodesic foliation on M if and only if*

$$
\begin{aligned}
&\frac{1}{\lambda^2}\{g_n(\nabla^{\pi}_X \pi_*(\omega\psi PY), \pi_*(V)) - g_n((\nabla\pi_*)(X, \omega\psi PY), \pi_*(V))\}\\
= \; &\cos^2\theta g_m(\mathcal{T}_X PY, V) + g_m(\mathcal{T}_X \omega Y, BV) + g_m(X, grad \ln \lambda) g_m(V, \omega\psi PY)
\end{aligned}
$$

for all $X, Y \in \Gamma(\ker \pi_*)$ and $V \in \Gamma(\ker \pi_*)^{\perp}$.

Proof. For $X, Y \in \Gamma(\ker \pi_*)$ and $V \in \Gamma(\ker \pi_*)^{\perp}$, using equations (1), (2), (6), (9), (10), (11) and Lemma 3.6, we have

$$
\begin{aligned}
&g_m(\nabla_X Y, V)\\
&= g_m(\nabla_X PY, V) + g_m(\nabla_X QY, V)\\
&= g_m(\nabla_X PY, V) + g_m(\nabla_X QY, V)\\
&= \; - g_m(\nabla_X \phi\psi PY, V) + g_m(\nabla_X \omega PY, \phi V) + g_m(\nabla_X \phi QY, \phi V)\\
&= \; \cos^2\theta g_m(\nabla_X PY, V) - g_m(\mathcal{H}\nabla_X \omega\psi PY, V)\\
&\quad + g_m(\nabla_X(\omega PY + \phi QY), CV) + g_m(\nabla_X(\omega PY + \phi QY), BV).
\end{aligned}
$$

Since $\omega PY + \phi QY = \omega Y$, we have

$$
\begin{aligned}
g_m(\nabla_X Y, V) = \; &\cos^2\theta g_m(\nabla_X PY, V) - g_m(\mathcal{H}\nabla_X \omega\psi PY, V)\\
&+ g_m(\mathcal{T}_X PY, V) + g_m(\mathcal{H}\nabla_X \omega Y, CV).
\end{aligned}
$$

Since π is a conformal submersion using (2) and Lemma 3.6, we have

$$
\begin{aligned}
&g_m(\nabla_X Y, V)\\
= \; &\cos^2\theta g_m(\mathcal{T}_X PY, V) + g_m(\mathcal{T}_X \omega Y, BV) - \frac{1}{\lambda^2} g_n(\nabla^{\pi}_X \pi_*(\omega\psi PY), \pi_*(V))\\
&+ \frac{1}{\lambda^2} g_n((\nabla\pi_*)(X, \omega\psi PY), \pi_*(V)) + \frac{1}{\lambda^2} g_n(\nabla^{\pi}_X \pi_*(\omega Y), \pi_*(CV))\\
&- \frac{1}{\lambda^2} g_n((\nabla\pi_*)(X, \omega Y), \pi_*(CV))\\
= \; &\cos^2\theta g_m(\mathcal{T}_X PY, V) + g_m(\mathcal{T}_X \omega Y, BV) - \frac{1}{\lambda^2} g_n(\nabla^{\pi}_X \pi_*(\omega\psi PY), \pi_*(V))\\
&+ \frac{1}{\lambda^2} g_n((\nabla\pi_*)(X, \omega\psi PY), \pi_*(V)) + g_m(X, grad \ln \lambda) g_m(V, \omega\psi PY)
\end{aligned}
$$

which is complete proof. □

Theorem 3.15. *Let π be a conformal hemi-slant submersion from a cosymplectic manifold $(M, \phi, \xi, \eta, g_m)$ onto a Riemannian manifold (N, g_n). Then the* *anti-invariant distribution $\mathcal{D}^{\perp}$ define a totally geodesic foliation on M if and only if*

$$g_m(\mathcal{T}_X Y, \omega\psi Z) = \frac{1}{\lambda^2} g_n(\nabla^{\pi}_X \pi_*(\phi Y), \pi_*(\omega Z)),$$

and

$$g_m(\mathcal{T}_X \phi Y, BV) = -\frac{1}{\lambda^2} g_n(\nabla^{\pi}_X \pi_*(\phi Y), \pi_*(CV))$$

for all $X, Y \in \Gamma(\mathcal{D}^{\perp})$, $Z \in \Gamma(\mathcal{D}^{\theta})$ and $V \in \Gamma(\ker \pi_*)^{\perp}$.

