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FUNCTIONS SUBORDINATE TO THE EXPONENTIAL FUNCTION

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To the memory of Professor Subhash Chander Arora

Abstract. We use the theory of differential subordination to explore various inequalities that are satisfied by an analytic function p defined on the unit disc so that the function p is subordinate to the function e^z. These results are applied to find sufficient conditions for the normalised analytic functions f defined on the unit disc to satisfy the subordination $zf'(z)/f(z) \prec e^z$.

1. Introduction and preliminaries

For $a \in \mathbb{C}$ and $n \in \mathbb{N}$, let $\mathcal{H}[a, n]$ denote the class of all analytic functions f, of the form $f(z) = a + \sum_{k=n}^{\infty} a_k z^k$, defined on the unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. With $\mathcal{H}_1 := \mathcal{H}[1, 1]$, we write $\mathcal{A} := \{zf : f \in \mathcal{H}_1\}$. The subclass of $\mathcal{A}$ consisting of functions univalent in $\mathbb{D}$ is denoted by $\mathcal{S}$. Geometric function theory, as the name suggests, is the study of functions in the class $\mathcal{A}$ that has specific geometric properties. A function $f \in \mathcal{A}$ is starlike if $f(\mathbb{D})$ is starlike with respect to the origin or equivalently if $\operatorname{Re}(zf'(z)/f(z)) > 0$ for all $z \in \mathbb{D}$. Similarly, a function $f \in \mathcal{A}$ is convex if $f(\mathbb{D})$ is convex or equivalently if $\operatorname{Re}(1 + zf''(z)/f'(z)) > 0$ for all $z \in \mathbb{D}$. The class of all starlike functions $f \in \mathcal{A}$ is denoted by $\mathcal{S}^*$ and that of all convex functions $f \in \mathcal{A}$ is denoted by $\mathcal{K}$. Another interesting subclass of functions that has geometric significance is the class of close-to-convex functions. A function $f \in \mathcal{A}$ is said to be close-to-convex if there exists a convex function g such that $\operatorname{Re}(f'(z)/g'(z)) > 0$ for all $z \in \mathbb{D}$. Every close-to-convex function maps the unit disc $\mathbb{D}$ onto a domain whose complement is union of a family of non-intersecting half-lines and is univalent. Geometrically, it is obvious that $\mathcal{K} \subset \mathcal{S}^*$. Analytically, this

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is equivalent to the implication $\operatorname{Re}(p(z) + zp'(z)/p(z)) > 0 \Longrightarrow \operatorname{Re} p(z) > 0$ where $p(z) := zf'(z)/f(z)$. The study of such implications leads to the theory of differential subordination.

For two functions f and g in $\mathcal{H}[a, n]$, we say that the function f is subordinate to the function g, written $f \prec g$ or $f(z) \prec g(z)$ $(z \in \mathbb{D})$, if there exists a function $w \in \mathcal{B}$ such that $f = g \circ w$, where $\mathcal{B}$ is the class of all analytic functions $w : \mathbb{D} \to \mathbb{D}$ with $w(0) = 0$. The functions in class $\mathcal{B}$ are precisely the functions that satisfy the familiar Schwarz lemma. If the function g is univalent, then it follows that $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. Let p be an analytic function defined in the unit disc $\mathbb{D}$ and let the function $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$. The theory of differential subordination deals with the following implication:

$$\{\psi(p(z), zp'(z), z^2p''(z); z) : z \in \mathbb{D}\} \subset \Omega \Rightarrow \{p(z) : z \in \mathbb{D}\} \subset \Delta, \tag{1.1}$$

where Ω and Δ are given domains in $\mathbb{C}$. If Ω and Δ are simply connected domains that are not the whole complex plane, then the Riemann mapping theorem guarantees the existence of univalent functions h and q defined on the unit disc $\mathbb{D}$ that maps $\mathbb{D}$ respectively onto Ω and Δ such that $h(0) = \psi(p(0), 0, 0; 0)$ and $q(0) = p(0)$. If $\psi(p(z), zp'(z), z^2p''(z); z)$ is analytic, the implication (1.1) can be rewritten as

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z) \Rightarrow p(z) \prec q(z).$$

The subordination $\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z)$ is known as second order differential subordination. An analytic function p is called a solution of the differential subordination

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z) \tag{1.2}$$

if p satisfies the second order differential subordination (1.2). A univalent function q is said to be a dominant of all solutions of the differential subordination (1.2) if the subordination $p \prec q$ holds for all p satisfying (1.2). A function q^* is the best dominant of (1.2) if q^* is a dominant of (1.2) and $q^* \prec q$ for all dominants q of (1.2). A function q will be the best dominant of (1.2) if q is both a dominant and a solution of (1.2). We recall some basic definitions and theorems that serve as a foundation for the theory of differential subordination.

Definition 1.1 ([10])**.** The class of all functions q that are analytic and univalent on $\overline{\mathbb{D}} \backslash E(q)$, where $E(q)$ consists of all points $\zeta \in \partial\mathbb{D}$ for which $q(z) \to \infty$ as $z \to \zeta$, is denoted by $\mathcal{Q}$, and is called the class of all functions with nice boundary.

Definition 1.2 ([10])**.** Let Ω be a subset of $\mathbb{C}$ and the function $q \in \mathcal{Q}$. The class of admissible functions, denoted by $\Psi_n(\Omega, q)$, consists of all functions $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ that satisfy the admissibility condition $\psi(r, s, t; z) \notin \Omega$ when $r = q(\zeta)$, $s = m\zeta q'(\zeta)$,

$$\operatorname{Re}\left(\frac{t}{s} + 1\right) \geq m \operatorname{Re}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right),$$

where $\zeta \in \partial\mathbb{D}\backslash E(q)$ and m is a real number such that $m \geq n$. Also $\Psi(\Omega, q) := \Psi_1(\Omega, q)$.

Theorem 1.3 ([10])**.** *Let the function ψ belong to $\Psi_n(\Omega, q)$ and let $q(0) = a$. If the function $p \in \mathcal{H}[a, n]$ satisfies the condition*

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega$$

for $z \in \mathbb{D}$, then the function p is subordinate to the function q.

Naz et al. [11] has investigated the properties of the class $\Psi(\Omega, q)$ where $q(z) = e^z$ and has derived the admissibility condition for the admissibility class $\Psi(\Omega, e^z)$. In this case, Definition 1.2 and Theorem 1.3 reduce to Definition 1.4 and Theorem 1.5, respectively.

