Commun. Korean Math. Soc. **38** (2023), No. 1, pp. 195–203
https://doi.org/10.4134/CKMS.c210255
pISSN: 1225-1763 / eISSN: 2234-3024

HOMOGENEOUS STRUCTURES ON FOUR-DIMENSIONAL LORENTZIAN DAMEK-RICCI SPACES

Assia Mostefaoui and Noura Sidhoumi

Abstract. Special examples of harmonic manifolds that are not symmetric, proving that the conjecture posed by Lichnerowicz fails in the non-compact case have been intensively studied. We completely classify homogeneous structures on Damek-Ricci spaces equipped with the left invariant metric.

1. Introduction and preliminaries

A pseudo-Riemannian manifold M is homogeneous provided that, given any points $p, q \in M$, there is an isometry ϕ of M such that $\phi(p) = q$. It is locally homogeneous if there is a local isometry mapping a neighborhood of p into a neighborhood of q. If M is homogeneous, then any geometrical properties at one point of M hold at every point. The characterization by E. Cartan of Riemannian locally symmetric spaces as those Riemannian manifolds whose curvature tensor is parallel was extended by Ambrose and Singer in [1] (see also [18]). They proved that a connected, simply connected and complete Riemannian manifold is homogeneous if and only if there exists a $(1, 2)$ tensor field T satisfying certain equations, see (1). In [12] Gadea and Oubiña have extended that characterization to pseudo-Riemannian manifolds. Specifically, let (M, g) be a connected pseudo-Riemannian manifold of dimension n and signature $(k, n-k)$. Let ∇ be the Levi-Civita connection of g and R its curvature tensor field. A homogeneous pseudo-Riemannian structure on (M, g) is a tensor field T of type $(1, 2)$ on M such that the connection $\tilde{\nabla} = \nabla - T$ satisfies

$$\tilde{\nabla} g = 0, \quad \tilde{\nabla} R = 0, \quad \tilde{\nabla} T = 0. \tag{1}$$

More explicitly, T is the solution of the following system of equations (known as Ambrose-Singer equations)

$$g\left(T_X Y, Z\right) + g\left(Y, T_X Z\right) = 0, \tag{2}$$

$$\left(\nabla_X R\right)_{YZ} = [T_X, R_{YZ}] - R_{T_X Y Z} - R_{Y T_X Z}, \tag{3}$$

Received July 22, 2021; Revised December 1, 2021; Accepted February 15, 2022.
2020 *Mathematics Subject Classification.* Primary 53C50, 53B30.
Key words and phrases. Damek-Ricci spaces, homogeneous structures, Lorentzian metrics, left-invariant metrics.

$$(\nabla_X T)_Y = [T_X, T_Y] - T_{T_X Y}, \tag{4}$$

for all vector fields X, Y, Z. If g is a Lorentzian metric ($k = 1$), we say that T is a homogeneous Lorentzian structure. The geometric meaning of the existence of a homogeneous pseudo-Riemannian structure is explained by the following:

Theorem 1.1 ([12])**.** *Let (M, g) be a connected, simply connected and complete pseudo-Riemannian manifold. Then, (M, g) admits a homogeneous pseudo-Riemannian structure if and only if it is a reductive homogeneous pseudo-Riemannian manifold.*

This means that $M = G/H$, where G is a connected Lie group acting transitively and effectively on M as a group of isometries, H is the isotropy group at a point $o \in M$, and the Lie algebra $\mathfrak{g}$ of G may be decomposed into a vector space direct sum of the Lie algebra $\mathfrak{h}$ of H and an $\mathrm{Ad}(H)$-invariant subspace m, that is $\mathfrak{g} = \mathfrak{h} \oplus m$, $\mathrm{Ad}(H)\, m \subset m$. It must be noted that any homogeneous Riemannian manifold is reductive, while a homogeneous pseudo-Riemannian manifold need not be reductive. Homogeneous Lorentzian structures have been investigated for several classes of Lorentzian manifolds.

For some special classes of metrics, the Ambrose-Singer equations can be completely solved, leading to a complete classification. Natural candidates for this kind of study are Lie groups. Some examples of complete classifications of homogeneous structures in a class of homogeneous pseudo-Riemannian may be found in [3–5, 7, 13, 14].

