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EXISTENCE THEOREMS FOR CRITICAL DEGENERATE EQUATIONS INVOLVING THE GRUSHIN OPERATORS

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Abstract. In this paper we prove the existence of nontrivial weak solutions to the boundary value problem

$$\begin{aligned} -G_1 u &= u^3 + f(x,y,u) \quad \text{in } \Omega, \\ u &\geq 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^3$, G_1 is a Grushin type operator, and $f(x,y,u)$ is a lower order perturbation of u^3 with $f(x,y,0) = 0$. The nonlinearity involved is of critical exponent, which differs from the existing results in [11, 12].

1. Introduction

In the last decades, the boundary value problem for regular semilinear elliptic equations on a bounded smooth domain $\Omega \subset \mathbb{R}^N$, $N \geq 3$,

$$\begin{cases} -\Delta u = g(x,u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

has interested many mathematicians. When the nonlinearity $g(x,u)$ is of subcritical growth, namely $|g(x,u)| \leq C|u|^{2^*}$ where $2^* = \frac{N+2}{N-2}$, one can apply standard variational methods to show the existence and multiple existence of weak solutions. When the nonlinearity consists of a critical exponent term solely, by Pohozaev identity, the problem does not have a solution. It was first showed in the seminal paper by Brezis–Nirenberg [4] that a lower order term could reverse the situation. This breakthrough result has initiated many studies on equations with critical exponent involving other types of operators or systems, for example in [1, 5, 7, 9]. The critical exponents depend on the operators under consideration and further variants of compactness theorems must be utilized to obtain existence and multiplicity results.

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On the other hand, concerning degenerate elliptic equations, the class of equations involving an operator of the Grushin type has been of great interest in recent years (see [6])

$$G_\alpha u = \Delta_x u + |x|^{2\alpha}\Delta_y u,$$

where

$$(x,y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} := \mathbb{R}^n,\ \Delta_x = \sum_{i=1}^{n_1} \frac{\partial^2}{\partial x_i^2},$$

$$\Delta_y = \sum_{j=1}^{n_2} \frac{\partial^2}{\partial y_j^2},\ \alpha \geq 0 \text{ and } |x|^2 = \sum_{i=1}^{n_1} x_i^2.$$

When $\alpha = 0$, $G_0 = \Delta$ is the classical Laplacian operator, and when $\alpha > 0$, G_α is not elliptic in domains intersecting the surface $x = 0$. In particular, G_1 plays an important role in the analysis on the Heisenberg group. Many aspects of the theory of degenerate elliptic differential operators such as analyticity and smoothness of solutions for linear degenerate equations, existence and multiplicity of solutions to boundary value problems for semilinear degenerate equations, degenerate evolution equations are presented in the monograph [17]; we also refer the readers to the papers [8,10,11,15,16] for more results and generations. Until recently almost all existing results concern the case when the nonlinear terms in those equations are of sub-critical growth, for instance [10,12,15]. The critical growth was observed for a model of the Grushin-type operators in [16], where the homogeneous dimension $Q = n_1 + 2n_2$ plays an important role.

In the present paper, first we study the existence of weak solutions for the boundary value problem

$$\begin{cases} -G_1 u = \lambda u + u^3 & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^2 \times \mathbb{R}$ and has nonempty intersection with the hyperplane $\{(x,y) \in \mathbb{R}^2 \times \mathbb{R} : x = 0\}$.

We denote by λ_1 the first eigenvalue of the Grushin operator $-G_1$ with homogeneous Dirichlet boundary condition on Ω (Theorem 2.2). We make use of the functional space $S^2_{1,0}(\Omega)$ (see Definition 2.1) and a weak solution is defined as follows.

Definition 1.1. A weak solution to the problem (1.1) is a nonnegative function $u \in S^2_{1,0}(\Omega)$ such that for all $\varphi \in S^2_{1,0}(\Omega)$, it holds

$$\int_\Omega \nabla_1 u \cdot \nabla_1 \varphi \mathrm{d}x\mathrm{d}y - \lambda \int_\Omega u\varphi \mathrm{d}x\mathrm{d}y - \int_\Omega u^3 \varphi \mathrm{d}x\mathrm{d}y = 0.$$

Then our first main result is about the existence of nontrivial nonnegative solutions.

Theorem 1.1. *For all $\lambda \in (0, \lambda_1)$, there exists a nontrivial nonnegative weak solution to the problem* (1.1).

In the case of larger parameter $\lambda \geq \lambda_1$, by using a Pohozaev identity we demonstrate the nonexistence of solutions in Remark 3.4.

We also deal with the following more general problem

$$\begin{cases} -G_1 u = u^3 + f(x, y, u) \text{ in } \Omega, \\ u \geq 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $f(x, y, u) \colon \Omega \times [0, +\infty) \to \mathbb{R}$ is measurable in (x, y), continuous in u, and satisfies

$$\sup_{\Omega\times[0,M]} |f(x, y, u)| < \infty \text{ for all } M > 0.$$

Moreover, we assume that $f(x, y, 0) = 0$ and $f(x, y, u)$ is of the form $a(x, y)u+ g(x, y, u)$, where $g(x, y, u)$ subjects to the following growth conditions:

(g1) $g(x, y, u) = o(u)$ as $u \to 0^+$ uniformly in (x, y).
(g2) $g(x, y, u) = o(u^3)$ as $u \to +\infty$ uniformly in (x, y).
(a1) $a(x, y) \in L^\infty(\Omega)$.

