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REPRESENTATIONS OF C^*-TERNARY RINGS

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Abstract. It is proved that there is a one to one correspondence between representations of C^*-ternary ring M and C^*-algebra $\mathcal{A}(M)$. We discuss primitive and modular ideals of a C^*-ternary ring and prove that a closed ideal I is primitive or modular if and only if so is the ideal $\mathcal{A}(I)$ of $\mathcal{A}(M)$. We also show that a closed ideal in M is primitive if and only if it is the kernel of some irreducible representation of M. Lastly, we obtain approximate identity characterization of strongly quasi-central C^*-ternary ring and the ideal structure of the TRO $V\otimes^{\mathrm{tmin}}B$ for a C^*-algebra B.

1. Introduction and preliminaries

For Hilbert spaces H and K, let $B(H,K)$ denote the space of all bounded linear operators from H to K. A (concrete) ternary ring of operator (TRO) between Hilbert spaces H and K is a norm closed subspace of $B(H,K)$, which is closed under the triple product $(x,y,z)\to xy^*z$. They were first introduced by Hestenes [10]. Tensor products, inductive limits, representation and ideal theory in category of TROs had been explored in a variety of papers including [2,5,7–9,13,14,20]. Let $V\subset B(H,K)$ be a TRO and $V^*=\{x^*:x\in V\}$ denote the conjugate space of V. Let $C(V)$ and $D(V)$ denote the C^*-algebras generated by VV^* and V^*V, respectively. Then, V is a non-degenerate and faithful Hilbert left-$C(V)$ and right-$D(V)$ bimodule such that $CV=V$ and $VD=V$. Also, we have the C^*-isomorphisms $C=K(V_D)$ and $D^{op}=K({}_CV)$, where we let $K(V_D)$ denote the space of all compact right-D module homomorphisms on V and $K({}_CV)$ denote the space of all compact left-C module homomorphisms on V. As given in [7], the C^*-algebra $A(V)$ generated by V, known as the linking C^*-algebra of V, is defined by

$$A(V)=\begin{bmatrix}C(V) & V\\ V^* & D(V)\end{bmatrix}.$$

A C^*-ternary ring is a complex Banach space M, equipped with a ternary product $(x,y,z)\to[x,y,z]$ of M^3 into M which is linear in the first and third

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variable, conjugate linear in the second variable, associative in the sense that $[[x, y, z], u, v] = [x, y, [z, u, v]] = [x, [u, z, y], v]$ and satisfies $||[x, x, x]|| = ||x||^3$, $||[x, y, z]|| \leq ||x||||y||||z||$. We refer to [1], [19] and [23] for all necessary background. Clearly, every TRO (in particular C^*-algebra, $M_{m\times n}(\mathbb{C})$, the space of all $m \times n$ matrices with entrices in $\mathbb{C}$ or $B(H, K)$) is a C^*-ternary ring. Recently, several mathematicians have been interested in studying different kind of maps and their stability between C^*-ternary rings. Hyers-Ulam stability of hom-derivations was proved in [11]. In [18], C. Park et al. studied partial multipliers in C^*-ternary rings. M. Moslehian established the Hyers-Ulam-Rassias stability of derivations in C^*-ternary rings [16]. The main subject of our paper is representations and ideals of C^*-ternary rings. We first recall some basic terminology related to C^*-ternary rings. A linear mapping ϕ between C^*-ternary rings is called a (ternary) homomorphism if ϕ preserves the ternary structure, i.e., $\phi([x, y, z]) = [\phi(x), \phi(y), \phi(z)]$. A norm-closed subspace I in a C^*-ternary rings M is called a right (left) ideal in M if $[I, M, M] \subset I$ ($[M, M, I] \subset I$). Throughout the paper, when we say that I is an ideal of M, we shall always assume that I is a two sided closed ideal of M. Pluta and Russo [19] extended the Hamana's notion of linking C^*-algebras to the category of C^*-ternary rings as follows: For a C^*-ternary ring M, let $\mathrm{End}(M)$ denote the set of all endomorphisms on M. Define,

$$E(M) = \mathrm{End}(M) \oplus \overline{\mathrm{End}(M)}^{\mathrm{op}},$$

where the scalar multiplication in $\overline{[\mathrm{End}(M)]}$ is defined as $(\lambda, f) \to \overline{\lambda} f$ and for $g, h \in M$, define $L(g, h) = [g, h, \cdot], R(g, h) = [\cdot, h, g]$,