The study of Damek-Ricci spaces includes a wide investigation in literature. That is particularly relevant, to give a negative answer, in high dimensions, to the famous question posed by Lichnrowicz: "Is a harmonic Riemannian manifold necessarily a symmetric space?"

Damek-Ricci spaces have been constructed by Damek and Ricci in [9]. These spaces are semidirect products of Heisenberg groups with the real line. Several results about these spaces have been investigated by many authors. In [10], Degla and Todjihounde proved the non existence of any proper (nongeodesic) biharmonic curve in the four-dimensional Damek-Ricci space although such curves exist in three-dimensional Heisenberg groups. In [2], they studied the dispersive properties of the linear wave equation on Damek-Ricci spaces and their application to nonlinear Cauchy problems. In [11], it was constructed uncountable many isoparametric families of hypersurfaces in Damek-Ricci spaces, by characterizing those of them that have constant principal curvatures. In [15], Koivogui and Todjihounde gave a setting for constructing Weierstrass representation formulas for simply connected minimal surfaces into four-dimensional Riemannian Damek-Ricci spaces. This was extended to the case of spacelike and timelike minimal surfaces in 4-dimensional Damek-Ricci spaces equipped with left-invariant Lorentzian metric [8]. In [17], Tan and Deng considered the four-dimensional Lorentzian Damek-Ricci spaces $(\mathbb{S}^4_\varepsilon, g_\varepsilon)$ and investigated other geometrical properties. In particular, they proved the non-existence of left-invariant Ricci solitons on these spaces. More recently, the second author generalized this result proving the non-existence of non invariant vector field for which the soliton equation is satisfied [16].

In this paper, we shall classify homogeneous structures of the four-dimensional Damek-Ricci spaces $(\mathbb{S}^4_\varepsilon, g_\varepsilon)$, equipped with the left-invariant Lorentzian metric g_ε.

The paper is organized in the following way. In Section 2, we shall report some basic information about four-dimensional Damek-Ricci space and its left-invariant metrics in global coordinates, we shall describe their Levi-Civita connection, the curvature and the Ricci tensor. In Section 3, homogeneous structures of four-dimensional Damek Ricci spaces $(\mathbb{S}^4_\varepsilon, g_\varepsilon)$ are considered, proving their classification.

2. Geometry of 4-dimensional Damek-Ricci spaces

For a brief introduction to Damek-Ricci spaces structures, we express the relevant definitions here. We start with a short description of four-dimensional Damek-Ricci spaces, referring to [6] and [9] for more details and further results. For this purpose, we need to recall the so-called generalized Heisenberg group, since Damek-Ricci space depends on it.

2.1. Generalized Heisenberg group

The generalized Heisenberg algebras are defined as follows. Let b and z be real vector spaces of dimension m and n, respectively, such that $\mathfrak{n}$ is the orthogonal sum $\mathfrak{n} = b \oplus z$. We define in $\mathfrak{n}$ the bracket

$$[U + X, V + Y] = \beta(U, V),$$

where $\beta : b \times b \to z$ is a skew-symmetric bilinear map. This product defines a Lie algebra structure on $\mathfrak{n}$.

We equip b with a positive inner product and z with a positive or Lorentzian inner product and let $\langle , \rangle_{\mathfrak{n}}$ denote the product metric. Define a linear map $J : Z \in z \to J_z \in \mathrm{End}(b)$ by

$$\langle J_Z U, V\rangle_{\mathfrak{n}} = \langle \beta(U, V), Z\rangle_{\mathfrak{n}} \text{ for all } U, V \in b \text{ and } Z \in z.$$

Then, $\mathfrak{n}$ is a two-step nilpotent Lie algebra with center z.

- If the inner product in z is positive and $J_Z^2 = -\langle Z, Z\rangle_{\mathfrak{n}} \mathrm{id}_b$ for all $Z \in z$, then the Lie algebra $\mathfrak{n}$ is called a generalized Riemannian Heisenberg algebra, and the associated simply connected nilpotent Lie group, endowed with the induced left-invariant Riemannian metric, is called a generalized Riemannian Heisenberg group.

- If the inner product in z is Lorentzian and

$$J_Z^2 = \begin{cases} -\langle Z, Z\rangle_{\mathfrak{n}} \mathrm{id}_b, & \text{when } Z \text{ is spacelike,} \\ \langle Z, Z\rangle_{\mathfrak{n}} \mathrm{id}_b, & \text{when } Z \text{ is timelike,} \end{cases}$$

then the Lie algebra $\mathfrak{n}$ is called a generalized Riemannian Heisenberg algebra, and the associated simply connected nilpotent Lie group, endowed with the induced left-invariant Lorentzian metric, is called a generalized Lorentzian Heisenberg group.