Nevertheless, we assume that the operator $-G_1 - a(x, y)$ has positive least eigenvalue, that is, there are $\alpha > 0$ and $\alpha_0 > 0$ such that

$$\int_\Omega \left(|\nabla_1 \phi|^2 - a\phi^2\right) \mathrm{d}x\mathrm{d}y \geq \alpha \int_\Omega \phi^2 \mathrm{d}x\mathrm{d}y \text{ for all } \phi \in S^2_{1,0}(\Omega),$$

or equivalently

$$\int_\Omega \left(|\nabla_1 \phi|^2 - a\phi^2\right) \mathrm{d}x\mathrm{d}y \geq \alpha_0 \int_\Omega |\nabla_1 \phi|^2 \mathrm{d}x\mathrm{d}y \text{ for all } \phi \in S^2_{1,0}(\Omega). \tag{1.3}$$

Definition 1.2. A weak solution to the equation in (1.2) is a function $u \in S^2_{1,0}(\Omega)$ such that for all $\varphi \in S^2_{1,0}(\Omega)$, it holds

$$\int_\Omega \nabla_1 u \cdot \nabla_1 \varphi \mathrm{d}x\mathrm{d}y - \int_\Omega u^3 \varphi \mathrm{d}x\mathrm{d}y - \int_\Omega f(x, y, u)\varphi \mathrm{d}x\mathrm{d}y = 0.$$

In this definition, we have not mentioned the sign of u. Therefore, it is understood that $f(x, y, u)$ is extended by 0 to the domain $u \leq 0$. We denote $F(x, y, u) = \int_0^u f(x, y, s)\mathrm{d}s$.

The existence theorem for the problem (1.2) is stated as follows.

Theorem 1.2. *Assume* (g1), (g2), (a1) *and* (1.3). *Suppose further that there exists some $v_0 \in S^2_{1,0}(\Omega)$, $v_0 \geq 0$ in Ω, $v_0 \not\equiv 0$, such that*

$$\sup_{t\geq 0} \Psi(tv_0) < \frac{S^2}{4}, \tag{1.4}$$

where S is the constant defined in Lemma 3.1, and

$$\Psi(u) = \frac{1}{2} \int_\Omega |\nabla_1 u|^2 \mathrm{d}x\mathrm{d}y - \frac{1}{4} \int_\Omega u^4 \mathrm{d}x\mathrm{d}y - \int_\Omega F(x, y, u)\mathrm{d}x\mathrm{d}y$$

is the Euler–Lagrange functional associated with the problem (1.2)*. Then there exists a nontrivial weak solution to the problem* (1.2).

2. Preliminaries

2.1. Function spaces and embedding theorem

Definition 2.1. We denote by $S_1^2(\Omega)$ the set of all functions $u \in L^2(\Omega)$ such that $\partial u/\partial x_1, \partial u/\partial x_2 \in L^2(\Omega)$, $|x|\partial u/\partial y \in L^2(\Omega)$. This space is furnished with the norm:

$$||u||_{S_1^2(\Omega)} = \left\{ \int_\Omega (|u|^2 + |\nabla_1 u|^2) \mathrm{d}x\mathrm{d}y \right\}^{\frac{1}{2}},$$

where

$$\mathrm{d}x = \mathrm{d}x_1 \mathrm{d}x_2, \qquad \nabla_1 u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, |x| \frac{\partial u}{\partial y} \right).$$

We can also define the scalar product in $S_1^2(\Omega)$ as follows:

$$(u, v)_{S_1^2(\Omega)} = (u, v)_{L^2(\Omega)} + (\nabla_1 u, \nabla_1 v)_{L^2(\Omega)}.$$

The space $S_{1,0}^2(\Omega)$ is defined as the completion of $C_0^1(\Omega)$ in $S_1^2(\Omega)$.

In general dimensions, the following embedding inequality was shown in [15]:

$$\left(\int_\Omega |u|^{q+1} \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{q+1}} \leq C(q, \Omega) ||u||_{S_{1,0}^2(\Omega)},$$

where $1 \leq q \leq \frac{Q+2}{Q-2}$, $C(q, \Omega) > 0$. It reads in our case that $1 \leq q \leq 3$. The embedding is compact if $1 \leq q < 3$.

We also know that $S_{1,0}^2(\Omega)$ is furnished with two equivalent norms

$$||u||_{S_1^2(\Omega)} \text{ and } |||u|||_{S_1^2(\Omega)} = \left(\int_\Omega |\nabla_1 u|^2 \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{2}}.$$

When $q = 3$, Beckner showed in [2] the following sharp estimate.

Theorem 2.1 ([2, Theorem 3])**.** *For* $f \in C^1(\mathbb{R}^3)$

$$\left[||f||_{L^4(\mathbb{R}^3)} \right]^2 \leq \frac{1}{2\pi} \int_{\mathbb{R}^2 \times \mathbb{R}} \left[|\nabla_x f|^2 + |x|^2 \left(\frac{\partial f}{\partial y} \right)^2 \right] \mathrm{d}x\mathrm{d}y.$$

This inequality is sharp, and an extremal function is given by $[(1 + |x|^2)^2 + 4|y|^2]^{-\frac{1}{2}}$.