Definition 1.4 ([11])**.** Let Ω be a domain in $\mathbb{C}$. Then $\Psi(\Omega, e^z)$ is defined to be the class of admissible functions consisting of all functions $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ that satisfy the admissibility condition $\psi(r, s, t; z) \notin \Omega$ whenever $r = e^{e^{i\theta}}$, $s = me^{i\theta}e^{e^{i\theta}}$ and $\operatorname{Re}(t/s + 1) \geq m(1 + \cos\theta)$ where $\theta \in [0, 2\pi)$, $m \geq 1$ and $z \in \mathbb{D}$.

Theorem 1.5 ([11])**.** *Let the function $p \in \mathcal{H}_1$. If the function $\psi \in \Psi(\Omega, e^z)$, then*

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega$$

for $z \in \mathbb{D}$, implies $p(z) \prec e^z$.

Applying Theorem 1.5, this paper focuses on finding various differential inequalities which when satisfied by functions in the class $\mathcal{H}_1$ imply that the functions are subordinate to e^z. Miller and Mocanu [10] and other authors in [1–4, 6–9, 12–14] and [15] have studied subordination and investigated several inequalities that serve as sufficient conditions for functions to be bounded, starlike, convex etc. For a function $p \in \mathcal{H}_1$, Naz et al. [11] had found the upper bound for the functions

$$p(z) + (1 + 2e)zp'(z), (1 + (1 + \sqrt{2})ezp'(z))^2 - 1, zp'(z), 2zp'(z) + z^2p''(z),$$
$$(p(z))^2 - p(z) + (1 + e)zp'(z), \frac{zp'(z)}{p(z)} \quad \text{and} \quad \frac{zp'(z)}{(p(z))^2},$$

so that $p(z) \prec e^z$. In this paper, for $p \in \mathcal{H}_1$ and under certain conditions on α, β, γ, n_1, n_2 and n_3, we determine the upper bound of functions like

$$\alpha p(z) + \beta zp'(z) + \gamma z^2p''(z), \alpha p(z) + \beta\frac{zp'(z)}{p(z)} + \gamma\frac{z^2p''(z)}{p(z)},$$
$$(p(z))^{n_1}(zp'(z))^{n_2}\left(\alpha + \beta\frac{zp''(z)}{p'(z)}\right) \quad \text{and} \quad (p(z))^{n_1} + \beta\frac{(zp'(z))^{n_2}}{(p(z))^{n_3}},$$

so that the function p is subordinate to e^z. Some results in [11] have been generalised in this paper. The results in this paper can be applied to functions $f \in \mathcal{A}$ which will then give sufficient conditions for the function f to belong to $\mathcal{S}_e^*$, the class of starlike functions associated with the exponential function; the class $\mathcal{S}_e^*$, introduced and studied by Mendiratta et al. [5], consists of all functions $f \in \mathcal{S}$ satisfying the subordination $zf'(z)/f(z) \prec e^z$. A function $f \in \mathcal{S}_e^*$ satisfies the inequality $|\log(zf'(z)/f(z))| < 1$ for z in the unit disc $\mathbb{D}$. The sufficient conditions will follow if we take $p(z) = zf'(z)/f(z)$ and so the details are omitted.

2. Main results

Naz et al. [11] proved that if the function $p \in \mathcal{H}_1$, then

$$\left|2zp'(z) + z^2p''(z)\right| < e^{-1}$$

for all $z \in \mathbb{D}$ implies that $p(z) \prec e^z$. Our next theorem gives a generalisation of the result.

Theorem 2.1. *Let α and β be complex numbers and γ be a non-negative real number such that $|\alpha| < (\operatorname{Re}\beta - \gamma)/(e^2 + e)$. If $p \in \mathcal{H}_1$ satisfies the inequality*

$$\left|\alpha p(z) + \beta zp'(z) + \gamma z^2p''(z)\right| < e^{-1}(\operatorname{Re}\beta - \gamma) - |\alpha|e \quad (z \in \mathbb{D}), \tag{2.1}$$

then the function p is subordinate to e^z.

Proof. Define the function $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ by

$$\psi(r, s, t; z) := \alpha r + \beta s + \gamma t$$

and let the set $\Omega \subset \mathbb{C}$ be defined by

$$\Omega := \{w \in \mathbb{C} : |w| < e^{-1}(\operatorname{Re}\beta - \gamma) - |\alpha|e\}.$$

By (2.1), we have $\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega$ for $z \in \mathbb{D}$. The theorem is proved by verifying the admissibility condition given in Definition 1.4, or in other words, by showing that $e^{-1}(\operatorname{Re}\beta - \gamma) - |\alpha|e$ is a lower bound for the term $|\alpha r + \beta s + \gamma t|$, where $r = e^{e^{i\theta}}$, $s = me^{i\theta}e^{e^{i\theta}}$, $\operatorname{Re}(t/s + 1) \geq m(1 + \cos\theta)$, $\theta \in [0, 2\pi)$, $m \geq 1$ and $z \in \mathbb{D}$. The following inequalities will be useful in the sequel:

$$e^{-n} \leq |s^n| \leq |r^n| \leq e^n \quad \text{and} \quad \operatorname{Re}\left(\frac{t}{s}\right) \geq -1 \quad (n \in \mathbb{N} \cup \{0\}).$$

The inequality $|r^n| \leq e^n$ follows since

$$|r^n| = \left|e^{ne^{i\theta}}\right| = e^{n\cos\theta} \leq e^n.$$

The inequality $|s^n| \geq |r^n| \geq e^{-n}$ follows as $m \geq 1$, $\left|e^{in\theta}\right| = 1$ and

$$|s^n| = |(me^{i\theta}r)^n| = |m^n||e^{in\theta}||r^n| \geq |r^n| = e^{n\cos\theta} \geq e^{-n}.$$

The inequality $\operatorname{Re}(t/s) \geq -1$ follows at once from the inequality $\operatorname{Re}(t/s + 1) \geq m(1 + \cos\theta)$.