2.2. Damek-Ricci spaces

Now, let $\varepsilon = \pm 1$ and $\mathfrak{a}_\varepsilon$ be a one-dimensional pseudo-Riemannian real vector space, which is Riemannian when $\varepsilon = 1$ and Lorentzian when $\varepsilon = -1$, and let $\mathfrak{n}_{-\varepsilon} = b \oplus z$ be a generalized Heisenberg algebra which is Lorentzian when $\varepsilon = 1$ and Riemannian when $\varepsilon = -1$.

Consider a new vector space $\mathfrak{a}_\varepsilon \oplus \mathfrak{n}_{-\varepsilon}$ as the vector space direct sum of $\mathfrak{a}_\varepsilon$ and $\mathfrak{n}_{-\varepsilon}$. Let $s, r \in \mathbb{R}$, $U, V \in b$ and $X, Y \in z$. We define the Lorentzian product $\langle \cdot, \cdot \rangle$ and a Lie bracket $[\cdot, \cdot]$ on $\mathfrak{a}_\varepsilon \oplus \mathfrak{n}_{-\varepsilon}$ by

$$\langle rA + U + X, sA + V + Y \rangle = \langle U + X, V + Y \rangle_{\mathfrak{n}_{-\varepsilon}} + \varepsilon rs,$$

$$[rA + U + X, sA + V + Y] = [U, V]_{\mathfrak{n}_{-\varepsilon}} + \frac{1}{2} rV - \frac{1}{2} sU + rY - sX$$

for a non zero vector A in $\mathfrak{a}_\varepsilon$. Therefore $\mathfrak{a}_\varepsilon \oplus \mathfrak{n}_{-\varepsilon}$ becomes a solvable Lie algebra. The corresponding simply connected Lie group, equipped with the induced left-invariant Lorentzian metric, is called a Lorentzian Damek-Ricci space and will be denoted by $\mathbb{S}_\varepsilon$.

2.3. Curvature of four-dimensional Damek-Ricci spaces

Consider the four-dimensional Damek-Ricci spaces $(\mathbb{S}^4_\varepsilon, g_\varepsilon)$, equipped with the left-invariant Lorentzian metric g_ε. Through the paper, we will denote the coordinate basis $\left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t} \right\}$ by $\{\partial_x, \partial_y, \partial_z, \partial_t\}$.

As it was pointed in [8], the left-invariant Lorentzian metric g_ε on the four-dimensional space $\mathbb{S}^4_\varepsilon$ is given by

$$g_\varepsilon = e^{-t} \mathrm{d}x^2 + e^{-t} \mathrm{d}y^2 + \varepsilon e^{-2t} (\mathrm{d}z + \frac{c}{2} y \mathrm{d}x - \frac{c}{2} x \mathrm{d}y)^2 - \varepsilon \mathrm{d}t^2, \tag{5}$$

where $c \in \mathbb{R}$.

Following [8], let us denote

$$e_1 = e^{\frac{t}{2}} \left(\frac{\partial}{\partial x} - \frac{cy}{2} \frac{\partial}{\partial z} \right), \; e_2 = e^{\frac{t}{2}} \left(\frac{\partial}{\partial y} + \frac{cx}{2} \frac{\partial}{\partial z} \right), \; e_3 = e^t \frac{\partial}{\partial z}, \; e_4 = \frac{\partial}{\partial t}. \tag{6}$$

Then, $\{e_1, e_2, e_3, e_4\}$ form an orthonormal basis of the Lie algebra $\mathfrak{s}^4$ of $\mathbb{S}^4_\varepsilon$ for which

$$g_\varepsilon(e_1, e_1) = g_\varepsilon(e_2, e_2) = 1, \quad g_\varepsilon(e_3, e_3) = -g_\varepsilon(e_4, e_4) = \varepsilon.$$

The bracket operation in $\mathfrak{s}^4$ is given by the formulas:

$$[e_1, e_2] = ce_3, \quad [e_1, e_3] = 0, \quad [e_1, e_4] = -\frac{1}{2} e_1,$$

$$[e_2, e_3] = 0, \qquad [e_2, e_4] = -\frac{1}{2} e_2, \quad [e_3, e_4] = -e_3.$$