$$l(g, h) = (L(g, h), L(h, g)) \in E(M)$$

and

$$r(g, h) = (R(h, g), R(g, h)) \in E(M)^{\mathrm{op}}.$$

Next, let $L = L(M)$ and $R = R(M)$ denote the closure of $\mathrm{span}\{l(g, h) : g, h \in M\}$ and $\mathrm{span}\{r(g, h) : g, h \in M\}$ in $B(M)$, respectively. Let $A = (A_1, A_2) \in E(M)$, $B = (B_1, B_2) \in E(M)^{\mathrm{op}}$, and $f \in M$. Then M is a left $E(M)$-module via $(A, f) \to A \cdot f = A_1 f$ and a right $E(M)^{\mathrm{op}}$-module via $(f, B) \to f \cdot B = B_1 f$. Let $\overline{M}$ denote the vector space M with the element f denoted by $\overline{f}$ and with the scalar multiplication defined by $(\lambda, \overline{f}) \to \lambda \circ \overline{f} = \overline{\overline{\lambda} f}$. Then $\overline{M}$ is a left $E(M)^{\mathrm{op}}$-module via $(B, \overline{f}) \to B \cdot \overline{f} = \overline{B_2 f}$ and a right $E(M)$-module via $(\overline{f}, A) \to \overline{f} \cdot A = \overline{A_2 f}$. Let

$$\mathcal{A} = \mathcal{A}(M) = \begin{bmatrix} L(M) & M \\ \overline{M} & R(M) \end{bmatrix} \subset B(M \oplus R)$$

and define multiplication and involution in $\mathcal{A}$ by

$$\begin{bmatrix} A & f \\ \overline{g} & B \end{bmatrix} \cdot \begin{bmatrix} A' & f' \\ \overline{g'} & B' \end{bmatrix} = \begin{bmatrix} A \cdot A' + l(f, g') & A \cdot f' + f \cdot B' \\ \overline{g} \cdot A' + B.\overline{g'} & r(g, f') + B \circ B' \end{bmatrix}$$

and

$$\begin{bmatrix} A & f \\ \overline{g} & B \end{bmatrix}^{\#} = \begin{bmatrix} \overline{A} & g \\ \overline{f} & \overline{B} \end{bmatrix}.$$

It is immediate to see that if M is a TRO, then $\mathcal{A}(M)$ is *-isomorphic to its linking C^*-algebra. Using ([19], Lemma 2.6), we obtain a functor $M \to \mathcal{A}(M)$, $(M \xrightarrow{\phi} N) \to (\mathcal{A}(M) \xrightarrow{\mathcal{A}(\phi)} \mathcal{A}(N))$ from the category of C^*-ternary rings to the category of C^*-algebras as follows: Given a surjective homomorphism $\phi : M \to N$, we have C^*-homomorphisms $L(\phi) : L(M) \to L(N)$ and $R(\phi) : R(M) \to R(N)$ by letting

$$L(\phi)\left(\sum_i([g_i, h_i, \cdot][h_1, g_i, \cdot])\right) = \sum_i([\phi(g_i), \phi(h_i), \cdot], [\phi(h_i), \phi(g_i), \cdot])$$

and

$$R(\phi)\left(\sum_i([\cdot, g_i, h_i][\cdot, h_i, g_i])\right) = \sum_i([\cdot, \phi(g_i), \phi(h_i)], [\cdot, \phi(h_i), \phi(g_i)]).$$

If the above ϕ is not surjective, then we can replace N by $\phi(N)$, which is a norm-closed sub-C^*-ternary ring.

It is well-known that there is a one to one correspondence between representations of a TRO and of its linking C^*-algebra ([2]). The construction of $\mathcal{A}(M)$ allows us to extend ([2], Proposition 3.1) from TROs to C^*-ternary rings. We study the connection between irreducible representations of C^*-ternary ring M and of C^*-algebra $\mathcal{A}(M)$.

Motivated by the ideal theory of C^*-algebras, we study primitive and modular ideals of C^*-ternary rings and relate them to corresponding primitive and modular ideals of C^*-algebra $\mathcal{A}(M)$. We prove that there is a homeomorphism between modular (primitive) ideals of M and $\mathcal{A}(M)$. We also show that an ideal I of M is primitive if and only if $I = \ker(\phi)$ for some irreducible representation ϕ of M. An approximate identity characterization is obtained for quasi central C^*-ternary rings. Finally, the ideal structure of injective tensor product of a TRO and a C^*-algebra has been discussed.

2. Representation theory of C^*-ternary rings

Representations and ideals of C^*-algebras have been well studied. The reader is referred to [22] and [17] for a detailed discussion on the same. In this section, we obtain a one to one correspondence between representations of a C^*-ternary ring M and of C^*-algebra $\mathcal{A}(M)$.

Definition. Let M be a C^*-ternary ring. A homomorphism $\phi : M \to B(H, K)$ is called a representation of M. If ϕ is injective it is called a faithful representation of M. ϕ is said to be nondegenerate if $\overline{\phi(M)H} = K$ and $\overline{\phi(M)^*K} = H$ (or equivalently, if $\zeta_1 \in H$, $\zeta_2 \in K$ are such that $\phi(M)\zeta_1 = 0$ and $\phi(M)^*\zeta_2 = 0$, then $\zeta_1 = 0$ and $\zeta_2 = 0$).