For $r = e^{e^{i\theta}}$, $s = me^{i\theta}e^{e^{i\theta}}$, $\operatorname{Re}(t/s+1) \geq m(1+\cos\theta)$, $\theta \in [0, 2\pi)$, $m \geq 1$ and $z \in \mathbb{D}$, we have

$$\begin{aligned}
|\psi(r,s,t;z)| &= |\alpha r + \beta s + \gamma t| \\
&\geq |\beta s + \gamma t| - |\alpha r| \\
&= |s| \left| \beta + \gamma \frac{t}{s} \right| - |\alpha r| \\
&\geq e^{-1} \left| \beta + \gamma \frac{t}{s} \right| - |\alpha r| \\
&\geq e^{-1} \left(\operatorname{Re}\beta + \gamma \operatorname{Re}\frac{t}{s} \right) - |\alpha r| \\
&\geq e^{-1}(\operatorname{Re}\beta - \gamma) - |\alpha| e.
\end{aligned}$$

Thus it is proved that the function ψ belongs to the class $\Psi(\Omega, e^z)$. The result hence follows from Theorem 1.5. □

Remark 2.2. For $\alpha = 0$, $\beta = 2$ and $\gamma = 1$, Theorem 2.1 reduces to [11, Ex 2.6].

Theorem 2.3. *Let α and β be complex numbers and γ be a non-negative real number such that $|\alpha| < (\operatorname{Re}\beta - \gamma)/(1+e)$. If the function $p \in \mathcal{H}_1$ satisfies the inequality*

$$\left| \alpha p(z) + \beta \frac{zp'(z)}{p(z)} + \gamma \frac{z^2 p''(z)}{p(z)} \right| < \operatorname{Re}\beta - \gamma - |\alpha| e \quad (z \in \mathbb{D}), \tag{2.2}$$

then the function p is subordinate to e^z.

Proof. Define the function $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ by

$$\psi(r,s,t;z) := \alpha r + \beta s/r + \gamma t/r$$

and the set $\Omega \subset \mathbb{C}$ by

$$\Omega := \{w \in \mathbb{C} : |w| < \operatorname{Re}\beta - \gamma - |\alpha| e\}.$$

For $r = e^{e^{i\theta}}$, $s = me^{i\theta}e^{e^{i\theta}}$, $\operatorname{Re}(t/s+1) \geq m(1+\cos\theta)$, $\theta \in [0, 2\pi)$, $m \geq 1$ and $z \in \mathbb{D}$, we have

$$\begin{aligned}
\left| \alpha r + \beta \frac{s}{r} + \gamma \frac{t}{r} \right| &\geq \left| \beta \frac{s}{r} + \gamma \frac{t}{r} \right| - |\alpha r| \\
&= \left| \frac{s}{r} \right| \left| \beta + \gamma \frac{t}{s} \right| - |\alpha r|.
\end{aligned}$$

Since $|s/r| = \left| me^{i\theta} \right| = m \geq 1$, it then follows that

$$\begin{aligned}
\left| \alpha r + \beta \frac{s}{r} + \gamma \frac{t}{r} \right| &\geq \left| \beta + \gamma \frac{t}{s} \right| - |\alpha r| \\
&\geq \operatorname{Re}\beta + \gamma \operatorname{Re}\frac{t}{s} - |\alpha r| \\
&\geq \operatorname{Re}\beta - \gamma - |\alpha| e,
\end{aligned}$$

which shows that the function $\psi \in \Psi(\Omega, e^z)$. The inequality (2.2) shows that $\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega$ for $z \in \mathbb{D}$. By Theorem 1.5, the result follows. □

For $\alpha = 0$, Theorem 2.3 reduces to the following corollary.

Corollary 2.4. *Let β be a complex number and γ be a non-negative real number with $\operatorname{Re}\beta > \gamma$. If $p \in \mathcal{H}_1$ satisfies the inequality*

$$\left|\frac{\beta zp'(z) + \gamma z^2 p''(z)}{p(z)}\right| < \operatorname{Re}\beta - \gamma \quad (z \in \mathbb{D}),$$

then the function p is subordinate to e^z.

Theorem 2.5. *Let n_1 be a non-negative integer, n_2 be a natural number, α be a complex number and β be a non-negative real number such that $\operatorname{Re}\alpha > \beta$. If the function $p \in \mathcal{H}_1$ satisfies the condition*

$$\left|(p(z))^{n_1}(zp'(z))^{n_2}\left(\alpha + \beta\frac{zp''(z)}{p'(z)}\right)\right| < e^{-(n_1+n_2)}(\operatorname{Re}\alpha - \beta) \quad (z \in \mathbb{D}), \tag{2.3}$$

then the function p is subordinate to e^z.

Proof. The theorem is proved by showing that the function $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ defined by

$$\psi(r, s, t; z) := r^{n_1} s^{n_2}(\alpha + \beta t/s)$$

belongs to the class $\Psi(\Omega, e^z)$, where

$$\Omega := \{w \in \mathbb{C} : |w| < e^{-(n_1+n_2)}(\operatorname{Re}\alpha - \beta)\}.$$

Observe that

$$\begin{aligned}\left|r^{n_1}s^{n_2}\left(\alpha + \beta\frac{t}{s}\right)\right| &= |r^{n_1}|\,|s^{n_2}|\left|\alpha + \beta\frac{t}{s}\right| \\ &\geq e^{-(n_1+n_2)}\left|\alpha + \beta\frac{t}{s}\right| \\ &\geq e^{-(n_1+n_2)}\left(\operatorname{Re}\alpha + \beta\operatorname{Re}\frac{t}{s}\right) \\ &\geq e^{-(n_1+n_2)}\left(\operatorname{Re}\alpha - \beta\right)\end{aligned}$$

for $r = e^{e^{i\theta}}$, $s = me^{i\theta}e^{e^{i\theta}}$, $\operatorname{Re}(t/s+1) \geq m(1+\cos\theta)$, $\theta \in [0, 2\pi)$, $m \geq 1$ and $z \in \mathbb{D}$. Thus the function $\psi \in \Psi(\Omega, e^z)$. By (2.3), we have $\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega$ for $z \in \mathbb{D}$, and so the result follows from Theorem 1.5. □

Theorem 2.6. *Let α and β be positive real numbers and let n be a non-negative integer. Then each of the following inequality, for all $z \in \mathbb{D}$, is sufficient for $p \in \mathcal{H}_1$ to be subordinate to e^z:*

(1) $\left|p(z) + \beta z^3 p'(z)p''(z)/p(z)\right| + \alpha\,|zp'(z)/p(z)| < \alpha - e^{-1}(\beta+1)$, $(e\alpha - \beta > 1 + e)$;

(2) $\operatorname{Re}((p(z))^n + \beta zp''(z)/p'(z)) + \alpha |zp'(z)/p(z)| < \alpha - \beta - e^n$, $(\alpha - \beta > 1 + e^n)$;
(3) $|p(z) + \beta z^3 p'(z)p''(z)/p(z)| + \alpha |zp'(z)| < e^{-1}(\alpha - \beta - 1)$, $(\alpha - \beta > 1 + e)$;
(4) $\operatorname{Re}((p(z))^n + \beta zp''(z)/p'(z)) + \alpha |zp'(z)| < \alpha e^{-1} - \beta - e^n$, $(\alpha e^{-1} - \beta > 1 + e^n)$.