We will denote by ∇ the Levi-Civita connection of $(\mathbb{S}^4_\varepsilon, g_\varepsilon)$, and by R its curvature tensor taken with the sign convention:

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

Using the Koszul formula to calculate the components of the Levi-Civita connection with respect to the orthonormal basis given by (6), we find

$$\nabla_{e_1} = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & -\frac{\varepsilon c}{2} & 0 \\ 0 & \frac{c}{2} & 0 & 0 \\ -\frac{\varepsilon}{2} & 0 & 0 & 0 \end{pmatrix}, \quad \nabla_{e_2} = \begin{pmatrix} 0 & 0 & \frac{c\varepsilon}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \\ -\frac{c}{2} & 0 & 0 & 0 \\ 0 & -\frac{\varepsilon}{2} & 0 & 0 \end{pmatrix},$$
$$\nabla_{e_3} = \begin{pmatrix} 0 & \frac{\varepsilon c}{2} & 0 & 0 \\ -\frac{\varepsilon c}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad \nabla_{e_4} = 0. \tag{7}$$

Denoting by R_{ij} the matrix describing $R(e_{i,}e_j)$ with respect to the orthonormal basis given by (6) and for $c^2 = 1$, we have

$$R_{12} = \begin{pmatrix} 0 & -\frac{\varepsilon}{2} & 0 & 0 \\ \frac{\varepsilon}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{c}{2} \\ 0 & 0 & \frac{c}{2} & 0 \end{pmatrix}, \qquad R_{13} = \begin{pmatrix} 0 & 0 & \frac{3}{4} & 0 \\ 0 & 0 & 0 & \frac{\varepsilon c}{4} \\ -\frac{3\varepsilon}{4} & 0 & 0 & 0 \\ 0 & \frac{c}{4} & 0 & 0 \end{pmatrix},$$
$$R_{14} = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{4} \\ 0 & 0 & -\frac{\varepsilon c}{4} & 0 \\ 0 & \frac{c}{4} & 0 & 0 \\ -\frac{\varepsilon}{4} & 0 & 0 & 0 \end{pmatrix}, \quad R_{23} = \begin{pmatrix} 0 & 0 & 0 & -\frac{\varepsilon c}{4} \\ 0 & 0 & \frac{3}{4} & 0 \\ 0 & -\frac{3\varepsilon}{4} & 0 & 0 \\ -\frac{c}{4} & 0 & 0 & 0 \end{pmatrix}, \tag{8}$$
$$R_{24} = \begin{pmatrix} 0 & 0 & \frac{\varepsilon c}{4} & 0 \\ 0 & 0 & 0 & -\frac{1}{4} \\ -\frac{c}{4} & 0 & 0 & 0 \\ 0 & -\frac{\varepsilon}{4} & 0 & 0 \end{pmatrix}, \quad R_{34} = \begin{pmatrix} 0 & \frac{\varepsilon c}{2} & 0 & 0 \\ -\frac{\varepsilon c}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

3. Homogeneous structures on four dimensional Damek-Ricci spaces $\mathbb{S}^4_\varepsilon$

A homogeneous Lorentzian structure T on the four dimensional Lorentzian Damek-Ricci spaces $(\mathbb{S}^4_\varepsilon, g_\varepsilon)$ is uniquely determined by its local components T^k_{ij} with respect to $\{e_1, e_2, e_3, e_4\}$. The smooth functions T^k_{ij} are defined by

$$T(e_i, e_j) = \sum_{k=1}^{4} T^k_{ij} e_k.$$

From (5) we prove that the first Ambrose-Singer equation (2) is satisfied if and only if

$$(9)\qquad \begin{cases} T^2_{i,1} = -T^1_{i,2} & \text{for } i = 1,\dots,4, \\ T^3_{i,j} = -\varepsilon T^j_{i,3} & \text{for } i = 1,\dots,4 \text{ and } j = 1,2, \\ T^k_{i,j} = T^j_{i,k} & \text{for } i = 1,\dots,4 \text{ and } j,k = 3,4, \\ T^j_{i,4} = \varepsilon T^4_{i,j} & \text{for } i = 1,\dots,4 \text{ and } j = 1,2, \\ T^1_{i,1} = 0 & \text{for } i = 1,\dots,4, \\ T^2_{i,2} = 0 & \text{for } i = 1,\dots,4, \\ T^3_{i,3} = 0 & \text{for } i = 1,\dots,4, \\ T^4_{i,4} = 0 & \text{for } i = 1,\dots,4. \end{cases}$$

