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DETECTABLE MEANS AND APPLICATIONS

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Abstract. In this paper, we introduce a new concept for bivariate means and we study its properties. Application of this concept for mean-inequalities is also discussed. Open problems are derived as well.

1. Introduction

Mean-theory arises in various contexts and contributes as a good tool for solving many scientific problems. It attracts many mathematicians by virtue of its nice properties and various applications. By (bivariate) mean we understand a binary map m between positive real numbers satisfying the following double inequality:

$$\forall a, b > 0 \qquad \min(a, b) \le m(a, b) \le \max(a, b).$$

Every mean satisfies $m(a, a) = a$ for each $a > 0$. Two trivial means are $(a, b) \longmapsto \min(a, b)$ and $(a, b) \longmapsto \max(a, b)$ and will be denoted by min and max, respectively. Let p be a real number. The binomial power mean is defined by

$$A_p =: A_p(a, b) =: \Big(\frac{a^p + b^p}{2}\Big)^{1/p}, \ \ p \neq 0.$$

This power mean includes some familiar means, namely (see [2] for instance and the related references cited therein),

$$A =: A(a, b) = \frac{a + b}{2} = A_1(a, b), \ G =: G(a, b) = \sqrt{ab} = \lim_{p\to 0} A_p(a, b),$$
$$H =: H(a, b) = \frac{2ab}{a + b} = A_{-1}(a, b), \ Q =: Q(a, b) = \sqrt{\frac{a^2 + b^2}{2}} = A_2(a, b),$$

which are known as the arithmetic mean, the geometric mean, the harmonic mean and the quadratic (or root-square) mean, respectively.

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There are other means of interest introduced in the literature. For instance, the following

$$L =: L(a,b) = \frac{b-a}{\ln\, b - \ln\, a},\ I =: I(a,b) = e^{-1}\left(\frac{b^b}{a^a}\right)^{1/(b-a)},$$

$$P =: P(a,b) = \frac{b-a}{2\arcsin\frac{b-a}{b+a}},\ T =: T(a,b) = \frac{b-a}{2\arctan\frac{b-a}{b+a}},$$

$$M =: M(a,b) = \frac{b-a}{2\,\text{arcsinh}\frac{b-a}{b+a}},$$

with $L(a,a) = I(a,a) = P(a,a) = T(a,a) = M(a,a) = a$, are known as the logarithmic mean, the identric mean [2], the first Seiffert mean [17], the second Seiffert mean [18] and the Neuman-Sándor mean [8], respectively. Recently, the three means P, T and M have been the subject of intensive research, because of their interesting properties and nice relationships. For more details about these means and their bounds in terms of the other familiar means, we refer the reader to [3, 4, 6, 7, 9] and the related references cited therein.

A mean m is symmetric if $m(a,b) = m(b,a)$ and homogeneous if $m(ka,kb) = k.m(a,b)$ for all $a,b > 0$ and $k > 0$. All the previous means are symmetric and homogeneous. We say that m is monotone if it is non-decreasing with respect to its each variables. For more detail about these concepts, see [13, 16] for instance.

For two means m_1 and m_2 we write $m_1 \leq m_2$ if and only if $m_1(a,b) \leq m_2(a,b)$ for every $a,b > 0$ and, $m_1 < m_2$ if and only if $m_1(a,b) < m_2(a,b)$ for all $a,b > 0$ with $a \neq b$. We say that m_1 and m_2 are comparable if one of the two mean inequalities $m_1 \leq m_2$ or $m_2 \leq m_1$ holds. The previous means are comparable with the following chain of inequalities

$$\min < H < G < L < P < I < A < M < T < Q < \max.$$

For a given mean m, we set $m^*(a,b) = \Big(m\big(a^{-1},b^{-1}\big)\Big)^{-1}$, and it is easy to see that m^* is also a mean, called the dual mean of m. If m is symmetric and homogeneous, then so is m^* and in this case we have $m^*(a,b) = ab/m(a,b)$. Every mean m satisfies $m^{**} := (m^*)^* = m$ and, if m_1 and m_2 are two means such that $m_1 \leq m_2$, then $m_1^* \geq m_2^*$. It is easy to see that $A_p^* = A_{-p}$ for any $p \in \mathbb{R}$. In particular, $\min^* = \max$ and $\max^* = \min$, the arithmetic and harmonic means are mutually dual, i.e., $A^* = H$, $H^* = A$, and the geometric mean is self-dual, i.e., $G^* = G$.