2.2. Eigenvalue problem

By variational principle, we can easily prove:

Theorem 2.2. *There exist $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n \leq \cdots \to \infty$, such that for each $k \geq 1$, the following Dirichlet problem*

$$\begin{cases} -G_1\varphi_k = \lambda_k\varphi_k \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

has a nontrivial solution in $S^2_{1,0}(\Omega)$. Moreover, $\{\varphi_k\}_{k\geq 1}$ constitutes an orthonormal basis of the Hilbert space $S^2_{1,0}(\Omega)$ and the first eigenfunction φ_1 is nonnegative.

2.3. Mountain Pass Theorem

We will use the following version of the Mountain Pass Theorem to study the existence problem.

Lemma 2.3 ([4]). *Let Φ be a C^1 functional on a Banach space E. Suppose that there exists a neighborhood U of 0 in E and a constant ρ such that*

(1) *$\Phi(u) \geq \rho$ for every $u \in \partial U$.*
(2) *$\Phi(0) < \rho$ and $\Phi(v) < \rho$ for some $v \notin U$.*

Set

$$c = \inf_{h\in\Gamma} \sup_{u\in h} \Phi(u),$$

where

$$\Gamma = \{h \in C([0,1]; E) : h(0) = 0, h(1) = v\}.$$

Then there is a sequence $\{u_j\}$ in E such that

$$\Phi(u_j) \to c \text{ and } \Phi'(u_j) \to 0 \text{ in } E^*.$$

This lemma guarantees the existence of a sequence which will converge weakly to the desired solution. We do not assume the $(\mathrm{PS})_c$ condition.

3. Proof of main results

3.1. Proof of Theorem 1.1

Let us define the Euler–Lagrange functional associated with the problem (1.1) as follows:

$$\Phi(u) = \frac{1}{2}\int_\Omega |\nabla_1 u|^2 \mathrm{d}x\mathrm{d}y - \frac{\lambda}{2}\int_\Omega u^2 \mathrm{d}x\mathrm{d}y - \frac{1}{4}\int_\Omega u^4 \mathrm{d}x\mathrm{d}y.$$

We can check that Φ is well defined on $S^2_{1,0}(\Omega)$ and $\Phi \in C^1(S^2_{1,0}(\Omega), \mathbb{R})$ with

$$\Phi'(u)(v) = \int_\Omega \nabla_1 u \cdot \nabla_1 v \mathrm{d}x\mathrm{d}y - \lambda\int_\Omega uv \mathrm{d}x\mathrm{d}y - \int_\Omega u^3 v \mathrm{d}x\mathrm{d}y.$$

A weak solution to (1.1) turns out to be a critical point of $\Phi(u)$.

Lemma 3.1. *Let denote*

$$S_\lambda = \inf_{\substack{u\in S^2_{1,0}\\ ||u||_{L^4}=1}} ||\nabla_1 u||^2_{L^2} - \lambda ||u||^2_{L^2}; \quad S = \inf_{\substack{u\in S^2_{1,0}\\ ||u||_{L^4}=1}} ||\nabla_1 u||^2_{L^2}.$$

Then $S_\lambda < S$ for all $\lambda > 0$.

Proof. Without loss of generality, assume that $O \in \Omega$. Let $\varphi(x,y) \in C^\infty_c(\Omega)$ be a cut-off function such that $\varphi \equiv 1$ in a neighborhood of O.

For a fixed $\varepsilon > 0$, let denote

$$U(x,y) = \left((1+|x|^2)^2 + 4|y|^2\right)^{-\frac{1}{2}} \text{ and } u_\varepsilon(x,y) = \varphi(x,y)\left((\varepsilon+|x|^2)^2 + 4|y|^2\right)^{-\frac{1}{2}}.$$

Consider the ratio

$$Q_\lambda(u) = \frac{||\nabla_1 u||^2_{L^2} - \lambda ||u||^2_{L^2}}{||u||^2_{L^4}}.$$

It is invariant under scaling and $S_\lambda = \inf_{u\in S^2_{1,0}\setminus\{0\}} Q_\lambda(u)$. Hence, we will estimate $Q_\lambda(u)$ at the function u_ε by verifying that

$$\begin{aligned} ||\nabla_1 u_\varepsilon||^2_{L^2} &= \frac{K_1}{\varepsilon} + O(1),\\ ||u_\varepsilon||^2_{L^4} &= \frac{K_2}{\varepsilon} + O(1),\\ ||u_\varepsilon||^2_{L^2} &\geq C|\log \varepsilon|, \end{aligned}$$

where K_1, K_2, K_3 are constants:

$$K_1 = ||\nabla_1 U||^2_{L^2(\mathbb{R}^3)}; \quad K_2 = ||U||^2_{L^4(\mathbb{R}^3)}; \quad K_3 = ||U||^2_{L^2(\mathbb{R}^3)}.$$

Moreover, $K_1 = K_2.S$.