Proof. We prove the theorem by verifying the admissibility condition given in Definition 1.4.

(1) Put $\Omega := \mathbb{C} \setminus [\alpha - e^{-1}(\beta + 1), \infty)$. We prove the theorem by showing that the function $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ defined by

$$\psi(r, s, t; z) := |r + \beta st/r| + \alpha |s/r|$$

belongs to the class $\Psi(\Omega, e^z)$. For $r = e^{e^{i\theta}}$, $s = me^{i\theta}e^{e^{i\theta}}$, $\operatorname{Re}(t/s + 1) \geq m(1 + \cos\theta)$, $\theta \in [0, 2\pi)$, $m \geq 1$ and $z \in \mathbb{D}$, we have

$$\begin{aligned}\left|r + \beta\frac{st}{r}\right| + \alpha\left|\frac{s}{r}\right| &= |r|\left|1 + \beta\frac{st}{r^2}\right| + \alpha\left|\frac{s}{r}\right| \\ &= |r|\left|1 + \beta\frac{me^{i\theta}t}{r}\right| + \alpha\left|\frac{s}{r}\right| \\ &\geq e^{-1}\left(\beta\left|\frac{t}{r}\right| - 1\right) + \alpha\left|me^{i\theta}\right| \\ &\geq e^{-1}\left(\beta\left|\frac{t}{r}\right| - 1\right) + \alpha.\end{aligned}$$

Since $|t/r| = |t/s|\,|s/r|$ and $|t/s| \geq \operatorname{Re}(t/s)$, it follows that

$$\begin{aligned}\left|r + \beta\frac{st}{r}\right| + \alpha\left|\frac{s}{r}\right| &\geq e^{-1}\left(\beta\left|\frac{t}{s}\right|\left|\frac{s}{r}\right| - 1\right) + \alpha \\ &\geq e^{-1}\left(\beta\operatorname{Re}\frac{t}{s} - 1\right) + \alpha \\ &\geq e^{-1}(-\beta - 1) + \alpha \\ &= \alpha - e^{-1}(\beta + 1),\end{aligned}$$

and hence $\psi \in \Psi(\Omega, e^z)$. Since $\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega$ for $z \in \mathbb{D}$, the result follows by Theorem 1.5.

(2) Define the set $\Omega \subset \mathbb{C}$ by $\Omega := \mathbb{C} \setminus [\alpha - \beta - e^n, \infty)$ and the function $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ by

$$\psi(r, s, t; z) := \operatorname{Re}(r^n + \beta t/s) + \alpha |s/r|$$

so that $\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega$ for $z \in \mathbb{D}$. Using $\operatorname{Re}(r^n) \geq -|r^n| \geq -e^n$, we get

$$\begin{aligned}\operatorname{Re}\left(r^n + \beta\frac{t}{s}\right) + \alpha\left|\frac{s}{r}\right| &= \operatorname{Re} r^n + \beta\operatorname{Re}\frac{t}{s} + \alpha\left|me^{i\theta}\right| \\ &\geq -e^n - \beta + \alpha = \alpha - \beta - e^n\end{aligned}$$

whenever $r = e^{e^{i\theta}}$, $s = me^{i\theta}e^{e^{i\theta}}$, $\operatorname{Re}(t/s+1) \geq m(1+\cos\theta)$, $\theta \in [0, 2\pi)$, $m \geq 1$ and $z \in \mathbb{D}$. Thus $p(z) \prec e^z$ by Theorem 1.5.

(3) Define the function $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ by

$$\psi(r, s, t; z) := |r + \beta st/r| + \alpha\,|s|$$

and the set $\Omega \subset \mathbb{C}$ by $\Omega := \mathbb{C} \setminus [e^{-1}(\alpha - \beta - 1), \infty)$. Then, for $r = e^{e^{i\theta}}$, $s = me^{i\theta}e^{e^{i\theta}}$, $\operatorname{Re}(t/s+1) \geq m(1+\cos\theta)$, $\theta \in [0, 2\pi)$, $m \geq 1$ and $z \in \mathbb{D}$, we get

$$\begin{aligned}
\left| r + \beta \frac{st}{r} \right| + \alpha\,|s| &= |r| \left| 1 + \beta \frac{st}{r^2} \right| + \alpha|s| \\
&\geq e^{-1} \left| 1 + \beta \frac{me^{i\theta}t}{r} \right| + \alpha|s| \\
&\geq e^{-1} \left(\beta \left| \frac{t}{r} \right| - 1 \right) + \alpha|s| \\
&\geq e^{-1}(-\beta - 1) + \alpha e^{-1} \\
&= e^{-1}(\alpha - \beta - 1),
\end{aligned}$$

which shows that the function $\psi \in \Psi(\Omega, e^z)$. Since

$$\left| p(z) + \beta z^3 p'(z) p''(z)/p(z) \right| + \alpha\,|zp'(z)| < e^{-1}(\alpha - \beta - 1)$$

for $z \in \mathbb{D}$, we get $\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega$ for z in the unit disc $\mathbb{D}$. Hence $p(z) \prec e^z$ by Theorem 1.5.

(4) In this case, the function $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ is defined by

$$\psi(r, s, t; z) := \operatorname{Re}(r^n + \beta t/s) + \alpha|s|.$$

If we let $\Omega := \mathbb{C} \setminus [\alpha e^{-1} - \beta - e^n, \infty)$, then by proceeding as in the proof of (2), we get

$$\operatorname{Re}\left(r^n + \beta \frac{t}{s} \right) + \alpha|s| \geq \alpha e^{-1} - \beta - e^n$$

whenever $r = e^{e^{i\theta}}$, $s = me^{i\theta}e^{e^{i\theta}}$, $\operatorname{Re}(t/s+1) \geq m(1+\cos\theta)$, $\theta \in [0, 2\pi)$, $m \geq 1$ and $z \in \mathbb{D}$. The result follows by Theorem 1.5 since $\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega$ for $z \in \mathbb{D}$. □

The next two theorems are generalisations of the following result in [11]: If $p \in \mathcal{H}_1$ satisfies the inequality

$$|p(z) + (1 + 2e)zp'(z)| < 2$$

for all $z \in \mathbb{D}$, then the function p is subordinate to e^z.