2. Sub-stabilizability and super-stabilizability

This section is devoted to state briefly some needed topics. For the sake of simplicity for the reader, we adopt throughout this paper the following definition for mean [5].

Definition. A continuous positive real function $m(a, b)$ for $a, b > 0$ is called a symmetric homogeneous mean (or simply a mean) if m satisfies the following conditions:

(i) $m(a, b) = m(b, a)$.
(ii) $m(ka, kb) = k.m(a, b)$ for any $k > 0$.
(iii) $m(a, b)$ is non-decreasing in a (and in b).
(iv) $\min(a, b) \le m(a, b) \le \max(a, b)$.

We now recall some basic concepts about means and some needed related results. For more details, we refer the reader to [10, 11, 14].

Let m_1, m_2, m_3 be three means. We set, for all $a, b > 0$,

$$\mathcal{R}\big(m_1, m_2, m_3\big)(a, b) = m_1\Big(m_2\big(a, m_3\big), m_2\big(m_3, b\big)\Big) \text{ with } m_3 =: m_3(a, b).$$

The mean m is called stable if $\mathcal{R}(m, m, m) = m$. The two trivial means min and max are stable.

If m_1 and m_2 are two nontrivial stable means such that $\mathcal{R}(m_1, m, m_2) = m$, then we say that m is (m_1, m_2)-stabilizable. Following [12], there exists one and only one (m_1, m_2)-stabilizable mean, provided that m_1 is a cross mean (see [11] for the definition and more details).

If moreover m_1 and m_2 are comparable, we say that m is (m_1, m_2)-sub-stabilizable if $\mathcal{R}(m_1, m, m_2) \le m$ and m is between m_1 and m_2, see [14]. If this latter mean inequality is strict, then we say that m is strictly (m_1, m_2)-sub-stabilizable. The super-stabilizability of m is defined in similar manner (by reversing the previous mean inequalities). In short, we can say that m is (m_1, m_2)-super-stabilizable if and only if m^* is (m_2^*, m_1^*)-sub-stabilizable. For more details about the concepts of sub-stabilizable and super-stabilizable means we refer the reader to [14].

The following results will be also needed in the sequel.

Theorem 2.1 ([10, 11])**.** *The following assertions hold*:

(i) *For any* $p \in \mathbb{R}$, A_p *is stable. In particular,* A, H, G *and* Q *are stable.*

(ii) L *is simultaneously* (A, G)*-stabilizable and* (H, A)*-stabilizable, while* I *is* (G, A)*-stabilizable.*

Theorem 2.2. *The following assertions hold true*:

(i) L *is strictly* (G, A)*-super-stabilizable and strictly* (A, H)*-sub-stabilizable, while* I *is strictly* (A, G)*-sub-stabilizable. See* [14]*.*

(ii) P *is strictly* (A, G)*-sub-stabilizable and strictly* (G, A)*-super-stabilizable. See* [14] *and* [1]*, respectively.*

(iii) M *is strictly* (A, Q)*-super-stabilizable. See* [15]*.*

(iv) T *is strictly* (A, A_3)*-super-stabilizable. See* [15]*.*

3. Detectable means

We preserve the same notation as in the previous sections. Let m be a mean. Assume that there exist two nontrivial stable means m_1 and m_2, with

$m_1 < m_2$, and two stabilizable means m_{21} and m_{12}, with $m_{21} < m_{12}$, such that

$$m_1 \le m_{21} \le m \le m_{12} \le m_2. \tag{1}$$

Let us further consider the following statements:

(i) m_{21} is (m_2, m_1)-stabilizable,

(ii) m_{12} is (m_1, m_2)-stabilizable,

(iii) m is (m_1, m_2)-super-stabilizable and (m_2, m_1)-sub-stabilizable. That is,

$$\mathcal{R}\big(m_2, m, m_1\big) \le m \le \mathcal{R}\big(m_1, m, m_2\big). \tag{2}$$

(iv) m is (m_1, m_2)-sub-stabilizable and (m_2, m_1)-super-stabilizable. That is,

$$\mathcal{R}\big(m_1, m, m_2\big) \le m \le \mathcal{R}\big(m_2, m, m_1\big). \tag{3}$$

With this, we now may state the following definition.

Definition. (a) If the mean m satisfies (i), (ii) and (iii), then we say that m is $\left[\begin{smallmatrix} m_1 & m_2 \\ m_2 & m_1 \end{smallmatrix}\right]$-detectable. If the inequalities (1) and (2) are strict, then we say that m is strictly $\left[\begin{smallmatrix} m_1 & m_2 \\ m_2 & m_1 \end{smallmatrix}\right]$-detectable.