Proof. For all $X, Y \in \Gamma(\mathcal{D}^{\perp})$, $Z \in \Gamma(\mathcal{D}^{\theta})$, and $V \in \Gamma(\ker \pi_*)^{\perp}$ using equations (1), (2), (5), (6), (10) and Lemma 3.6, we have

$$\begin{aligned} g_m(\nabla_X Y, Z) &= g_m(\nabla_X \phi Y, \psi Z) + g_m(\nabla_X \phi Y, \omega Z) \\ &= \cos^2\theta g_m(\nabla_X Y, Z) - g_m(\mathcal{T}_X Y, \omega\psi Z) + g_m(\mathcal{H}\nabla_X \phi Y, \omega Z). \end{aligned}$$

Now, we have

$$\sin^2\theta g_m(\nabla_X Y, Z) = -g_m(\mathcal{T}_X Y, \omega\psi Z) + g_m(\mathcal{H}\nabla_X \phi Y, \omega Z).$$

Since π is a conformal submersion using (2) and Lemma 3.6, we have

$$\begin{aligned} \sin^2\theta g_m(\nabla_X Y, Z) = &- g_m(\mathcal{T}_X Y, \omega\psi Z) + \frac{1}{\lambda^2} g_n(\nabla^{\pi}_X \pi_*(JY), \pi_*(\omega Z)) \\ &- \frac{1}{\lambda^2} g_n((\nabla\pi_*)(X, \phi Y), \pi_*(\omega Z)) \\ = &- g_m(\mathcal{T}_X Y, \omega\psi Z) + \frac{1}{\lambda^2} g_n(\nabla^{\pi}_X \pi_*(\phi Y), \pi_*(\omega Z)) \\ &- \frac{1}{\lambda^2} g_n(X(\ln\lambda)\pi_*(\phi Y) + \phi Y(\ln\lambda)\pi_*(X) \\ &- g_m(X, \phi Y)\pi_*(grad \ln\lambda), \pi_*(\omega Z)) \\ = &- g_m(\mathcal{T}_X Y, \omega\psi Z) + \frac{1}{\lambda^2} g_n(\nabla^{\pi}_X \pi_*(\phi Y), \pi_*(\omega Z)). \end{aligned}$$

Next, using equations (1), (2), (6) and (11), we have

$$\begin{aligned} g_m(\nabla_X Y, V) &= g_m(\nabla_X \phi Y, \phi V) \\ &= g_m(\mathcal{T}_X \phi Y, BV) + g_m(\mathcal{H}\nabla_X \phi Y, CV). \end{aligned}$$

Since π is a conformal submersion using (2) and Lemma 3.6, we have

$$\begin{aligned} &g_m(\nabla_X Y, V) \\ &= g_m(\mathcal{T}_X \phi Y, BV) + \frac{1}{\lambda^2} g_n(\nabla^{\pi}_X \pi_*(\phi Y), \pi_*(CV)) \\ &\quad - \frac{1}{\lambda^2} g_n((\nabla\pi_*)(X, \phi Y), \pi_*(CV)) \\ &= g_m(\mathcal{T}_X \phi Y, BV) + \frac{1}{\lambda^2} g_n(\nabla^{\pi}_X \pi_*(\phi Y), \pi_*(CV)) \\ &\quad - \frac{1}{\lambda^2} g_n(X(\ln\lambda)\pi_*(\phi Y) + \phi Y(\ln\lambda)X - g_m(X, \phi Y)\pi_*(grad \ln\lambda), \pi_*(CV)) \end{aligned}$$

$$= g_m(\mathcal{T}_X \phi Y, BV) + \frac{1}{\lambda^2} g_n(\nabla^\pi_X \pi_*(\phi Y), \pi_*(CV))$$

which is complete proof. □

Theorem 3.16. *Let π be a conformal hemi-slant submersion from a cosymplectic manifold $(M, \phi, \xi, \eta, g_m)$ onto a Riemannian manifold (N, g_n). Then the slant distribution $\mathcal{D}^\theta$ define a totally geodesic foliation on M if and only if*

$$g_m(\mathcal{T}_Z \omega\psi W, X) = \frac{1}{\lambda^2} g_n(\nabla^\pi_Z \pi_*(\omega W), \pi_*(\phi X)),$$

and

$$g_m(\mathcal{T}_Z \omega\psi W, V) - g_m(\mathcal{T}_Z \omega W, BV) = \frac{1}{\lambda^2} g_n(\nabla^\pi_Z \pi_*(\omega W), \pi_*(CV))$$

for all $Z, W \in \Gamma(\mathcal{D}^\theta)$, $X \in \Gamma(\mathcal{D}^\perp)$ and $V \in \Gamma(\ker \pi_*)^\perp$.