Next, using (7), (8) and (9), a straight computation leads to prove that the second Ambrose-Singer equation (3) is satisfied if and only if

$$(10)\qquad \begin{cases} T^4_{i,i} = -\frac{\varepsilon}{2} \quad \text{for } i = 1,2, \\ T^3_{1,2} = \frac{c}{2}, \\ T^3_{2,1} = -\frac{c}{2}, \\ T^3_{3,4} = -1, \\ T^1_{3,1} = 0, \\ T^3_{1,1} = T^3_{1,4} = T^3_{2,2} = T^3_{2,4} = T^3_{3,1} = T^3_{3,2} = T^3_{4,1} = T^3_{4,2} = T^3_{4,4} = 0, \\ T^4_{1,2} = T^4_{2,1} = T^4_{3,1} = T^4_{3,2} = T^4_{4,1} = T^4_{4,2} = 0. \end{cases}$$

Finally, from (7), (9), (10) and after computations we can prove that the third Ambrose-Singer equation (4) is satisfied if and only if

$$(11)\qquad \begin{cases} e^{\frac{t}{2}}\left(\partial_x T^1_{1,2} - \frac{cy}{2}\partial_z T^1_{1,2}\right) - T^1_{1,2}T^1_{2,2} = 0, \\ e^{\frac{t}{2}}\left(\partial_x T^1_{2,2} - \frac{cy}{2}\partial_z T^1_{2,2}\right) + \left(T^1_{1,2}\right)^2 = 0, \\ \partial_x T^1_{3,2} - \frac{cy}{2}\partial_z T^1_{3,2} = 0, \\ \partial_x T^1_{4,2} - \frac{cy}{2}\partial_z T^1_{4,2} = 0, \\ e^{\frac{t}{2}}\left(\partial_y T^1_{1,2} + \frac{cx}{2}\partial_z T^1_{1,2}\right) - \left(T^1_{2,2}\right)^2 = 0, \\ e^{\frac{t}{2}}\left(\partial_y T^1_{2,2} + \frac{cx}{2}\partial_z T^1_{2,2}\right) + T^1_{1,2}T^1_{2,2} = 0, \\ \partial_y T^1_{3,2} + \frac{cx}{2}\partial_z T^1_{3,2} = 0, \\ \partial_y T^1_{4,2} + \frac{cx}{2}\partial_z T^1_{4,2} = 0, \\ e^t\partial_z T^1_{1,2} - T^1_{2,2}\left(T^1_{3,2} - \frac{1}{2}c\varepsilon\right) = 0, \\ e^t\partial_z T^1_{2,2} - T^1_{1,2}\left(-T^1_{3,2} + \frac{1}{2}c\varepsilon\right) = 0, \\ \partial_t T^1_{2,2} + T^1_{1,2}T^1_{4,2} = 0, \\ \partial_t T^1_{1,2} - T^1_{4,2}T^1_{2,2} = 0, \\ T^1_{3,2} = k_1, \\ T^1_{4,2} = k_2, \end{cases}$$

where k_1, k_2 are real constants. Now, we put $F = T^1_{1,2}$, $G = T^1_{2,2}$, which are smooth functions on $\mathbb{S}^4_\varepsilon$.

Therefore, (11) becomes

$$(12)\qquad \begin{cases} e^{\frac{t}{2}}\left(\partial_x F - \frac{cy}{2}\partial_z F\right) - FG = 0, \\ e^{\frac{t}{2}}\left(\partial_x G - \frac{cy}{2}\partial_z G\right) + F^2 = 0, \\ e^{\frac{t}{2}}\left(\partial_y F + \frac{cx}{2}\partial_z F\right) - G^2 = 0, \\ e^{\frac{t}{2}}\left(\partial_y G + \frac{cx}{2}\partial_z G\right) + FG = 0, \\ e^t\partial_z F - G\left(k_1 - \frac{1}{2}c\varepsilon\right) = 0, \\ e^t\partial_z G + F\left(k_1 - \frac{1}{2}c\varepsilon\right) = 0, \\ \partial_t G + k_2 F = 0, \\ \partial_t F - k_2 G = 0. \end{cases}$$

Deriving the seventh equation in (12) with respect to t and using the last equation in (12), we get

$$G = h_1\cos(k_2 t) + h_2\sin(k_2 t)$$

for some smooth functions $h_1 = h_1(x, y, z)$ and $h_2 = h_2(x, y, z)$.