(b) If m satisfies (i), (ii) and (iv), then we say that m is $\left[\begin{smallmatrix} m_1 & m_2 \\ m_2 & m_1 \end{smallmatrix}\right]$-detectable. If (1) and (3) are strict, then we say that m is strictly $\left[\begin{smallmatrix} m_1 & m_2 \\ m_2 & m_1 \end{smallmatrix}\right]$-detectable.

Example 3.1. (i) The logarithmic mean L is (A, G)-stabilizable and (G, A)-super-stabilizable. Further, we can write

$$m_1 = G < L = m_{21} \le L = m < I = m_{12} < A = m_2,$$

with I is (G, A)-stabilizable. We can then say that L is $\left[\begin{smallmatrix} G & A \\ A & G \end{smallmatrix}\right]$-detectable, but not strictly.

(ii) The identric mean I is (G, A)-stabilizable and (A, G)-sub-stabilizable. Here, we write

$$m_1 = G < L = m_{21} < I = m \le I = m_{12} < A = m_2,$$

with L is (A, G)-stabilizable. Then, I is $\left[\begin{smallmatrix} G & A \\ A & G \end{smallmatrix}\right]$-detectable, but not strictly.

Example 3.2. The mean L is (H, A)-stabilizable and (A, H)-sub-stabilizable. Writing

$$m_1 = H < L^* = m_{21} < L = m \le L = m_{12} < A = m_2,$$

with L^* is (A, H)-stabilizable, we then deduce that L is also $\left[\begin{smallmatrix} H & A \\ A & H \end{smallmatrix}\right]$-detectable, but not strictly.

Other more interesting examples of detectable means will be presented later. The next result is immediate from the previous definition and its proof is therefore omitted here for the reader.

Proposition 3.3. *The next statements are equivalent*:
(i) m *is* (*strictly*) $\left[\begin{smallmatrix} m_1 & m_2 \\ m_2 & m_1 \end{smallmatrix}\right]$*-detectable,*
(ii) m^* *is* (*strictly*) $\begin{bmatrix} m_2^* & m_1^* \\ m_1^* & m_2^* \end{bmatrix}$*-detectable.*

Combining this latter proposition with Example 3.1 and Example 3.2, we immediately deduce the following.

Example 3.4. L^* is simultaneously (not strictly) $\left[\begin{smallmatrix} H & G \\ G & H \end{smallmatrix}\right]$-detectable and $\left[\begin{smallmatrix} H & A \\ A & H \end{smallmatrix}\right]$-detectable, while I^* is (not strictly) $\left[\begin{smallmatrix} H & G \\ G & H \end{smallmatrix}\right]$-detectable. It follows that L and L^* are both (not strictly) $\left[\begin{smallmatrix} H & A \\ A & H \end{smallmatrix}\right]$-detectable.

Remark 3.5. The previous example brings us good and new information. In fact, it asserts the two following assertions:
(i) A given mean could be $\left[\begin{smallmatrix} m_1 & m_2 \\ m_2 & m_1 \end{smallmatrix}\right]$-detectable and $\left[\begin{smallmatrix} m_3 & m_4 \\ m_4 & m_3 \end{smallmatrix}\right]$-detectable, for some stable means $m_1 \neq m_3$ and $m_2 \neq m_4$.
(ii) Given two stable means m_1 and m_2, two different means could be both $\left[\begin{smallmatrix} m_1 & m_2 \\ m_2 & m_1 \end{smallmatrix}\right]$-detectable.

In [14], the authors proved that if a certain mean m is (A,G)-sub-stabilizable, then $L \leq m \leq A$. We then deduce that the three means L, I and P could be $\left[\begin{smallmatrix} G & A \\ A & G \end{smallmatrix}\right]$-detectable, which is confirmed by Example 3.1 for L and I. For the mean P, we have the following result.

Theorem 3.6. *The first Seiffert mean P is strictly* $\left[\begin{smallmatrix} G & A \\ A & G \end{smallmatrix}\right]$*-detectable.*

Proof. First, we know that

$$m_1 = G < L = m_{21} < P = m < I = m_{12} < A = m_2,$$

with L is (A, G)-stabilizable and I is (G, A)-stabilizable. Further, following Theorem 2.2(ii), P is strictly (A, G)-sub-stabilizable and strictly (G, A)-super-stabilizable. The desired result follows. □

Now, a natural question arises from the above. Does exist a mean m which is strictly $\left[\begin{smallmatrix} A & G \\ G & A \end{smallmatrix}\right]$-detectable? Before giving an affirmative answer to this question, we introduce a general point of view recited in the following definition.