First, let note that $\nabla_1\varphi = 0$ near the origin and its derivatives and itself are bounded on their supports, we obtain

$$\begin{aligned} \nabla_1 u_\varepsilon &= \frac{\nabla_1\varphi(x,y)}{((\varepsilon+|x|^2)^2+4|y|^2)^{\frac{1}{2}}} - \frac{1}{2}\frac{\varphi(x,y)\left(4(\varepsilon+|x|^2)x, 8|x|y\right)}{((\varepsilon+|x|^2)^2+4|y|^2)^{\frac{3}{2}}}\\ \Rightarrow ||\nabla_1 u_\varepsilon||^2_{L^2} &= O(1) + 4\int_\Omega \frac{|x|^2}{\left((\varepsilon+|x|^2)^2+4|y|^2\right)^2}\mathrm{d}x\mathrm{d}y\\ &= O(1) + 4\int_{\mathbb{R}^3} \frac{|x|^2}{\left((\varepsilon+|x|^2)^2+4|y|^2\right)^2}\mathrm{d}x\mathrm{d}y. \end{aligned}$$

By the change of variables $x = \sqrt{\varepsilon}x'$ and $y = \varepsilon y'$, the integral becomes

$$\int_{\mathbb{R}^3} \frac{\varepsilon|x'|^2}{\left((\varepsilon+\varepsilon|x'|^2)^2+4|\varepsilon y'|^2\right)^2}\varepsilon^2\mathrm{d}x'\mathrm{d}y' = \int_{\mathbb{R}^3} \frac{|x'|^2}{\varepsilon\left((1+|x'|^2)^2+4|y'|^2\right)^2}\mathrm{d}x'\mathrm{d}y'.$$

Hence, we derive

$$||\nabla_1 u_\varepsilon||^2_{L^2} = O(1) + \frac{K_1}{\varepsilon}. \tag{3.1}$$

Second, we estimate

$$
\begin{aligned}
||u_\varepsilon||_{L^4}^4 &= \int_\Omega |u_\varepsilon|^4 \mathrm{d}x\mathrm{d}y = \int_\Omega \frac{\varphi^4}{\left((\varepsilon+|x|^2)^2+4|y|^2\right)^2}\mathrm{d}x\mathrm{d}y \\
&= \int_\Omega \frac{\varphi^4-1}{\left((\varepsilon+|x|^2)^2+4|y|^2\right)^2}\mathrm{d}x\mathrm{d}y + \int_\Omega \frac{1}{\left((\varepsilon+|x|^2)^2+4|y|^2\right)^2}\mathrm{d}x\mathrm{d}y \\
&= O(1) + \int_{\mathbb{R}^3} \frac{1}{\left((\varepsilon+|x|^2)^2+4|y|^2\right)^2}\mathrm{d}x\mathrm{d}y \\
&= O(1) + \int_{\mathbb{R}^3} \frac{1}{\varepsilon^2\left((1+|x'|^2)^2+4|y'|^2\right)^2}\mathrm{d}x'\mathrm{d}y' \\
&= O(1) + \frac{K_2'}{\varepsilon^2},
\end{aligned}
$$

where $K_2' = ||U||_{L^4}^4$. Therefore, we gain

$$
||u_\varepsilon||_{L^4}^2 = O(1) + \frac{K_2}{\varepsilon}. \tag{3.2}
$$

Similarly, we obtain

$$
\begin{aligned}
||u_\varepsilon||_{L^2}^2 &= \int_\Omega |u_\varepsilon|^2 \mathrm{d}x\mathrm{d}y = \int_\Omega \frac{\varphi^2}{(\varepsilon+|x|^2)^2+4|y|^2}\mathrm{d}x\mathrm{d}y \\
&= \int_\Omega \frac{\varphi^2-1}{(\varepsilon+|x|^2)^2+4|y|^2}\mathrm{d}x\mathrm{d}y + \int_\Omega \frac{1}{(\varepsilon+|x|^2)^2+4|y|^2}\mathrm{d}x\mathrm{d}y. \qquad (3.3)
\end{aligned}
$$

There exists $R>0$ such that $B(R) = \{|x|^4+2|y|^2 \le R^2\} \subset \Omega$. Therefore, the latter integral in (3.3) can be estimated by

$$
\int_\Omega \frac{1}{(\varepsilon+|x|^2)^2+4|y|^2}\mathrm{d}x\mathrm{d}y \ge \int_{B(R)} \frac{1}{2\varepsilon^2+2|x|^4+4|y|^2}\mathrm{d}x\mathrm{d}y.
$$

Changing the variables

$$
\begin{cases} x_1 = \sqrt{r\sin\theta}\cos\varphi, \\ x_2 = \sqrt{r\sin\theta}\sin\varphi, \\ y = \dfrac{r}{\sqrt{2}}\cos\theta, \end{cases} \qquad \text{where } \begin{cases} 0 < r \le R, \\ 0 \le \varphi < 2\pi, \\ 0 \le \theta \le \pi. \end{cases}
$$

The Jacobian $J = \frac{r}{2\sqrt{2}}$. Therefore,

$$
\begin{aligned}
\int_{B(R)} \frac{1}{2\varepsilon^2+2|x|^4+4|y|^2}\mathrm{d}x\mathrm{d}y &= \int_0^{2\pi} d\varphi \int_0^\pi d\theta \int_0^R \frac{rdr}{4\sqrt{2}(\varepsilon^2+r^2)} \\
&= \frac{\pi^2}{4\sqrt{2}} \log(\varepsilon^2+r^2)\,\Big|_0^R = C|\log\varepsilon| + O(1).
\end{aligned}
$$

Hence, we obtain

$$
||u_\varepsilon||_{L^4}^2 \ge C|\log\varepsilon|. \tag{3.4}
$$

Combining (3.1), (3.2), (3.4), we conclude that

$$Q_\lambda(u_\varepsilon) \leq \frac{K_1}{K_2} - \lambda \frac{C}{K_2} \varepsilon |\log \varepsilon| + O(\varepsilon)$$

which implies that $S_\lambda < \frac{K_1}{K_2} = S$ for all $\lambda > 0$. $\square$

Remark 3.2. $S = \frac{1}{\sqrt{2\pi}}$ corresponds to the best constant of the Sobolev type embedding

$$S^2_{1,0}(\Omega) \hookrightarrow\hookrightarrow L^4(\Omega).$$

The ratio $\frac{||\nabla_1 u||_{L^2}}{||u||_{L^4}}$ is invariant under scaling. When $\Omega = \mathbb{R}^3$, the infimum is achieved by the function $U(x,y) = \left((1+|x|^2)^2 + 4|y|^2\right)^{-\frac{1}{2}}$.