Theorem 2.7. *Let n_1 be a non-negative integer, n_2 be a natural number, n_3 be an integer and β be a complex number.*

(1) *If* $n_2 - n_1 - n_3 \geq 0$ *and* $|\beta| > e^{(n_2-n_3)}(1 + e^{-n_1})$ *and* $p \in \mathcal{H}_1$ *satisfies the inequality*

$$\left|(p(z))^{n_1} + \beta \frac{(zp'(z))^{n_2}}{(p(z))^{n_3}}\right| < |\beta| e^{(n_3-n_2)} - e^{-n_1} \quad (z \in \mathbb{D}),$$

then the function p *is subordinate to* e^z;

(2) *If* $n_2 - n_1 - n_3 \leq 0$ *and* $|\beta| > e^{(n_1+n_3-n_2)}(e^{n_1} + 1)$ *and* $p \in \mathcal{H}_1$ *satisfies the inequality*

$$\left|(p(z))^{n_1} + \beta \frac{(zp'(z))^{n_2}}{(p(z))^{n_3}}\right| < |\beta| e^{(n_2-n_3-2n_1)} - e^{-n_1} \quad (z \in \mathbb{D}),$$

then the function p *is subordinate to* e^z.

Proof. Consider the function $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ defined by

$$\psi(r, s, t; z) := r^{n_1} + \beta s^{n_2}/r^{n_3}.$$

Observe that for $r = e^{e^{i\theta}}$, $s = me^{i\theta}e^{e^{i\theta}}$, $\mathrm{Re}(t/s+1) \geq m(1+\cos\theta)$, $\theta \in [0, 2\pi)$, $m \geq 1$ and $z \in \mathbb{D}$, we have

$$\begin{aligned}\left|r^{n_1} + \beta \frac{s^{n_2}}{r^{n_3}}\right| &= |r^{n_1}| \left|1 + \beta m^{n_2} e^{in_2\theta} r^{(n_2-n_3-n_1)}\right| \\ &= e^{n_1 \cos\theta} \left|1 + \beta m^{n_2} e^{in_2\theta} r^{(n_2-n_3-n_1)}\right| \\ &\geq e^{-n_1} \left(|\beta| \left|r^{(n_2-n_1-n_3)}\right| - 1\right) \\ &= e^{-n_1} \left(|\beta| e^{(n_2-n_1-n_3)\cos\theta} - 1\right).\end{aligned}$$

(1) If $n_2 - n_1 - n_3 \geq 0$, then

$$(n_2 - n_1 - n_3)\cos\theta \geq -(n_2 - n_1 - n_3) = n_1 + n_3 - n_2$$

and hence

$$e^{(n_2-n_1-n_3)\cos\theta} \geq e^{(n_1+n_3-n_2)}.$$

Therefore

$$\left|r^{n_1} + \beta \frac{s^{n_2}}{r^{n_3}}\right| \geq e^{-n_1} \left(|\beta| e^{(n_1+n_3-n_2)} - 1\right) = |\beta| e^{(n_3-n_2)} - e^{-n_1}.$$

If we define the set $\Omega \subset \mathbb{C}$ by

$$\Omega := \{w \in \mathbb{C} : |w| < |\beta| e^{(n_3-n_2)} - e^{-n_1}\}$$

for $|\beta| > e^{(n_2-n_3)}(1 + e^{-n_1})$, then the above calculations show that $\psi \in \Psi(\Omega, e^z)$. The result follows by Theorem 1.5 as

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega$$

for $z \in \mathbb{D}$.

(2) In this case, $n_2 - n_1 - n_3 \leq 0$. Then

$$(n_2 - n_1 - n_3)\cos\theta \geq (n_2 - n_1 - n_3)$$

and
$$e^{(n_2-n_1-n_3)\cos\theta} \geq e^{(n_2-n_1-n_3)}.$$
Hence
$$\begin{aligned}\left|r^{n_1} + \beta\frac{s^{n_2}}{r^{n_3}}\right| &\geq e^{-n_1}\left(|\beta|e^{(n_2-n_1-n_3)} - 1\right)\\ &= |\beta|e^{(n_2-n_3-2n_1)} - e^{-n_1}.\end{aligned}$$
Define the set $\Omega \subset \mathbb{C}$ in this case by
$$\Omega := \{w \in \mathbb{C} : |w| < |\beta|e^{(n_2-n_3-2n_1)} - e^{-n_1}\}$$
for $|\beta| > e^{(n_1+n_3-n_2)}(e^{n_1}+1)$. Then $\psi \in \Psi(\Omega, e^z)$ and
$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega$$
for $z \in \mathbb{D}$. Hence $p(z) \prec e^z$. □

Theorem 2.8. *Let α and β be complex numbers with $|\beta| > |\alpha| + |1-\alpha|e + e$. If the function $p \in \mathcal{H}_1$ satisfies the inequality*
$$\left|(1-\alpha)(p(z))^2 + \alpha p(z) + \beta zp'(z)\right| < e^{-1}(|\beta| - |\alpha| - |1-\alpha|e) \quad (z \in \mathbb{D}),$$
then the function p is subordinate to e^z.