Definition. Let m be a strictly $\left[\begin{smallmatrix} m_1 & m_2 \\ m_2 & m_1 \end{smallmatrix}\right]$-detectable mean. We say that m is reversible if there exists a mean which is strictly $\left[\begin{smallmatrix} m_2 & m_1 \\ m_1 & m_2 \end{smallmatrix}\right]$-detectable.

It is easy to see that if m is (m_1, m_2)-reversible, then m^* is (m_2^*, m_1^*)-reversible. A detectable mean m is not (m_1, m_2)-reversible, with $m_1 < m_2$, means that one of the two following statements holds true:
- m is strictly $\left[\begin{smallmatrix} m_1 & m_2 \\ m_2 & m_1 \end{smallmatrix}\right]$-detectable and there is no mean which is strictly $\left[\begin{smallmatrix} m_2 & m_1 \\ m_1 & m_2 \end{smallmatrix}\right]$-detectable.
- m is strictly $\left[\begin{smallmatrix} m_2 & m_1 \\ m_1 & m_2 \end{smallmatrix}\right]$-detectable and there is no mean which is strictly $\left[\begin{smallmatrix} m_1 & m_2 \\ m_2 & m_1 \end{smallmatrix}\right]$-detectable.

The following result explains more this latter situation and answers affirmatively the previous question.

Theorem 3.7. *The mean P is not (G,A)-reversible. That is, there is no mean m which is strictly $\left[\begin{smallmatrix} A & G \\ G & A \end{smallmatrix}\right]$-detectable.*

Proof. Assume that P is (G,A)-reversible. Since P is strictly $\left[\begin{smallmatrix} G & A \\ A & G \end{smallmatrix}\right]$-detectable we then must show that there is no mean m which is strictly $\left[\begin{smallmatrix} A & G \\ G & A \end{smallmatrix}\right]$-detectable. Assume that such mean m exists. By definition, m should satisfies

$$L < m < I \text{ and } \mathcal{R}\big(G,m,A\big) < m < \mathcal{R}\big(A,m,G\big).$$

The inequalities $m < \mathcal{R}\big(A,m,G\big)$ means that

$$m(a,b) < \mathcal{R}\big(A,m,G\big)(a,b) = A\big(\sqrt{a},\sqrt{b}\big)m\big(\sqrt{a},\sqrt{b}\big)$$

for all $a,b > 0, a \neq b$. By a mathematical induction we then deduce (for $a,b > 0, a \neq b$)

$$\begin{aligned} m(a,b) &< m\Big(a^{1/N},b^{1/N}\Big)\prod_{n=1}^{N} A\Big(a^{1/2^n},b^{1/2^n}\Big) \\ &\leq \max\Big(a^{1/N},b^{1/N}\Big)\prod_{n=1}^{N} A\Big(a^{1/2^n},b^{1/2^n}\Big) \end{aligned}$$

for all integer $N \geq 1$. Letting $N \to \infty$ in this latter inequality we then obtain

$$m(a,b) \leq \prod_{n=1}^{\infty} A\Big(a^{1/2^n},b^{1/2^n}\Big).$$

This, with the relationship, [13]

$$L(a,b) = \prod_{n=1}^{\infty} A\Big(a^{1/2^n},b^{1/2^n}\Big),$$

yields $m \leq L$. This contradicts $L < m$ and the proof is complete. □

Corollary 3.8. *P^* is strictly $\left[\begin{smallmatrix} H & G \\ G & H \end{smallmatrix}\right]$-detectable and P^* is not (H,G)-reversible.*

Proof. It follows from Proposition 3.3 when combined with Theorem 3.6 and Theorem 3.7, respectively. □

From the previous study it is natural to arise the following question:

Question. Are the means T and M strictly detectable? Are T and/or M reversible?

About this, no affirmative answer at the moment and we state the following conjecture.

Conjecture. *The means T and M are mutually (A,Q)-reversible. Precisely, M is strictly $\begin{bmatrix} A & Q \\ Q & A \end{bmatrix}$-detectable while T is strictly $\begin{bmatrix} Q & A \\ A & Q \end{bmatrix}$-detectable.*

4. Application for mean-inequalities

In this section, we will show that how the detectability concept can be used as tool for refining some mean-inequalities. For the sake of simplicity, we denote by $m := m_{12}$ (resp. $m = m_{21}$) the unique mean m which is (m_1, m_2)-stabilizable $\big($resp. m is (m_2, m_1)-stabilizable$\big)$. We first state the following result which is of interest.