Proof of Theorem 1.1. Let $\{u_j\}_{j\in\mathbb{N}} \subset S^2_{1,0}(\Omega)$ be a minimizing sequence for S_λ, that is

$$||u_j||_{L^4} = 1, \quad ||\nabla_1 u_j||^2_{L^2} - \lambda ||u_j||^2_{L^2} = S_\lambda + o(1). \tag{3.5}$$

Using the embedding $L^4(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$ on the bounded domain Ω, $\{u_j\}$ is also bounded in $S^2_{1,0}(\Omega)$. We may extract a subsequence which is still denoted by $\{u_j\}$ such that

$$\begin{aligned} &u_j \rightharpoonup u \text{ weakly in } S^2_{1,0}(\Omega), \\ &u_j \to u \text{ strongly in } L^q(\Omega),\ 1 < q < 4, \\ &u_j \rightharpoonup u \text{ weakly in } L^4(\Omega), \\ &u_j \to u \text{ a.e. in } \Omega. \end{aligned}$$

Moreover, $||u_j||_{L^4} = 1$ implies that $||u||_{L^4} \leq 1$.

By definition of S, we have $||\nabla_1 u_j||_{L^2} \geq S$. It follows from (3.5) that

$$\lambda ||u||^2_{L^2} \geq S - S_\lambda > 0,$$

therefore, $u \not\equiv 0$. Furthermore, as $u_j \rightharpoonup u$ weakly in $S^2_{1,0}(\Omega)$ and $u_j \to u$ strongly in $L^2(\Omega)$, from (3.5) we get

$$||\nabla_1 u||^2_{L^2} + ||\nabla_1 (u_j - u)||^2_{L^2} - \lambda ||u||^2_{L^2} = S_\lambda + o(1). \tag{3.6}$$

Using the weak convergence in $L^4(\Omega)$ and almost everywhere convergence in Ω of $\{u_j\}$ to u, we deduce from a result of Brezis–Loss [3] that

$$||u_j||^4_{L^4} = ||u||^4_{L^4} + ||u - u_j||^4_{L^4} + o(1).$$

This in turn yields that

$$\begin{aligned} 1 = ||u_j||^4_{L^4} &= ||u||^4_{L^4} + ||u - u_j||^4_{L^4} + o(1) \\ &\leq ||u||^2_{L^4} + ||u - u_j||^2_{L^4} + o(1) \\ &\leq ||u||^2_{L^4} + \frac{1}{S} ||\nabla_1 (u - u_j)||^2_{L^2} + o(1). \end{aligned}$$

Because $\lambda < \lambda_1$, we know that $S_\lambda > 0$. The last estimate implies

$$S_\lambda \leq S_\lambda ||u||_{L^4}^2 + \frac{S_\lambda}{S} ||\nabla_1 (u - u_j)||_{L^2}^2 + o(1).$$

Combining with (3.6), we have

$$\begin{aligned} ||\nabla_1 u||_{L^2}^2 - \lambda ||u||_{L^2}^2 &= S_\lambda - ||\nabla_1 (u_j - u)||_{L^2}^2 + o(1) \\ &\leq S_\lambda ||u||_{L^4}^2 + \left(\frac{S_\lambda}{S} - 1 \right) ||\nabla_1 (u - u_j)||_{L^2}^2 + o(1) \leq S_\lambda. \end{aligned}$$

This fact means that u is a minimizer to S_λ, which is also a minimizer of the functional

$$\bar{E}(w) = \int_\Omega \left(|\nabla_1 w|^2 + |w|^2 \right) \mathrm{d}x\mathrm{d}y$$

under the constraint $G(w) = \int_\Omega |w|^4 \mathrm{d}x\mathrm{d}y = 1$. It is obvious that $DG(u) \neq 0$ as $\langle u, DG(u) \rangle = 4 \int_\Omega |u|^4 \mathrm{d}x\mathrm{d}y = 4$. Therefore, by the Lagrange multiplier rule, we obtain the existence of a multiplier μ such that $D\bar{E}(u) = \mu DG(u)$, or equivalently, for all $\varphi \in S_{1,0}^2(\Omega)$, it holds

$$\int_\Omega \nabla_1 u \cdot \nabla_1 \varphi \mathrm{d}x\mathrm{d}y - \lambda \int_\Omega u \varphi \mathrm{d}x\mathrm{d}y - \mu \int_\Omega u^3 \varphi \mathrm{d}x\mathrm{d}y = 0.$$

On the one hand, take $\varphi = u$, we see that $\mu = S_\lambda$. On the other hand, this identity implies that u is a weak solution to the equation

$$-G_1 u = \lambda u + S_\lambda u^3.$$

As this solution is a minimizer for S_λ, $|u|$ is also a nonnegative weak solution to the aforementioned equation.