Let ϕ be a representation of M. Applying the functor $\mathcal{A}$, we obtain a C^*-algebra homomorphism $\mathcal{A}(\phi) : \mathcal{A}(M) \to \mathcal{A}(B(H,K))$. Let σ be a *-isomorphism from linking C^*-algebra $B(K \oplus H)$ of $B(H,K)$ to $\mathcal{A}(B(H,K))$. Every representation of M induces a representation $\pi_\phi = \sigma \circ \mathcal{A}(\phi)$ of C^*-algebra $\mathcal{A}(M)$.

Proposition 2.1. *The map $\phi \to \pi_\phi$ is a bijection from the set of all representations of a C^*-ternary ring M onto the set of all representations C^*-algebra $\mathcal{A}(M)$.*

Proof. Let $\pi : \mathcal{A}(M) \to B(H)$ be a representation of $\mathcal{A}(M)$. We identify $L(M)$, $R(M)$ and M with their images in $\mathcal{A}(M)$ and put $H_1 = \overline{\pi(\mathcal{A}(M))H}$ and $H_2 = H_1^\perp$ then $H = H_2 \oplus H_1$. Let σ be a canonical isomorphism between $B(H)$ and $A(B(H_1, H_2))$. Then $\rho = \sigma \circ \pi$ is a representation of $\mathcal{A}(M)$ satisfying

$$\rho(L(M)) \subset \begin{bmatrix} B(H_1) & 0 \\ 0 & 0 \end{bmatrix}, \ \rho(R(M)) \subset \begin{bmatrix} 0 & 0 \\ 0 & B(H_2) \end{bmatrix}.$$

Let $x \in M$ and write

$$\rho = \begin{bmatrix} \rho_1 & \rho_2 \\ \rho_3 & \rho_4 \end{bmatrix}.$$

Observe that

$$\begin{aligned} \rho \begin{bmatrix} l(x,x) & 0 \\ 0 & 0 \end{bmatrix} &= \rho \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \rho \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}^{\#} \\ &= \rho \begin{bmatrix} \rho_1(x)\rho_1(x)^* + \rho_2(x)\rho_2(x)^* & \rho_1(x)\rho_3(x)^* + \rho_2(x)\rho_4(x)^* \\ \rho_3(x)\rho_1(x)^* + \rho_4(x)\rho_2(x)^* & \rho_3(x)\rho_3(x)^* + \rho_4(x)\rho_4(x)^* \end{bmatrix} \end{aligned}$$

and therefore $\rho_3 = 0 = \rho_4$. Similarly, we get

$$\begin{aligned} \rho \begin{bmatrix} 0 & 0 \\ 0 & r(x,x) \end{bmatrix} &= \rho \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}^{\#} \rho \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \\ &= \rho \begin{bmatrix} \rho_1(x)^*\rho_1(x) & \rho_1(x)^*\rho_2(x) \\ \rho_2(x)^*\rho_1(x) & \rho_2(x)^*\rho_2(x) \end{bmatrix} \end{aligned}$$

and thus $\rho_1 = 0$. Therefore we get a representation ρ_2 of M with $\mathcal{A}(\rho_2) = \rho$. Finally, setting $\phi = \sigma^{-1} \circ \rho_2$ we are done. □

Definition. Let $\phi : M \to B(H,K)$ be a representation of M. For closed subspaces $H_1 \subset H$ and $K_1 \subset K$, the pair (H_1, K_1) is said to be ϕ-invariant if $\phi(M)H_1 \subset K_1$ and $\phi(M)^*K_1 \subset H_1$. A representation ϕ is said to be irreducible if $(0,0)$ and (H,K) are the only invariant subspaces.

The proof of the next proposition is along the similar lines to the proof of ([2], Lemma 3.5), so we omit the proof.

Proposition 2.2. *Let M be a C^*-ternary ring, and let $\phi : M \to B(H,K)$ be a representation of M and let $\pi = \pi_\phi$. Then the followings are equivalent:*

(1) $\phi \neq 0$ *is irreducible.*

(2) $L(\phi) \neq 0$ *and* $R(\phi) \neq 0$ *are irreducible representations of* $L(M)$ *and* $R(M)$*, respectively.*
(3) $\pi \neq 0$ *is irreducible.*

Definition. Let M be a C^*-ternary ring and $\phi_i : M \to B(H_i, K_i)$ be a family of representations. The sum representation $\phi : M \to B(H, K)$ is the homomorphism with $H := \oplus H_i$, $K := \oplus K_i$ and $\phi(x)((h_i)_i) := (\phi_i(x)(h_i))_i$ for $(h_i)_i \in H$ and $x \in M$.

The following corollary is an immediate consequence of the last proposition.