Proof. Consider the function $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ defined by
$$\psi(r, s, t; z) := (1-\alpha)r^2 + \alpha r + \beta s.$$
Then, for $r = e^{e^{i\theta}}$, $s = me^{i\theta}e^{e^{i\theta}}$, $\mathrm{Re}(t/s+1) \geq m(1+\cos\theta)$, $\theta \in [0, 2\pi)$, $m \geq 1$ and $z \in \mathbb{D}$, we get the following inequality:
$$\begin{aligned}\left|(1-\alpha)r^2 + \alpha r + \beta s\right| &= |r|\left|(1-\alpha)r + \alpha + \beta me^{i\theta}\right|\\ &\geq e^{-1}\left|(1-\alpha)r + \alpha + \beta me^{i\theta}\right|\\ &\geq e^{-1}\left(|\beta| - |\alpha + (1-\alpha)r|\right)\\ &\geq e^{-1}\left(|\beta| - |\alpha| - |1-\alpha||r|\right)\\ &\geq e^{-1}\left(|\beta| - |\alpha| - |1-\alpha|e\right).\end{aligned}$$
Now define the set $\Omega \subset \mathbb{C}$ by
$$\Omega := \{w \in \mathbb{C} : |w| < e^{-1}\left(|\beta| - |\alpha| - |1-\alpha|e\right)\}$$
so that the above inequality and the hypothesis of the theorem respectively shows that $\psi \in \Psi(\Omega, e^z)$ and $\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega$ for $z \in \mathbb{D}$. The result then follows by Theorem 1.5. □

Remark 2.9. The result in [11, Ex 2.2] is obtained either by substituting $n_1 = 1$, $n_2 = 2$, $n_3 = 0$ and $\beta = 1 + 2e$ in Theorem 2.7 or by substituting $\alpha = 1$ and $\beta = 1 + 2e$ in Theorem 2.8.

Theorem 2.10. *Let n_1 be a non-negative integer, n_2 be a natural number and α and β be complex numbers. If the function $p \in \mathcal{H}_1$ satisfies any of the following conditions, then the function p is subordinate to e^z:*

(1) $|(zp'(z))^{n_2}/(\alpha - (p(z))^{n_1})| < e^{-n_2}/(|\alpha| + e^{n_1})$ $(z \in \mathbb{D})$;
(2) $|(zp'(z))^{n_2}/(\alpha(1 + (p(z))^{n_1}))| < e^{-n_2}/(|\alpha|(1 + e^{n_1}))$ $(z \in \mathbb{D})$;
(3) $|(p(z))^{n_1} + (\alpha(zp'(z))^{n_2})/(1 + \beta(p(z))^{n_1})| < |\alpha|e^{-n_2}/(1+|\beta|e^{n_1}) - e^{n_1}$, $(|\alpha| > e^{n_2}(1 + e^{n_1})(1 + |\beta|e^{n_1}))$, $(z \in \mathbb{D})$.

Proof. (1) Let us consider the function $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ defined by

$$\psi(r, s, t; z) := s^{n_2}/(\alpha - r^{n_1})$$

and the set $\Omega \subset \mathbb{C}$ defined by

$$\Omega := \{w \in \mathbb{C} : |w| < e^{-n_2}/(|\alpha| + e^{n_1})\}.$$

Then it can easily be seen that

$$|\alpha - r^{n_1}| \le |\alpha| + |r^{n_1}| \le |\alpha| + e^{n_1}$$

and thus

$$\left|\frac{s^{n_2}}{\alpha - r^{n_1}}\right| \ge \frac{e^{-n_2}}{|\alpha| + e^{n_1}}$$

whenever $r = e^{e^{i\theta}}$, $s = me^{i\theta}e^{e^{i\theta}}$, $\mathrm{Re}(t/s+1) \ge m(1+\cos\theta)$, $\theta \in [0, 2\pi)$, $m \ge 1$ and $z \in \mathbb{D}$. Thus the result follows by Theorem 1.5.

(2) Here the function $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ is defined by

$$\psi(r, s, t; z) := s^{n_2}/(\alpha(1 + r^{n_1})).$$

As in the proof of (1), it can be shown that the function $\psi \in \Psi(\Omega, e^z)$ and $\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega$ for $z \in \mathbb{D}$, where $\Omega \subset \mathbb{C}$ is defined by

$$\Omega := \{w \in \mathbb{C} : |w| < e^{-n_2}/(|\alpha|(1 + e^{n_1}))\}.$$

(3) Since $|1 + \beta r^{n_1}| \le 1 + |\beta|e^{n_1}$, $|r^{n_1}| \le e^{n_1}$ and $|s^{n_2}| \ge e^{-n_2}$, we get

$$\left|r^{n_1} + \frac{\alpha s^{n_2}}{1 + \beta r^{n_1}}\right| \ge \frac{|\alpha||s^{n_2}|}{1 + |\beta|e^{n_1}} - |r^{n_1}| \ge \frac{|\alpha|e^{-n_2}}{1 + |\beta|e^{n_1}} - e^{n_1}$$

whenever $r = e^{e^{i\theta}}$, $s = me^{i\theta}e^{e^{i\theta}}$, $\mathrm{Re}(t/s+1) \ge m(1+\cos\theta)$, $\theta \in [0, 2\pi)$, $m \ge 1$ and $z \in \mathbb{D}$. Therefore by defining $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ by

$$\psi(r, s, t; z) := r^{n_1} + \alpha s^{n_2}/(1 + \beta r^{n_1})$$

and the set $\Omega \subset \mathbb{C}$ by

$$\Omega := \{w \in \mathbb{C} : |w| < |\alpha|e^{-n_2}/(1 + |\beta|e^{n_1}) - e^{n_1}\},$$

the required result is obtained by applying Theorem 1.5. □

Theorem 2.11. *Let the function p belongs to the class $\mathcal{H}_1$, n_1 be a non-negative integer, n_2 be a natural number and α be a complex number. Then the following holds*:

(1) *If $|\alpha| < e^{-n_1}$, then*

$$|((p(z))^{n_1} - \alpha)(zp'(z))^{n_2}| < e^{-n_2}(e^{-n_1} - |\alpha|) \ (z \in \mathbb{D})$$

implies the function p is subordinate to e^z.

(2) *If* $|\alpha| > e^{n_1}$, *then*

$$|((p(z))^{n_1} - \alpha)(zp'(z))^{n_2}| < e^{-n_2}(|\alpha| - e^{n_1}) \ (z \in \mathbb{D})$$

implies the function p *is subordinate to* e^z.

(3) *If* $|\alpha| < e^{-n_1}$, *then*

$$\left|((p(z))^{n_1} - \alpha)\left(\frac{zp'(z)}{p(z)}\right)^{n_2}\right| < e^{-n_1} - |\alpha| \ (z \in \mathbb{D})$$

implies the function p *is subordinate to* e^z.

(4) *If* $|\alpha| > e^{n_1}$, *then*

$$\left|((p(z))^{n_1} - \alpha)\left(\frac{zp'(z)}{p(z)}\right)^{n_2}\right| < |\alpha| - e^{n_1} \ (z \in \mathbb{D})$$

implies the function p *is subordinate to* e^z.

Proof. The proof is similar to the proofs of the earlier theorems. We will see the choice of the function $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ and the set $\Omega \subset \mathbb{C}$ in each cases separately.