Proof. For all $Z, W \in \Gamma(\mathcal{D}^\theta), X \in \Gamma(\mathcal{D}^\perp)$ and $V \in \Gamma(\ker \pi_*)^\perp$, using equations (1), (2), (10) and Lemma 3.6, we have

$$\begin{aligned} g_m(\nabla_Z W, X) &= g_m(\nabla_Z \psi W, \phi X) + g_m(\nabla_Z \omega W, \phi X) \\ &= \cos^2\theta g_m(\nabla_Z W, X) - g_m(\nabla_Z \omega\psi W, X) + g_m(\nabla_Z \omega W, \phi X). \end{aligned}$$

Now, we have

$$\sin^2\theta g_m(\nabla_Z W, X) = -g_m(\mathcal{T}_Z \omega\psi W, X) + g_m(\mathcal{H}\nabla_Z \omega W, \phi X).$$

Since π is a conformal submersion using (2) and Lemma 3.6, we have

$$\begin{aligned} \sin^2\theta g_m(\nabla_Z W, X) = & - g_m(\mathcal{T}_Z \omega\psi W, X) + \frac{1}{\lambda^2} g_n(\nabla^\pi_Z \pi_*(\omega W), \pi_*(\phi X)) \\ & - \frac{1}{\lambda^2} g_n((\nabla \pi_*)(Z, \omega W), \pi_*(\phi X)) \\ = & - g_m(\mathcal{T}_Z \omega\psi W, X) + \frac{1}{\lambda^2} g_n(\nabla^\pi_Z \pi_*(\omega W), \pi_*(\phi X)) \\ & - \frac{1}{\lambda^2} g_n(Z(\ln\lambda)\pi_*(\omega W) + \omega W(\ln\lambda)\pi_*(Z) \\ & - g_m(Z, \omega W)(\pi_*(grad \ln\lambda))\pi_*(\phi X)) \\ = & - g_m(\mathcal{T}_Z \omega\psi W, X) + \frac{1}{\lambda^2} g_n(\nabla^\pi_Z \pi_*(\omega W), \pi_*(\phi X)). \end{aligned}$$

Next, using equations (1), (2), (6) and Lemma 3.6, we have

$$\begin{aligned} g_m(\nabla_Z W, V) &= g_m(\nabla_Z \psi W, \phi V) + g_m(\nabla_Z \omega W, \phi V) \\ &= \cos^2\theta g_m(\nabla_Z W, V) - g_m(\nabla_Z \omega\psi W, V) \\ &\quad + g_m(\mathcal{T}_Z \omega W + \mathcal{H}\nabla_Z \omega W, BV + CV). \end{aligned}$$

Now, we have

$$\sin^2\theta g_m(\nabla_Z W, V) = -g_m(\nabla_Z \omega\psi W, V) + g_m(\mathcal{T}_Z \omega W, BV) + g_m(\mathcal{H}\nabla_Z \omega W, CV).$$

Since π is a conformal submersion using (2) and Lemma 3.6, we have

$$\begin{aligned}
&\sin^2\theta g_m(\nabla_Z W, V)\\
=&-g_m(\mathcal{T}_Z\omega\psi W, V)+g_m(\mathcal{T}_Z\omega W, BV)\\
&+\frac{1}{\lambda^2}g_n(\nabla^\pi_Z\pi_*(\omega W),\pi_*(CV))-\frac{1}{\lambda^2}g_n((\nabla\pi_*)(Z,\omega W),\pi_*(CV))\\
=&-g_m(\mathcal{T}_Z\omega\psi W, V)+g_m(\mathcal{T}_Z\omega W, BV)+\frac{1}{\lambda^2}g_n(\nabla^\pi_Z\pi_*(\omega W),\pi_*(CV))\\
&-\frac{1}{\lambda^2}g_n(Z(\ln\lambda)\pi_*(\omega W)+\omega W(\ln\lambda)\pi_*(Z)\\
&-g_m(Z,\omega W)(\pi_*(grad\ln\lambda)),\pi_*(CV))\\
=&-g_m(\mathcal{T}_Z\omega\psi W, V)+g_m(\mathcal{T}_Z\omega W, BV)+\frac{1}{\lambda^2}g_n(\nabla^\pi_Z\pi_*(\omega W),\pi_*(CV))
\end{aligned}$$

which is complete proof. □

4. Examples

Example 4.1. Let $\mathbb{R}^7$ have a cosymplectic structure as in Example 2.1. Define a map $h:\mathbb{R}^7\to\mathbb{R}^3$ by,

$$h(x_1,x_2,x_3,y_1,y_2,y_3,z)=e^\pi\left(x_1,\frac{\sqrt{3}x_2+x_3}{2},x_6\right).$$