Since $S_\lambda > 0$, by scaling, we obtain a nontrivial nonnegative weak solution, which we also denoted by u, to (1.1). □

Remark 3.3. If $\lambda \geq \lambda_1$, then (1.1) does not possess a nontrivial solution. Indeed, take the first eigenfunction $\varphi_1 \in S_{1,0}^2(\Omega)$ as the test function, we obtain

$$\begin{aligned} \int_\Omega (-G_1 u) \varphi_1 \mathrm{d}x\mathrm{d}y &= \int_\Omega u (-G_1 \varphi_1) \mathrm{d}x\mathrm{d}y = \lambda_1 \int_\Omega u \varphi_1 \mathrm{d}x\mathrm{d}y \\ &= \lambda \int_\Omega u \varphi_1 \mathrm{d}x\mathrm{d}y + \int_\Omega u^3 \varphi_1 \mathrm{d}x\mathrm{d}y. \end{aligned}$$

Moreover, the Grushin operator is elliptic except on the set $x = 0$, the first eigenfunction φ_1 must be positive outside that surface. Besides that, u is a nontrivial solution, which implies the positivity of u in a certain set E with nonzero measure. Hence,

$$\begin{aligned} \int_\Omega u^3 \varphi_1 \mathrm{d}x\mathrm{d}y &\geq \int_E u^3 \varphi_1 \mathrm{d}x\mathrm{d}y > 0; \\ \int_\Omega u \varphi_1 \mathrm{d}x\mathrm{d}y &\geq \int_E u \varphi_1 \mathrm{d}x\mathrm{d}y > 0. \end{aligned}$$

Combining these facts, we conclude that

$$\lambda_1 \int_\Omega u\varphi_1 \mathrm{d}x\mathrm{d}y > \lambda \int_\Omega u\varphi_1 \mathrm{d}x\mathrm{d}y.$$

Therefore, $\lambda_1 > \lambda$.

Remark 3.4. Following the work [8], we show that in the case $\lambda \leq 0$ and Ω is δ_t-starshaped with respect to the origin ($O \in \Omega$), the equation (1.1) also does not possess a nontrivial solution.

Let us recall the framework involving the operator Δ_λ. Theorem 2.2 in [8] states that, if u is a solution to the problem

$$\begin{aligned} \Delta_\lambda u + f(u) &= 0 \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \end{aligned}$$

then the following Pohozaev type identity holds

$$\int_\Omega \left[F(u) + \left(\frac{1}{Q} - \frac{1}{2}\right) uf(u)\right] \mathrm{d}x = \frac{1}{2Q} \int_{\partial\Omega} \left(\frac{\partial u}{\partial \nu}\right)^2 |\nu_\lambda|^2 \langle T, \nu\rangle ds,$$

where $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$ is the outward normal vector, $\nu_\lambda = (\lambda_1\nu_1, \lambda_2\nu_2, \ldots, \lambda_n\nu_n)$, and T is the vector field determined by $T = \sum_{i=1}^n \varepsilon_i x_i \frac{\partial}{\partial x_i}$. Furthermore, if $f(u)$ is locally Lipschitz, then Theorem 2.6 in [8] states a nonexistence result.

In our present consideration of the Grushin type operator G_1 in $\mathbb{R}^2 \times \mathbb{R}$, we have

$$\varepsilon_1 = \varepsilon_2 = 1, \varepsilon_3 = 2; \quad \lambda_1 = \cdots = \lambda_2 = 1, \lambda_3 = 2|x|.$$

The domain Ω is said to be δ_t-starshaped with respect to O if $\langle T, \nu\rangle = x_1\nu_1 + x_2\nu_2 + 2y\nu_3 \geq 0$ on $\partial\Omega$.

Replacing the nonlinearity $f(u) = \lambda u + u^3$ which is locally Lipschitz and

$$F(u) + \left(\frac{1}{4} - \frac{1}{2}\right) f(u)u = \frac{\lambda}{4} u^2 \leq 0 \text{ for all } u > 0 \text{ and } \lambda \leq 0.$$

Therefore, Theorem 2.6 in [8] concludes that there does not exist a nontrival weak solution to (1.1).

3.2. Proof of Theorem 1.2

Let us recall the Euler–Lagrange functional associated with the problem (1.2) mentioned in Theorem 1.2 as follows:

$$\Psi(u) = \frac{1}{2} \int_\Omega |\nabla_1 u|^2 \mathrm{d}x\mathrm{d}y - \frac{1}{4} \int_\Omega |u|^4 \mathrm{d}x\mathrm{d}y - \int_\Omega F(x, y, u) \mathrm{d}x\mathrm{d}y.$$

We can check that Ψ is well defined on $S^2_{1,0}(\Omega)$ and $\Psi \in C^1(S^2_{1,0}(\Omega), \mathbb{R})$ with

$$\Psi'(u)(v) = \int_\Omega \nabla_1 u \cdot \nabla_1 v \mathrm{d}x\mathrm{d}y - \int_\Omega u^3 v \mathrm{d}x\mathrm{d}y - \int_\Omega f(x, y, u) v \mathrm{d}x\mathrm{d}y.$$

A weak solution to (1.2) turns out to be a critical point of $\Psi(u)$.