(1) $\psi(r, s, t; z) := (r^{n_1} - \alpha)s^{n_2}$ and $\Omega := \{w \in \mathbb{C} : |w| < e^{-n_2}(e^{-n_1} - |\alpha|)\}$.
(2) $\psi(r, s, t; z) := (r^{n_1} - \alpha)s^{n_2}$ and $\Omega := \{w \in \mathbb{C} : |w| < e^{-n_2}(|\alpha| - e^{n_1})\}$.
(3) $\psi(r, s, t; z) := (r^{n_1} - \alpha)(s/r)^{n_2}$ and $\Omega := \{w \in \mathbb{C} : |w| < e^{-n_1} - |\alpha|\}$.
(4) $\psi(r, s, t; z) := (r^{n_1} - \alpha)(s/r)^{n_2}$ and $\Omega := \{w \in \mathbb{C} : |w| < |\alpha| - e^{n_1}\}$. □

Theorem 2.12. *Let* n *be a natural number and* β *be a complex number. Then for the function* $p \in \mathcal{H}_1$, *any of the following conditions, for all* $z \in \mathbb{D}$, *are sufficient for the function* p *to be subordinate to* e^z:

(1) $|(zp'(z))^n/(\beta(p(z) - 1) + 1)| < e^{-n}/(|\beta|(1 + e) + 1)$;
(2) $\left|(zp'(z))^n/(\beta(p(z) - 1) + 1)^2\right| < e^{-n}/(\beta^2(e^2 + 2e + 1) + 2\beta(e - 1) + 1)$, ($\beta$ *is a positive real number*);
(3) $\left|(zp'(z)/p(z))^n/(\beta(p(z) - 1) + 1)\right| < 1/(|\beta|(1 + e) + 1)$;
(4) $\left|(zp'(z)/p(z))^n/(\beta(p(z) - 1) + 1)^2\right| < 1/(\beta^2(e^2+2e+1)+2\beta(e-1)+1)$, ($\beta$ *is a positive real number*).

Proof. For $r = e^{e^{i\theta}}$, $\theta \in [0, 2\pi)$, observe that

$$\begin{aligned}|\beta(r - 1) + 1| &\le |\beta||r - 1| + 1 \le |\beta||r| + |\beta| + 1\\ &\le |\beta|e + |\beta| + 1 = |\beta|(1 + e) + 1.\end{aligned}$$

If β is a positive real number, then, for $r = e^{e^{i\theta}}$, $\theta \in [0, 2\pi)$, using $\operatorname{Re}(r^n) \le |r^n| \le e^n$, we get

$$\begin{aligned}|\beta(r - 1) + 1|^2 &= (\beta(r - 1) + 1)(\beta(\overline{r} - 1) + 1)\\ &= \beta^2(r - 1)(\overline{r} - 1) + \beta(r - 1) + \beta(\overline{r} - 1) + 1\\ &= \beta^2(|r|^2 - 2\operatorname{Re} r + 1) + 2\beta\operatorname{Re}(r - 1) + 1\\ &\le \beta^2(e^2 + 2e + 1) + 2\beta(e - 1) + 1.\end{aligned}$$

Also if $s = me^{i\theta}e^{e^{i\theta}}$ and $r = e^{e^{i\theta}}$, $\theta \in [0, 2\pi)$, then $|s^n| \geq e^{-n}$ and $|(s/r)^n| \geq 1$. Hence all the statements in the theorem is proved by defining different functions $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ and sets $\Omega \subset \mathbb{C}$ as follows:

(1) $\psi(r, s, t; z) := s^n/(\beta(r-1)+1)$ and

$$\Omega := \{w \in \mathbb{C} : |w| < e^{-n}/(|\beta|(1+e)+1)\}.$$

(2) $\psi(r, s, t; z) := s^n/(\beta(r-1)+1)^2$ and

$$\Omega := \{w \in \mathbb{C} : |w| < e^{-n}/(\beta^2(e^2+2e+1)+2\beta(e-1)+1)\}.$$

(3) $\psi(r, s, t; z) := (s/r)^n/(\beta(r-1)+1)$ and

$$\Omega := \{w \in \mathbb{C} : |w| < 1/(|\beta|(1+e)+1)\}.$$

(4) $\psi(r, s, t; z) := (s/r)^n/(\beta(r-1)+1)^2$ and

$$\Omega := \{w \in \mathbb{C} : |w| < 1/(\beta^2(e^2+2e+1)+2\beta(e-1)+1)\}. \qquad \square$$

Theorem 2.13. *Let the function $p \in \mathcal{H}_1$. Let α be a complex number, β be a positive real number and γ be a non-negative real number. Then any of the following conditions, for all $z \in \mathbb{D}$, are sufficient for the function p to be subordinate to the function e^z:*

(1) $\left|\alpha p(z) + \gamma z^2 p''(z)/p(z)\right| + \beta\left|zp'(z)/p(z)\right| < \beta - \gamma - |\alpha|e$, $\left(|\alpha| < (\beta-\gamma)/(1+e)\right)$;

(2) $\left|\alpha p(z) + \gamma z^2 p''(z)/p(z)\right| + \beta\left|zp'(z)\right| < \beta e^{-1} - \gamma - |\alpha|e$, $\left(|\alpha| < (\beta e^{-1}-\gamma)/(1+e)\right)$.

Proof. (1) To prove this part of the theorem we define the set $\Omega \subset \mathbb{C}$ by $\Omega := \mathbb{C} \setminus [\beta - \gamma - |\alpha|e, \infty)$. Then

$$\begin{aligned} \left|\alpha r + \gamma\frac{t}{r}\right| + \beta\left|\frac{s}{r}\right| &\geq \left|\gamma\frac{t}{r}\right| - |\alpha r| + \beta \\ &= \gamma\left|\frac{t}{s}\right|\left|\frac{s}{r}\right| - |\alpha r| + \beta \\ &\geq -\gamma - |\alpha r| + \beta \\ &\geq \beta - \gamma - |\alpha|e, \end{aligned}$$

whenever $r = e^{e^{i\theta}}$, $s = me^{i\theta}e^{e^{i\theta}}$, $\mathrm{Re}(t/s+1) \geq m(1+\cos\theta)$, $\theta \in [0, 2\pi)$, $m \geq 1$ and $z \in \mathbb{D}$. Hence, if $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ is defined by

$$\psi(r, s, t; z) := |\alpha r + \gamma t/r| + \beta\,|s/r|\,,$$

then we get $\psi \in \Psi(\Omega, e^z)$ and $\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega$ for $z \in \mathbb{D}$ which proves that $p(z) \prec e^z$.