Corollary 2.3. *Every non degenerate representation of a finite dimensional* C^**-ternary ring* M *is the direct sum of irreducible representations.*

Proof. Let $\phi : M \to B(H, K)$ be a nondegenerate representation of M. Let $\pi = \pi_\phi$. Then π is a nondegenerate representation of $\mathcal{A}(M)$. By ([4], Theorem I.10.7), π splits into a direct sum of irreducible representations say $\pi = \oplus \pi_i$. By the last proposition, there exist irreducible representations $\phi_i : M \to B(H, K)$ such that $\pi_i = \pi_{\phi_i}$ thus we obtain $\phi = \oplus \phi_i$. □

For an ideal I of a C^*-ternary ring M, it is immediate from ([19], Lemma 1.1) that $\mathcal{A}(I)$ is an ideal of C^*-algebra $\mathcal{A}(M)$. Moreover, it is not difficult to see that the map θ defined by $I \to \mathcal{A}(I)$ is a one-to-one correspondence between closed ideals of M and $\mathcal{A}(M)$. Let $\mathrm{Id}(M)$ denote the space of all closed ideals of M. We define a topology on $\mathrm{Id}(M)$ which we will call τ_w-topology. A subbasis for τ_w-topology is given by the sets of the form $U(J) = \{I \in \mathrm{Id}(M) : I \not\supseteq J\}$ where $J \in \mathrm{Id}(M)$. It is not difficult to verify that $\mathrm{Id}(M)$ with τ_w-topology is T_0-space.

Proposition 2.4. *The map* $\theta : \mathrm{Id}(M) \to \mathrm{Id}(\mathcal{A}(M))$ *is a homeomorphism.*

Proof. It is enough to prove that θ and its inverse are continuous. Let

$$U(\mathcal{A}(J)) = \{\mathcal{A}(I_1) \in \mathrm{Id}(\mathcal{A}(M)) : \mathcal{A}(I_1) \not\supseteq \mathcal{A}(J)\}$$

be an basic open set of $\mathrm{Id}(\mathcal{A}(M))$. Note that, $\theta^{-1}(U(\mathcal{A}(J))) = U(J)$ so the map θ is continuous. Similarly, note that $\theta(U(I)) = U(\mathcal{A}(I))$ so the map θ is an open map. In particular, θ^{-1} is continuous and therefore θ is a homeomorphism. □

It may be noted that for a left ideal I of M, $(I : M) = \{a \in M : [a, M, M] \subset I\}$ is an ideal of M.

Definition. A left ideal I is called modular if there exist e and f in M such that $a - [a, e, f] \in I$ for every $a \in M$. An ideal I is called a maximal modular ideal if it is modular and also a maximal proper ideal.

Evidently if I is a modular ideal, then $(I : M)$ is the largest ideal of M contained in I. By definition, every ideal containing a modular ideal is itself modular. Therefore an ideal I of M is maximal modular if and only if it is maximal within the set of all modular proper ideal.

Definition. An ideal I of M is called primitive if it is quotient of a maximal modular left ideal, i.e., $I = (J : M)$ for some maximal modular left ideal J of M.

Proposition 2.5. *A maximal modular ideal is primitive.*

Proof. Let I be a proper modular ideal of M and let e and f in M be such that $a - [a, e, f] \in I$ for all $a \in M$. Let J be a maximal modular left ideal containing I. Then, $[I, M, M] \subset I \subset J$ and therefore, $I \subset (J : M)$. Hence, the maximality of I implies that $I = (J : M)$. □

If M is commutative ($[x, y, z] = [z, y, x]$), then the converse is also true, that is, if I is a primitive ideal of M, then I is maximal modular. For a C^*-ternary ring M, let $\mathrm{Max}(M)$ and $\mathrm{Prim}(M)$ denote the set of all closed maximal and primitive ideals of M, respectively.

Theorem 2.6. *Let M be a C^*-ternary ring and I a closed ideal of M. Then the following statements hold*:

(1) *θ maps* $\mathrm{Max}(M)$ *homeomorphically onto* $\mathrm{Max}(\mathcal{A}(M))$.
(2) *I is modular if and only if $\mathcal{A}(I)$ is modular.*
(3) *If I' is a modular closed ideal of M and $J = (I' : M)$, then $\mathcal{A}(J) = \mathcal{A}(I' : M) = (\mathcal{A}(I') : \mathcal{A}(M))$.*
(4) *θ maps* $\mathrm{Prim}(M)$ *homeomorphically onto* $\mathrm{Prim}(\mathcal{A}(M))$.

Proof. (1) Suppose I is maximal and $\mathcal{A}(I) \subset J$ for some ideal J of $\mathcal{A}(M)$. But $J = \mathcal{A}(I')$ so $\mathcal{A}(I) \subset \mathcal{A}(I')$ and therefore $I \subset I'$, contradiction as I is maximal. The converse is obvious as $I \subset I'$ implies $\mathcal{A}(I) \subset \mathcal{A}(I')$.

(2) Suppose that I is modular so there exist $e, f \in M$ such that $x - [x, e, f] \in I$ for all $x \in M$. Let

$$u = \begin{bmatrix} l(e, f) & 0 \\ 0 & r(e, f) \end{bmatrix}.$$

Note that $\mathcal{A}(M) - \mathcal{A}(M)u \in \mathcal{A}(I)$ and $\mathcal{A}(M) - u\mathcal{A}(M) \in \mathcal{A}(I)$ so $\mathcal{A}(I)$ is modular.