Again, by deriving the fifth and the sixth equations in (12) with respect to t and using the seventh and the eighth equations in (12), we obtain

$$(13)\qquad \begin{aligned} &e^t\partial_z F + k_2 e^t\partial_z G + k_2\left(k_1 - \tfrac{1}{2}c\varepsilon\right)F = 0, \\ &e^t\partial_z G - k_2 e^t\partial_z F + \left(k_1 - \tfrac{1}{2}c\varepsilon\right)\partial_t F = 0. \end{aligned}$$

Now, we derive the first equation in (13) with respect to t and taking into account the derivative of the fifth and sixth equations in (12) with respect to t, we get

$$k_2\left(k_1 - \frac{1}{2}c\varepsilon\right)F = 0.$$

Thus, we need to consider the following different cases.

1) If $F = 0$, then, from the third equation in (12), we find $G = 0$.

2) Let $k_1 = \frac{1}{2}c\varepsilon$. In this case, adding the first and fourth equations of (12) and using the fifth and the sixth ones, we get

$$\partial_x F = -\partial_y G.$$

Now, deriving the third equation of (12) with respect to x, we obtain

$$\begin{aligned} &\partial_y^2 G = 0, \\ &G\partial_x G = 0. \end{aligned}$$

For $G = 0$, we have immediately $F = 0$, while if $\partial_x G = 0$, the second and the third equations in (12) give $F = 0$ and $G = 0$.

3) $\boldsymbol{k_2} = 0$. That is to say $K = 0$, thus $G = h_1$. Now, deriving the sixth equation in (12) with respect to z and using the fifth one, we obtain

$$e^t\partial_z^2 h_1 + \left(k_1 - \frac{1}{2}c\varepsilon\right)^2 e^{-t}h_1 = 0.$$

Therefore,

$$\begin{cases} \partial_z^2 h_1 = 0, \\ \left(k_1 - \frac{1}{2}c\varepsilon\right)^2 h_1 = 0. \end{cases}$$

If $h_1 = 0$, that is to say $G = 0$, then from the second equation in (12) we have $F = 0$.

Theorem 3.1. *Consider the four-dimensional Damek-Ricci spaces* $(\mathbb{S}^4_\varepsilon, g_\varepsilon)$ *equipped with the left-invariant Lorentzian metric* g_ε*. Then* $(\mathbb{S}^4_\varepsilon, g_\varepsilon)$ *is locally reductive homogeneous space.*

Furthermore, all homogeneous structures of $(\mathbb{S}^4_\varepsilon, g_\varepsilon)$ *are determined* (*through their local components* T^k_{ij}) *with respect to the orthonormal basis given by* (6) *and dual basis* $\left(\omega^i\right)_{i=1,\dots,4}$ *as follows*:

$$\begin{aligned} T = {} & \alpha\left(\omega^3\otimes\omega^2\otimes e_1 - \omega^3\otimes\omega^1\otimes e_2\right) + \beta\left(\omega^4\otimes\omega^2\otimes e_1 - \omega^4\otimes\omega^1\otimes e_2\right) \\ & + \frac{c\varepsilon}{2}\left(\omega^2\otimes\omega^3\otimes e_1 - \omega^1\otimes\omega^3\otimes e_2\right) - \frac{1}{2}\left(\omega^1\otimes\omega^4\otimes e_1 + \omega^2\otimes\omega^4\otimes e_2\right) \\ & - \frac{\varepsilon}{2}\left(\omega^1\otimes\omega^1\otimes e_4 + \omega^2\otimes\omega^2\otimes e_4\right) - \omega^3\otimes\omega^4\otimes e_3 - \omega^3\otimes\omega^3\otimes e_4 \\ & + \frac{c}{2}\left(\omega^1\otimes\omega^2\otimes e_3 - \omega^2\otimes\omega^1\otimes e_3\right). \end{aligned}$$

References

[1] W. Ambrose and I. M. Singer, *On homogeneous Riemannian manifolds*, Duke Math. J. **25** (1958), 647–669. `http://projecteuclid.org/euclid.dmj/1077468199`

[2] J.-P. Anker, V. Pierfelice, and M. Vallarino, *The wave equation on Damek-Ricci spaces*, Ann. Mat. Pura Appl. (4) **194** (2015), no. 3, 731–758. `https://doi.org/10.1007/s10231-013-0395-x`

[3] A. B. Ardakani and A. H. Zaeim, *Homogeneous structures on special classes of space-times*, Int. J. Nonlinear Sci. **14** (2012), no. 1, 95–99.