We aim at showing the existence of a nonnegative solution. Therefore, we introduce an auxiliary functional $\Phi(u)$ as follows. By the structure of $f(x, y, u)$, let us fix a constant $\mu \geq 0$ such that

$$-f(x, y, u) \leq \mu u + u^3 \quad \text{for all } u \geq 0. \tag{3.7}$$

We denote $u^+ = \max\{u, 0\}$ and $u^- = \min\{u, 0\}$ and then define

$$\Phi(u) = \int_\Omega \left\{ \frac{1}{2}|\nabla_1 u|^2 + \frac{1}{2}\mu u^2 - F(x, y, u^+) - \frac{1}{4}(u^+)^4 - \frac{1}{2}\mu(u^+)^2 \right\} \mathrm{d}x\mathrm{d}y.$$

$\Phi(u)$ is also a C^1 functional, $\Phi(0) = 0$ and $\Phi(u) = \Psi(u)$ when $u \geq 0$.

In the following, we verify that $\Phi(u)$ satisfies the assumptions of Lemma 2.3.

Lemma 3.5. *There exist $\rho > 0$ and $R > 0$ such that if $||u||_{S^2_{1,0}} = R$, then $\Phi(u) \geq \rho$.*

Proof. By the condition (g1), for any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$g(x, y, u) \leq \varepsilon u \text{ for a.a. } (x, y) \in \Omega \text{ and for all } 0 \leq u \leq \delta.$$

By the condition (g2), there exists an $M > 0$ such that

$$g(x, y, u) \leq \varepsilon u^3 \text{ for a.a. } (x, y) \in \Omega \text{ and for all } u \geq M.$$

By evaluating $g(x, y, u)$ for $\delta \leq u \leq M$, we obtain that for a certain value $C_1 = C_1(\varepsilon) > 0$, it holds

$$g(x, y, u) \leq \varepsilon u + C_1 u^3,$$

and consequently,

$$F(x, y, u) \leq \frac{1}{2}a(x, y)u^2 + \frac{\varepsilon}{2}u^2 + C_1\frac{u^4}{4}$$

for a.a. $(x, y) \in \Omega$ and for all $u \geq 0$.

Therefore, we can estimate

$$\Phi(u) \geq \int_\Omega \left\{ \frac{1}{2}|\nabla_1 u|^2 - \frac{1}{2}a(x, y)(u^+)^2 - \frac{\varepsilon}{2}(u^+)^2 - \frac{C_1 + 1}{4}(u^+)^4 \right\} \mathrm{d}x\mathrm{d}y.$$

Using the spectrum condition (1.3), with ε small enough, we have

$$\Phi(u) \geq \frac{1}{2}|||u^-|||^2_{S^2_{1,0}} + \frac{\alpha_0}{2}|||u^+|||^2_{S^2_{1,0}} - \frac{C_1 + 1}{4S^2}|||u^+|||^4_{S^2_{1,0}}.$$

This estimate implies that there exists some small positive R such that for $|||u|||_{S^2_{1,0}} = R$, it holds $\Phi(u) > \rho > 0$ for a certain ρ. □

Lemma 3.6. *There exists $v \in S^2_{1,0}(\Omega)$ such that $||v||_{S^2_{1,0}} > R$ and $\Phi(v) < 0 < \rho$.*

Proof. Let us consider the special function v_0 in the assumption (1.4). $v_0 \geq 0$ so $\Phi(tv_0) = \Psi(tv_0)$ for all $t \geq 0$. We will show a stronger statement that $\lim\limits_{t\to+\infty} \Phi(tv_0) = -\infty$. For each $t > 0$, we have

$$\Phi(tv_0) = \frac{t^2}{2}||\nabla_1 v_0||^2_{L^2} - \frac{t^4}{4}||v_0||^4_{L^4} - \int_\Omega F(x, y, tv_0)\mathrm{d}x\mathrm{d}y. \tag{3.8}$$

By condition (g2) and the boundedness of f on bounded sets, we have for a fixed $\varepsilon > 0$, there is a constant $C_2 = C_2(\varepsilon)$ such that

$$|f(x, y, u)| \leq \varepsilon u^3 + C_2 \text{ for a.a. } (x, y) \in \Omega, \text{ for all } u \geq 0. \tag{3.9}$$

Hence,

$$|F(x, y, u)| \leq \frac{\varepsilon}{4}u^4 + C_2 u \text{ for a.a. } (x, y) \in \Omega, \text{ for all } u \geq 0. \tag{3.10}$$

Hence, for t large enough, the second term in (3.8) is prominent, which implies that

$$\lim_{t\to+\infty} \Phi(tv_0) = -\infty.$$

It implies that choosing t_0 large enough, one obtains $||t_0 v_0||_{S^2_{1,0}} > R$ and $\Phi(t_0 v_0) < 0 < \rho$. □

Applying Lemma 2.3, we obtain a sequence $\{u_j\} \subset S^2_{1,0}(\Omega)$ such that $\Phi(u_j) \to c$ and $\Phi'(u_j) \to 0$ in $(S^2_{1,0}(\Omega))^*$, where

$$c = \inf_{h\in\Gamma} \sup_{u\in h} \Phi(u); \quad \Gamma = \{h \in C([0, 1]; S^2_{1,0}(\Omega)) : h(0) = 0, h(1) = t_0 v_0\}.$$

Moreover, by condition (1.4), we know that $c < \frac{S^2}{2}$.

We now show that $\{u_j\}$ is a bounded sequence in $S^2_{1,0}(\Omega)$.