Conversely, assume that $\mathcal{A}(I)$ is modular so there exists u such that $\mathcal{A}(M) - \mathcal{A}(M)u \in \mathcal{A}(I)$ and $\mathcal{A}(M) - u\mathcal{A}(M) \in \mathcal{A}(I)$. Let

$$u = \begin{bmatrix} a & b \\ \bar{c} & d \end{bmatrix}.$$

Since $\mathcal{A}(I)$ is modular so we must have,

$$\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -l(x, c) & x - xd \\ 0 & 0 \end{bmatrix} \in \mathcal{A}(I).$$

In particular, $x - xd \in I$ for all x. Similarly, $x - dx \in I$. As $d \in \mathcal{A}(I)$ so we may assume $d = r(e, f)$ and therefore I is modular.

(3) Assume $J = (I' : M)$. As I' is modular therefore there exist $e, f \in M$ such that $x - [x, e, f] \in M$ and $x - [f, e, x] \in M$ for all $x \in M$. Suppose $a \in (I' : M)$, note that as I' is modular so we have $a = [a, e, f] - ([a, e, f] - a) \in I'$ and

therefore $(I' : M) \subset I'$ which implies $J \subset I'$ so $\mathcal{A}(I) \subset \mathcal{A}(I')$. Since $\mathcal{A}(J)$ is an ideal of $\mathcal{A}(M)$ we have $\mathcal{A}(J)\mathcal{A}(M) \subset \mathcal{A}(J) \subset \mathcal{A}(I')$ so $\mathcal{A}(J) \subset (\mathcal{A}(I') : \mathcal{A}(M))$. Now we need to show that $(\mathcal{A}(I') : \mathcal{A}(M)) \subset \mathcal{A}(J)$. Since $(\mathcal{A}(I') : \mathcal{A}(M))$ is an ideal of $\mathcal{A}(M)$ so assume $(\mathcal{A}(I') : \mathcal{A}(M)) = \mathcal{A}(L)$ for some ideal L of M. On the contrary assume $\mathcal{A}(L)$ is not a subset of $\mathcal{A}(J)$ therefore there exists $x \in L$ such that $x \notin J$. Note that

$$\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & r(v,v) \end{bmatrix} = \begin{bmatrix} 0 & [x,v,v] \\ 0 & 0 \end{bmatrix} \in \mathcal{A}(L)\mathcal{A}(M) \subset \mathcal{A}(I').$$

So $[x,v,v] \in I'$ for all $v \in M$ which implies $x \in (I' : M) = J$, a contradiction.

(4) Suppose I is primitive so $I = (I' : M)$ for some maximal modular ideal I' of M. As I' is maximal modular so $\mathcal{A}(M)$ is also maximal modular. Also by last part, $\mathcal{A}(I) = (\mathcal{A}(I') : \mathcal{A}(M))$ so $\mathcal{A}(I)$ is primitive. Conversely, assume that $\mathcal{A}(I)$ is modular so $\mathcal{A}(I) = (J : \mathcal{A}(M))$ for some maximal modular ideal J of $\mathcal{A}(M)$. As J is maximal modular therefore $J = \mathcal{A}(I')$ for some maximal modular ideal I' of M. So, $\mathcal{A}(I) = (\mathcal{A}(I') : \mathcal{A}(M))$. Again using last part, we have $(\mathcal{A}(I') : \mathcal{A}(M)) = \mathcal{A}(I' : M)$ so $\mathcal{A}(I) = \mathcal{A}(I' : M)$ and therefore $I = (I' : M)$ so I is primitive. □

We say that a C^*-ternary ring M has the Wiener property if every proper closed ideal of M is annihilated by some irreducible representation of M. By Theorem 2.6, it is easy to see that every closed TRO ideal is contained in some primitive ideal. Now we shall show that every C^*-ternary ring has Wiener property. First we need a lemma:

Lemma 2.7. *Let $\phi : M \to N$ be a homomorphism of C^*-ternary rings. Then* $\mathrm{Ker}(\mathcal{A}(\phi)) = \mathcal{A}(\mathrm{Ker}(\phi))$.

Proof. Suppose $x = \left[\begin{smallmatrix} A & f \\ g & B \end{smallmatrix}\right] \in \mathrm{Ker}(\mathcal{A}(\phi))$. Then $L(\phi)(A) = 0$, $\phi(f) = 0$, $\overline{\phi(g)} = 0$ and $R(\phi)(B) = 0$ thus f and g belongs to $\mathrm{Ker}(\phi)$. Note that $\phi(f' \cdot B) = 0$ for all $f' \in M$, that is, $M\,\mathrm{Ker}(R(\phi)) \subset \mathrm{Ker}(\phi)$ so $\mathrm{Ker}(L(\phi)) \subset L(\mathrm{Ker}(\phi))$. Similarly, $\mathrm{Ker}(R(\phi)) \subset R(\mathrm{Ker}(\phi))$ thus $\mathrm{Ker}(\mathcal{A}(\phi)) \subset \mathcal{A}(\mathrm{Ker}(\phi))$. On the other hand, it is clear that $\mathcal{A}(\mathrm{Ker}(\phi)) \subset \mathrm{Ker}(\mathcal{A}(\phi))$. □