Theorem 4.1. *Let $m_1, m^{'}, m, m^{''}, m_2$ be strictly non-decreasing means such that*

$$m_1 < m^{'} < m < m^{''} < m_2. \tag{4}$$

Assume that further the following assertions are satisfied:

(i) *m_1 and m_2 are non trivial stable means.*
(ii) *m is strictly (m_2, m_1)-sub-stabilizable and strictly (m_1, m_2)-super-stabilizable.*

Then we have

$$\begin{aligned} m_1 &< \mathcal{R}\big(m_1, m^{'}, m_1\big) < \mathcal{R}\big(m_2, m^{'}, m_1\big) < \mathcal{R}\big(m_2, m, m_1\big) < m \\ &< \mathcal{R}\big(m_1, m, m_2\big) < \mathcal{R}\big(m_1, m^{''}, m_2\big) < \mathcal{R}\big(m_2, m^{"}, m_2\big) < m_2. \end{aligned} \tag{5}$$

Proof. Since m_1 is stable and all involved means are strictly non-decreasing then we have, with (4),

$$m_1 = \mathcal{R}\big(m_1, m_1, m_1\big) < \mathcal{R}\big(m_1, m^{'}, m_1\big) < \mathcal{R}\big(m_2, m^{'}, m_1\big) < \mathcal{R}\big(m_2, m, m_1\big).$$

This, with the fact that m is strictly (m_2, m_1)-sub-stabilizable, yields the mean-inequalities at left of m in (5). The mean-inequalities at right of m in (5) can be proved in a similar manner. □

Theorem 4.1 stems its importance in the fact that the means $m^{'}$ and $m^{''}$ can be chosen in a large spectrum, since they only satisfy (4) without any more condition. In particular, we have the following corollary.

Corollary 4.2. *Let $m^{'}$ and $m^{''}$ be two strictly non-decreasing means such that*

$$G < m^{'} < P < m^{''} < A.$$

Then the following chain of inequalities holds

$$\begin{aligned} G &< \mathcal{R}\big(G, m^{'}, G\big) < \mathcal{R}\big(A, m^{'}, G\big) < \mathcal{R}\big(A, P, G\big) < P \\ &< \mathcal{R}\big(G, P, A\big) < \mathcal{R}\big(G, m^{''}, A\big) < \mathcal{R}\big(A, m^{''}, A\big) < A. \end{aligned}$$

Using the detectability concept, the following corollary is immediate from the above.

Corollary 4.3. *Let m be a strictly $\left[\begin{smallmatrix} m_1 & m_2 \\ m_2 & m_1 \end{smallmatrix}\right]$-detectable mean. Assume that m_1 and m_2 are strictly non-decreasing. Then we have*

$$m_1 < \mathcal{R}\big(m_1, m_{21}, m_1\big) < m_{21} < \mathcal{R}\big(m_2, m, m_1\big) < m$$

$$< \mathcal{R}\big(m_1, m, m_2\big) < m_{12} < \mathcal{R}\big(m_2, m_{12}, m_2\big) < m_2,$$

which refine the mean-inequalities $m_1 < m_{21} < m < m_{12} < m_2$.

In order to illustrate the previous theoretical results we present the following example.

Example 4.4. The following chain of inequalities is well-known, see [3, 4, 14] for instance.

$$G < L_2 := (AL)^{1/2} < P < A_{2/3} < A.$$

According to Corollary 4.2 we have

$$G < \mathcal{R}\big(G, L_2, G\big) < \mathcal{R}\big(A, L_2, G\big) < \mathcal{R}\big(A, P, G\big) < P$$
$$< \mathcal{R}\big(G, P, A\big) < \mathcal{R}\big(G, A_{2/3}, A\big) < \mathcal{R}\big(A, A_{2/3}, A\big) < A.$$

We can compute all sides of the previous inequalities by referring to the definition of $\mathcal{R}$. For instance, simple computations lead to

$$\mathcal{R}\big(G, L_2, G\big) = \big(GL\big)^{1/2}, \;\; \mathcal{R}\big(A, L_2, G\big) = \big(A_{1/2}L\big)^{1/2}$$

and

$$\mathcal{R}\big(G, A_{2/3}, A\big) = \left(\frac{G^{4/3} + A^{4/3} + \big(A.A_{2/3}\big)^{2/3}}{4}\right)^{3/4}.$$

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