According to the convergence property of the sequence $\{u_j\}$ itself, we can write

$$\begin{aligned}&\int_\Omega \left\{\frac{1}{2}|\nabla_1 u_j|^2 + \frac{1}{2}\mu u_j^2 - F(x, y, u_j^+) - \frac{1}{4}(u_j^+)^4 - \frac{1}{2}\mu(u_j^+)^2\right\} \mathrm{d}x\mathrm{d}y \\ &\quad = c + o(1),\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}&\langle\Phi'(u_j), u_j\rangle \\ &= \int_\Omega \left\{|\nabla_1 u_j|^2 + \mu u_j^2 - f(x, y, u_j^+)u_j^+ - (u_j^+)^4 - \mu(u_j^+)^2\right\} \mathrm{d}x\mathrm{d}y.\end{aligned} \tag{3.12}$$

From these estimates, we obtain

$$\begin{aligned}\frac{1}{4}\int_\Omega (u_j^+)^4\mathrm{d}x\mathrm{d}y =&\ c + o(1) + \int_\Omega \{F(x, y, u_j^+) - \frac{1}{2}f(x, y, u_j^+)u_j^+\}\mathrm{d}x\mathrm{d}y \\ &- \frac{1}{2}\langle\Phi'(u_j), u_j\rangle \\ \leq&\ c + o(1) + \int_\Omega \{F(x, y, u_j^+) - \frac{1}{2}f(x, y, u_j^+)u_j^+\}\mathrm{d}x\mathrm{d}y\end{aligned} \tag{3.13}$$

$$+\frac{1}{2}||\Phi'(u_j)||_{(S^2_{1,0})^*}||u_j||_{S^2_{1,0}}.$$

Estimating the last integral by (3.9), (3.10), choosing ε small enough, (3.13) then yields

$$\int_\Omega (u_j^+)^4 \mathrm{d}x\mathrm{d}y \le C + C||u_j||_{S^2_{1,0}}.$$

Taking the relation (3.11) into account, noticing the exponent of $||u_j||_{S^2_{1,0}}$ on each side, we conclude the boundedness of $||u_j||_{S^2_{1,0}}$. We can extract a subsequence, still denoted by $\{u_j\}$, which satisfy the followings

$$\begin{aligned}&u_j \rightharpoonup u \text{ weakly in } S^2_{1,0}(\Omega),\\ &u_j \to u \text{ in } L^q(\Omega) \text{ for all } q < 4,\\ &u_j \to u \text{ a.e. in } \Omega.\end{aligned}$$

Since Ω is bounded, using the growth condition of f, as in [14], these convergence properties imply

$$\begin{aligned}(u_j^+)^3 &\to (u^+)^3 \text{ a.e. and weakly in } L^{\frac{4}{3}}(\Omega),\\ f(x,y,u_j^+) &\to f(x,y,u^+) \text{ a.e. and weakly in } L^{\frac{4}{3}}(\Omega).\end{aligned}$$

Therefore, passing to the limit in the expression of $\Phi'(u_j)$, we conclude that

$$-G_1 u + \mu u = (u^+)^3 + f(x,y,u^+) + \mu u^+ \text{ in } (S^2_{1,0}(\Omega))^*.$$

By (3.7), the right hand side of the estimate above is nonnegative. By the Maximum principle in [13], we get that $u \ge 0$. Therefore, we deduce that u is a solution to the equation (1.2).

We shall now verify that u is a nontrivial solution. Indeed, suppose that $u \equiv 0$. We aim at passing to the limit in (3.11), (3.12).

By (3.9), (3.10), we have

$$\left|\int_\Omega f(x,y,u_j^+)u_j^+ \mathrm{d}x\mathrm{d}y\right| \le \varepsilon \int_\Omega (u_j^+)^4 \mathrm{d}x\mathrm{d}y + C_2\int_\Omega u_j^+ \mathrm{d}x\mathrm{d}y,$$

$$\left|\int_\Omega F(x,y,u_j^+)\mathrm{d}x\mathrm{d}y\right| \le \frac{\varepsilon}{4}\int_\Omega (u_j^+)^4 \mathrm{d}x\mathrm{d}y + C_2\int_\Omega u_j^+ \mathrm{d}x\mathrm{d}y. \tag{3.14}$$

Because $\{u_j\}$ is bounded in $S^2_{1,0}(\Omega)$, it is also bounded in $L^4(\Omega)$. By compactness embedding, $u_j \to 0$ in $L^2(\Omega)$. Therefore, the estimates (3.14) imply

$$\int_\Omega f(x,y,u_j^+)u_j^+ \mathrm{d}x\mathrm{d}y \to 0, \ \int_\Omega F(x,y,u_j^+)\mathrm{d}x\mathrm{d}y \to 0.$$

Again $\{u_j\}$ is bounded in $S^2_{1,0}(\Omega)$, after extracting a subsequence which still is denoted by $\{u_j\}$, we may assume that

$$\int_\Omega |\nabla_1 u_j|^2 \mathrm{d}x\mathrm{d}y \to l, \ l \in \mathbb{R}.$$

Applying these limits in (3.12) then (3.11), we get

$$\int_{\Omega} (u_j^+)^4 \mathrm{d}x\mathrm{d}y \to l; \text{ and then } \frac{l}{4} = c.$$

Now we recall the sharp estimate

$$\|\nabla_1 u_j\|_{L^2}^2 \geq S\|u_j\|_{L^4}^2 \geq S\|u_j^+\|_{L^4}^2,$$

which implies after taking the limit that

$$l \geq S l^{\frac{1}{2}} \Rightarrow c \geq \frac{S^2}{4},$$

which contradicts to the fact $c < \frac{S^2}{4}$. Hence, u is a nontrivial nonnegative weak solution to (1.2).

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