Theorem 2.8. *Suppose M is a C^*-ternary ring and I is a closed ideal of M. Then I is a primitive ideal of M if and only if I is kernel of some non-zero irreducible representation of M.*

Proof. Let I be a primitive ideal of M. As I is primitive therefore $\mathcal{A}(I)$ is also a primitive ideal of $\mathcal{A}(M)$. Since $\mathcal{A}(M)$ is a C^*-algebra so $\mathcal{A}(I)$ is kernel of some irreducible representation θ of $\mathcal{A}(M)$. By Proposition 2.2, we may assume that $\theta = \pi_\phi$ for some representation $\phi : M \to B(H, K)$. By the last lemma, $\mathcal{A}(\mathrm{Ker}(\phi)) = \mathrm{Ker}(\theta) = \mathcal{A}(I)$. Since $\mathrm{Ker}(\phi)$ is a closed ideal of M, thus $I = \mathrm{Ker}(\phi)$.

Now conversely assume that $I = \mathrm{Ker}(\phi)$, where ϕ is some nonzero irreducible representation of M. Let $\pi = \pi_\phi$. By Proposition 2.2, π is irreducible and by the last lemma, $\ker(\pi) = \mathcal{A}(I)$. As $\mathcal{A}(M)$ is a C^*-algebra so $\mathcal{A}(I)$ being kernel of irreducible must be primitive and therefore I is primitive by Theorem 2.6. □

Let I be an ideal of C^*-ternary ring M. Then M/I is also a C^*-ternary ring. Following is an immediate consequence of the last theorem:

Corollary 2.9. *Let M be a C^*-ternary ring and I be a closed ideal of M. Let $\mathrm{Prim}_{\mathrm{I}}(M)$ and $\mathrm{Prim}^{\mathrm{I}}(M)$ be the sets of primitive ideals of M containing I and not containing I, respectively. Then*

(1) *The map $J \to J/I$ is a homeomorphism of $\mathrm{Prim}_{\mathrm{I}}(M)$ onto $\mathrm{Prim}(M/I)$.*
(2) *The map $J \to J \cap I$ is a homeomorphism of $\mathrm{Prim}^{\mathrm{I}}(M)$ onto $\mathrm{Prim}(I)$.*

Proof. (1) It follows immediately from Theorem 2.8.

(2) Let $J \in \mathrm{Prim}^{\mathrm{I}}(M)$. Then J is kernel of an irreducible representation ϕ of M such that $\phi(I) \neq 0$. Since ϕ is irreducible, $\phi|_I$ is also irreducible and $J \cap I = \mathrm{Ker}(\phi|_I) \in \mathrm{Prim}(I)$ therefore the map $J \to J \cap I$ is well defined. Now assume $J' \in \mathrm{Prim}(I)$, then there exists an irreducible representation ϕ' of I such that $J' = \mathrm{Ker}(\pi')$ so there is an irreducible representation ϕ of M such that $\phi' = \phi|_I$ and therefore $J' = \mathrm{Ker}(\phi) \cap I$. Finally, we need to show that the map is injective. For this, suppose $J_1 \cap I = J_2 \cap I$. Applying the functor $\mathcal{A}$, we get $\mathcal{A}(J_1) \cap \mathcal{A}(I) = \mathcal{A}(J_2) \cap \mathcal{A}(I) \subset \mathcal{A}(J_2)$. Note that

$$\mathcal{A}(J_1)\mathcal{A}(I) = \mathcal{A}(J_1) \cap \mathcal{A}(I) \subset \mathcal{A}(J_2)$$

and $\mathcal{A}(J_2) \not\supseteq \mathcal{A}(I)$. Since J_2 is primitive therefore $\mathcal{A}(J_2)$ is primitive. Since a primitive ideal is prime, $\mathcal{A}(J_2)$ is prime and therefore $\mathcal{A}(J_1) \subset \mathcal{A}(J_2)$ which implies $J_1 \subset J_2$. Similarly, $J_2 \subset J_1$ so $J_1 = J_2$. It is routine to check that the map is in fact a homeomorphism. □

Remark 2.10. We may define a C^*-ternary ring M to be primitive if its zero ideal is primitive. By Theorem 2.8, M is primitive if and only if M has a faithful nonzero irreducible representation. Moreover, M is primitive if and only if $\mathcal{A}(M)$ is primitive. Recall that every nonzero simple C^*-algebra is primitive and thus every nonzero simple C^*-ternary ring is also primitive but converse need not be true. An easy counterexample is provided by $B(H, K)$, where H and K are infinite dimensional Hilbert spaces. Also, one can define an ideal I of M to be essential if I intersects with every ideal of M non trivially. It is easy to verify that I is essential if and only if ideal $\mathcal{A}(I)$ of $\mathcal{A}(\mathcal{M})$ is essential.