Then, by direct calculations, we obtain the Jacobian matrix of h as

$$\begin{pmatrix}1&0&0&0&0&0&0\\0&\frac{\sqrt{3}}{2}&\frac{1}{2}&0&0&0&0\\0&0&0&0&0&1&0\end{pmatrix}.$$

Since, the rank of above Jacobian matrix is 3, therefore the map h is a submersion. After some straightforward computations, we obtain

$$(\ker h_*)=span\{X_1=\frac{\partial}{\partial y_1},X_2=\frac{\partial}{\partial y_2},X_3=\frac{1}{2}\left(\frac{\partial}{\partial x_2}-\sqrt{3}\frac{\partial}{\partial x_3}\right),X_4=\frac{\partial}{\partial z}\}$$

and

$$(\ker h_*)^\perp=span\{V_1=\frac{\partial}{\partial x_1},V_2=\frac{\partial}{\partial y_2},V_3=\frac{1}{2}\left(\sqrt{3}\frac{\partial}{\partial x_2}+\frac{\partial}{\partial x_3}\right)\}.$$

Then it follows that,

$$\mathcal{D}^\theta=span\{X_1=\frac{\partial}{\partial y_2},X_2=\frac{1}{2}\left(\frac{\partial}{\partial x_2}-\sqrt{3}\frac{\partial}{\partial x_3}\right)\},$$
$$\mathcal{D}^\perp=span\{V_1=\frac{\partial}{\partial y_1}\}.$$

Thus the map h is a conformal hemi-slant submersion with the hemi-slant angle $\theta=\frac{\pi}{3}$ and dialation $\lambda=e^\pi$.

Example 4.2. Let $\mathbb{R}^7$ have a cosymplectic structure as in Example 2.1. Define a map $h : \mathbb{R}^7 \to \mathbb{R}^3$ by,

$$h(x_1, x_2, x_3, y_1, y_2, y_3, z) = e^7 \left(x_2, x_1 \sin\alpha - y_2 \cos\alpha, y_3\right).$$

Then, by direct calculations, we obtain the jacobian matrix of h as

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \sin\alpha & 0 & 0 & 0 & -\cos\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Since, the rank of above Jacobian matrix is 3, therefore the map h is a submersion. After some straightforward computations, we obtain

$$(\ker h_*) = span\{X_1 = \frac{\partial}{\partial x_3}, X_2 = \frac{\partial}{\partial y_1}, X_3 = cos\alpha\frac{\partial}{\partial x_1} + sin\alpha\frac{\partial}{\partial y_2}, X_4 = \frac{\partial}{\partial z}\}$$

and

$$(\ker h_*)^{\perp} = span\{V_1 = \frac{\partial}{\partial x_2}, V_2 = \frac{\partial}{\partial y_3}, V_3 = sin\alpha\frac{\partial}{\partial x_1} - cos\alpha\frac{\partial}{\partial y_2}\}.$$

Then, it follows that

$$\begin{aligned} \mathcal{D}^{\theta} &= span\{X_2 = \frac{\partial}{\partial y_1}, X_3 = cos\alpha\frac{\partial}{\partial x_1} + sin\alpha\frac{\partial}{\partial y_2}\} \\ \mathcal{D}^{\perp} &= span\{V_1 = \frac{\partial}{\partial x_3}\}. \end{aligned}$$

Thus the map h is a conformal hemi-slant submersion with the hemi-slant angle $\theta = \alpha$ and dialation $\lambda = e^7$.

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Vinay Kumar
Department of Mathematics and Astronomy
University of Lucknow
Lucknow 226020, India
Email address: vinaykumarbbau@gmail.com

Rajendra Prasad
Department of Mathematics and Astronomy
University of Lucknow
Lucknow 226020, India
Email address: rp.manpur@rediffmail.com

Sandeep Kumar Verma
Department of Mathematics and Astronomy
University of Lucknow
Lucknow 226020, India
Email address: skverma1208@gmail.com