(2) The proof of this part of the theorem is similar to the proof of (1). Here the function ψ is defined by

$$\psi(r, s, t; z) := |\alpha r + \gamma t/r| + \beta\,|s|$$

and the set $\Omega \subset \mathbb{C}$ is defined by

$$\Omega := \mathbb{C} \setminus [\beta e^{-1} - \gamma - |\alpha| e, \infty). \qquad \square$$

Theorem 2.14. *Let the function $p \in \mathcal{H}_1$. Let n be a natural number and α, β and γ be real numbers such that $\alpha \geq 0$, $\beta > 0$ and $\gamma \geq 0$. Then, following are some of the sufficient conditions for the function p to be subordinate to the function e^z for all $z \in \mathbb{D}$:*

(1) $\alpha|(p(z))^n| + \beta|(zp'(z))^n| < (\alpha+\beta)e^{-n}$, $(\beta > \alpha(e^n - 1))$;
(2) $\alpha|p(z)| + \beta|zp'(z)| + \gamma|z^2p''(z)| < e^{-1}(\alpha+\beta-\gamma)$, $(\alpha < (\beta-\gamma)/(e-1))$;
(3) $\alpha|(p(z))^n| + \beta\,|zp'(z)/p(z)| + \gamma\left|z^2p''(z)/p(z)\right| < \alpha e^{-n} + \beta - \gamma$, $(\alpha < (\beta-\gamma)/(1-e^{-n}))$;
(4) $\alpha|(p(z))^n| + \beta\,|zp'(z)/p(z)| + \gamma\,|zp''(z)/p'(z)| < \alpha e^{-n} + \beta - \gamma$, $(\alpha < (\beta-\gamma)(1-e^{-n}))$.

Proof. (1) For $r = e^{e^{i\theta}}$, $s = me^{i\theta}e^{e^{i\theta}}$, $\operatorname{Re}(t/s+1) \geq m(1+\cos\theta)$, $\theta \in [0, 2\pi)$, $m \geq 1$ and $z \in \mathbb{D}$, observe that

$$\alpha|r^n| + \beta|s^n| \geq \alpha e^{-n} + \beta e^{-n} = (\alpha+\beta)e^{-n}.$$

Therefore, by defining the function $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ by

$$\psi(r, s, t; z) := \alpha|r^n| + \beta|s^n|$$

and the set $\Omega \subset \mathbb{C}$ by

$$\Omega := \mathbb{C} \setminus [(\alpha+\beta)e^{-n}, \infty),$$

the result follows.

(2) Consider the expression $\alpha|r| + \beta|s| + \gamma|t|$. Note that

$$\begin{aligned} \alpha|r| + \beta|s| + \gamma|t| &\geq \alpha e^{-1} + |s|\left(\beta + \gamma\left|\frac{t}{s}\right|\right) \\ &\geq \alpha e^{-1} + e^{-1}(\beta - \gamma) \\ &= e^{-1}(\alpha+\beta-\gamma) \end{aligned}$$

if $r = e^{e^{i\theta}}$, $s = me^{i\theta}e^{e^{i\theta}}$, $\operatorname{Re}(t/s+1) \geq m(1+\cos\theta)$, $\theta \in [0, 2\pi)$, $m \geq 1$ and $z \in \mathbb{D}$. In this case, the set Ω is defined by

$$\Omega := \mathbb{C} \setminus [e^{-1}(\alpha+\beta-\gamma), \infty).$$

Defining the function $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ by

$$\psi(r, s, t; z) := \alpha|r| + \beta|s| + \gamma|t|$$

gives the required result.

(3) Since $|r^n| \geq e^{-n}$, $|s/r| \geq 1$ and $|t/r| = |s/r|\,|t/s|$, whenever $r = e^{e^{i\theta}}$, $s = me^{i\theta}e^{e^{i\theta}}$, $\operatorname{Re}(t/s+1) \geq m(1+\cos\theta)$, $\theta \in [0, 2\pi)$, $m \geq 1$ and $z \in \mathbb{D}$, we get

$$\begin{aligned} \alpha|r^n| + \beta\left|\frac{s}{r}\right| + \gamma\left|\frac{t}{r}\right| &\geq \alpha e^{-n} + \left|\frac{s}{r}\right|\left(\beta + \gamma\left|\frac{t}{s}\right|\right) \\ &\geq \alpha e^{-n} + \beta - \gamma. \end{aligned}$$

The result follows by thus considering the function $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ defined by

$$\psi(r, s, t; z) := \alpha|r^n| + \beta|s/r| + \gamma|t/r|$$

and the set Ω defined by

$$\Omega := \mathbb{C} \setminus [\alpha e^{-n} + \beta - \gamma, \infty).$$

(4) For $r = e^{e^{i\theta}}$, $s = me^{i\theta}e^{e^{i\theta}}$, $\operatorname{Re}(t/s+1) \geq m(1+\cos\theta)$, $\theta \in [0, 2\pi)$, $m \geq 1$ and $z \in \mathbb{D}$, since $|r^n| \geq e^{-n}$, $|s/r| \geq 1$ and $|t/s| \geq \operatorname{Re}(t/s) \geq -1$, it can be shown by a simple calculation that $\psi \in \Psi(\Omega, e^z)$, where $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ is defined by

$$\psi(r, s, t; z) := \alpha|r^n| + \beta|s/r| + \gamma|t/s|$$

and $\Omega \subset \mathbb{C}$ is defined by

$$\Omega := \mathbb{C} \setminus [\alpha e^{-n} + \beta - \gamma, \infty).$$

The result thus follows from Theorem 1.5. □

Remark 2.15. For $\alpha = 0$, $\beta = 1$ and $n = 1$, Theorem 2.14(1) reduces to the following result in [11, Ex 2.4]: If $p \in \mathcal{H}_1$ satisfies the inequality

$$|zp'(z)| < e^{-1}$$

for $z \in \mathbb{D}$, then the function p is subordinate to e^z.

Remark 2.16. For $f \in \mathcal{A}$, by substituting

$$p(z) := \frac{zf'(z)}{f(z)}$$

in all the theorems proved in this paper, we get the sufficient conditions for the function f to belong to the class $\mathcal{S}^*_e$.

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