Let M be a C^*-ternary ring. We define strong center of M as

$$\mathrm{Z}_{\mathrm{C^*\text{-}tring}}(M) = \{x \in M : [a, x, c] = [c, x, a]\ \forall a, c \in M\}.$$

Observe that if $\mathbf{A}$ is C^*-algebra and $\mathrm{Z}(\mathbf{A})$ denotes center of $\mathbf{A}$, then using approximate identity it is not difficult to see that $\mathrm{Z}_{\mathrm{C^*\text{-}tring}}(A) \subset \mathrm{Z}(\mathbf{A})$ but in general $\mathrm{Z}_{\mathrm{C^*\text{-}tring}}(\mathbf{A})$ may be much smaller than $Z(\mathbf{A})$. For instance, if $A = M_n$ it is not difficult to see that $\mathrm{Z}_{\mathrm{C^*\text{-}tring}}(A) = \{0\}$ while $Z(M_n) = cI_n$. This is the reason we call $\mathrm{Z}_{\mathrm{C^*\text{-}tring}}(M)$, a strong center of M. The following is an easy consequence of the definition.

Proposition 2.11. *The closed linear subspace* $\mathrm{Z}_{\mathrm{C^*\text{-}tring}}(M)$ *is a commutative sub-C^*-ternary ring of M. Moreover, if M is commutative, then* $\mathrm{Z}_{\mathrm{C^*\text{-}tring}}(M) = M$.

Definition. A C^*-ternary ring M is said to be strongly quasi-central if no primitive ideal of M contains its strong center.

It is easy to verify that if $\mathbf{A}$ is a strongly quasi-central C^*-algebra, then $\mathbf{A}$ is quasi central.

Definition. Let M be a C^*-ternary ring. A bounded net (e_λ, f_λ) is a left (resp. right)-bounded approximate identity for M if there exists $n > 0$ such that $||e_\lambda|| \leq n$, $||f_\lambda|| \leq n$ and the net $[e_\lambda, f_\lambda, x]$ (resp. the net $[x, e_\lambda, f_\lambda]$) converges to x, for all $x \in M$.

Proposition 2.12. *Let M be a C^*-ternary ring having an approximate identity each element of which belongs to* $\mathrm{Z}_{\mathrm{C^*\text{-}tring}}(M)$. *Then M is strongly quasi-central. Conversely, if M is strongly quasi-central, then any bounded approximate identity of* $\mathrm{Z}_{\mathrm{C^*\text{-}tring}}(M)$ *is an approximate identity of M.*

Proof. Let (e_λ, f_λ) be an approximate identity of M such that $e_\lambda \in \mathrm{Z}_{\mathrm{C^*\text{-}tring}}(M)$ and $f_\lambda \in \mathrm{Z}_{\mathrm{C^*\text{-}tring}}(M)$ for all λ. Let P be a primitive ideal of M which contains $\mathrm{Z}_{\mathrm{C^*\text{-}tring}}(M)$. Then for any $x \in M$ we have $[e_\lambda, f_\lambda, x] \in P$ and $x = \lim_\lambda [e_\lambda, f_\lambda, x] \in P$. Hence, $P = M$ which is a contradiction. Thus, M is strongly quasi-central.

Conversely, let (e_λ, f_λ) be an approximate identity of $\mathrm{Z}_{\mathrm{C^*\text{-}tring}}(M)$. Define

$$I = \{x \in M : \lim_\lambda [e_\lambda, f_\lambda, x] = x\}.$$

Note that $\mathrm{Z}_{\mathrm{C^*\text{-}tring}}(M) \subset I$ and I is an ideal of M. Also since (e_λ, f_λ) is bounded, I is closed. Now if (e_λ, f_λ) is not an approximate identity of M, then $I \neq M$ so I is a proper ideal of M. Let P be a primitive ideal of M which contains I and hence $\mathrm{Z}_{\mathrm{C^*\text{-}tring}}(M)$. Consequently, M is not strongly quasi-central. ☐

3. Injective tensor product for TROs

In this section all our C^*-ternary rings are TROs. This section is devoted to applications of results obtained in the previous section. Since for a TRO V, $\mathcal{A}(V)$ is *-isomorphic to $A(V)$ so we will identify $\mathcal{A}(V)$ with $A(V)$. Given TROs $V \subset B(H_1, K_1)$ and $W \subset B(H_2, K_2)$, there is a canonical triple product on the algebraic tensor product $V \otimes W$ given by

$$(v_1 \otimes w_1)(v_2 \otimes w_2)^*(v_3 \otimes w_3) = v_1 v_2^* v_3 \otimes w_1 w_2^* w_3.$$

We let $V \otimes^{\mathrm{tmin}} W$ denote the closure of $V \otimes W$ in $B(H_1 \overline{\otimes} H_2, K_1 \overline{\otimes} K_2)$. By the very construction itself we see that $V \otimes^{\mathrm{tmin}} W$ is a TRO. It is clear that if V

and W are C^*-algebras, then $V \otimes^{\mathrm{tmin}} W$ is also a C^*-algebra and is same as $V \otimes^{\min} W$. For more details the reader is referred to [14]. It is worth noting that if V is an exact TRO and B is an exact C^*-algebra, then $V \otimes^{\mathrm{tmin}} B$ is also an exact TRO.