[4] W. Batat, G. Calvaruso, and B. De Leo, *Homogeneous Lorentzian* 3*-manifolds with a parallel null vector field*, Balkan J. Geom. Appl. **14** (2009), no. 1, 11–20.

[5] W. Batat and S. Rahmani, *Homogeneous Lorentzian structures on the generalized Heisenberg group*, Differ. Geom. Dyn. Syst. **12** (2010), 12–17.

[6] J. Berndt, F. Tricerri, and L. Vanhecke, *Generalized Heisenberg groups and Damek-Ricci harmonic spaces*, Lecture Notes in Mathematics, 1598, Springer-Verlag, Berlin, 1995. `https://doi.org/10.1007/BFb0076902`

[7] G. Calvaruso, *Homogeneous structures on three-dimensional Lorentzian manifolds*, J. Geom. Phys. **57** (2007), no. 4, 1279–1291. `https://doi.org/10.1016/j.geomphys.2006.10.005`

[8] A. Cintra, F. Mercuri, and I. Onnis, *Minimal surfaces in* 4*-dimensional Lorentzian Damek-Ricci spaces*, arXiv:1501.03427.

[9] E. Damek and F. Ricci, *A class of nonsymmetric harmonic Riemannian spaces*, Bull. Amer. Math. Soc. (N.S.) **27** (1992), no. 1, 139–142. `https://doi.org/10.1090/S0273-0979-1992-00293-8`

[10] S. Degla and L. Todjihounde, *Biharmonic curves in four-dimensional Damek-Ricci spaces*, J. Math. Sci. Adv. Appl. **5** (2010), no. 1, 19–27.

[11] J. C. Díaz-Ramos and M. Domínguez-Vázquez, *Isoparametric hypersurfaces in Damek-Ricci spaces*, Adv. Math. **239** (2013), 1–17. `https://doi.org/10.1016/j.aim.2013.02.010`

[12] P. M. Gadea and J. A. Oubiña, *Homogeneous pseudo-Riemannian structures and homogeneous almost para-Hermitian structures*, Houston J. Math. **18** (1992), no. 3, 449–465.

[13] P. M. Gadea and J. A. Oubiña, *Homogeneous Lorentzian structures on the oscillator groups*, Arch. Math. (Basel) **73** (1999), no. 4, 311–320. https://doi.org/10.1007/s000130050403

[14] P. M. Gadea and A. P. Ramos, *Homogeneous Lorentzian structures on some Gödel-Levichev's spacetimes, and associated reductive decompositions*, Balkan J. Geom. Appl. **8** (2003), no. 1, 45–51.

[15] M. Koivogui and L. Todjihounde, *Weierstrass representation for minimal immersions into Damek-Ricci spaces*, Int. Electron. J. Geom. **6** (2013), no. 1, 1–7.

[16] N. Sidhoumi, *Ricci solitons of four-dimensional Lorentzian Damek–Ricci spaces*, J. Math. Phys. Anal. Geom. **16** (2020), no. 2, 1–10.

[17] J. Tan and S. Deng, *Some geometrical properties of four-dimensional Lorentzian Damek-Ricci spaces*, Publ. Math. Debrecen **89** (2016), no. 1-2, 105–124.

[18] F. Tricerri and L. Vanhecke, *Homogeneous structures on Riemannian manifolds*, London Mathematical Society Lecture Note Series, 83, Cambridge University Press, Cambridge, 1983. https://doi.org/10.1017/CBO9781107325531

Assia Mostefaoui
École Nationale Polytechnique d'Oran-Maurice Audin
B.P 1523 El M'naouar Oran 31000, Algeria
Email address: assia.mostefaoui@enp-oran.dz

Noura Sidhoumi
École Nationale Polytechnique d'Oran-Maurice Audin
B.P 1523 El M'naouar Oran 31000, Algeria
Email address: noura.sidhoumi@enp-oran.dz