Proposition 3.1. *Let V_1, W_1, V_2, W_2 be TROs. Let $f_i : V_i \to W_i$ be homomorphisms for $i = 1, 2$. Then $f_1 \otimes f_2$ continuously extends to a homomorphism $f_1 \otimes f_2 : V_1 \otimes^{\mathrm{tmin}} V_2 \to W_1 \otimes^{\mathrm{tmin}} W_2$. Moreover, $f_1 \otimes f_2$ is injective if f_1 and f_2 are so.*

Proof. By ([9], Proposition 3.4) each f_i is a contraction. Also, for each $n \in \mathbb{N}$,

$$(f_i)_n : M_n(V_i) \to M_n(W_i) : [v_{i,j}] \to [f_i(v_{i,j}]$$

is also a homomorphism, and thus is a contraction, so that f_i is a complete contraction. Since injective tensor product of operator spaces is injective ([6, Proposition 8.1.5]) therefore $f_1 \otimes f_2$ continuously extends by density to a completely bounded map $f_1 \otimes f_2 : V_1 \otimes^{\mathrm{tmin}} V_2 \to W_1 \otimes^{\mathrm{tmin}} W_2$. The extended map $f_1 \otimes f_2$ is also a homomorphism. Moreover, if each f_i is injective, then f_i is a complete isometry and therefore $f_1 \otimes f_2$ is also a complete isometry. □

Proposition 3.2. *Let I and J be ideals of TROs V and W, respectively. Then $I \otimes^{\mathrm{tmin}} J$ is an ideal of $V \otimes^{\mathrm{tmin}} W$.*

Proof. Let $i : I \to V$ and $j : J \to W$ be natural embeddings. By above proposition, we get an injective (complete isometric) homomorphism $i \otimes j : I \otimes^{\mathrm{tmin}} J \to V \otimes^{\mathrm{tmin}} W$ therefore, $I \otimes^{\mathrm{tmin}} J$ can be treated as a sub-TRO of $V \otimes^{\mathrm{tmin}} W$. Finally, using density result follows easily. □

Proposition 3.3. *Let V and W be strongly quasi-central TROs. Then $V \otimes^{\mathrm{tmin}} W$ is also a strongly quasi-central TRO.*

Proof. Using Proposition 2.12, let (a_λ, b_λ) and (c_λ, d_λ) be the central bounded approximate identities of V and W, respectively. Note that each element of the net $(a_\lambda \otimes c_\lambda, b_\lambda \otimes d_\lambda)$ belongs to $\mathrm{Z}_{\mathrm{C^*\text{-}tring}}(V \otimes^{\mathrm{tmin}} W)$ and $(a_\lambda \otimes c_\lambda, b_\lambda \otimes d_\lambda)$ is an approximate identity of $V \otimes^{\mathrm{tmin}} W$. □

Let X be a locally compact Hausdorff topological space and V be a TRO. Let $f : X \to V$ be a continuous function. Recall that f is said to vanish at infinity if for each $\epsilon > 0$, there exists a compact subset K of X such that $||f(x)|| < \epsilon$ whenever $x \notin K$. Denote,

$$C_0(X, V) := \{f : X \to V : f \text{ is continuous and vanishes at infinity}\}.$$

Let f_1, f_2 and $f_3 \in C_0(X, V)$. We define $f_1 f_2^* f_3(x) = f_1(x) f_2(x)^* f_3(x)$. It is easy to see that $f_1 f_2^* f_3$ vanishes at infinity and therefore we get a map

$$C_0(X, V) \times C_0(X, V) \times C_0(X, V) \to C_0(X, V)$$

defined by

$$(f_1, f_2, f_3) \to f_1 f_2^* f_3.$$

Let μ be a Borel measure with full support on X and $V \subset B(H,K)$. Let $H' = L^2(X,\mu,H)$ and $K' = L^2(X,\mu,K)$. For $f \in C_0(X,V)$, let $\pi(f)$ be the operator acting as $\pi(f)g(x) = f(x)g(x)$. Here $g(x) \in H$ and $f(x) \in V \subset B(H,K)$ and therefore $f(x)g(x) \in K$. As μ has full support therefore the map π is an injective *-morphism. Thus $C_0(X,V)$ is also a TRO. Also, note that the map $\phi : A(C_0(X,V)) \to C_0(X,A(V))$ defined by

$$\phi\left(\begin{bmatrix} f_1f_2^* & f \\ g^* & f_3^*f_4 \end{bmatrix}\right)(x) = \begin{bmatrix} f_1(x)f_2(x)^* & f(x) \\ g(x)^* & f_3^*(x)f_4(x) \end{bmatrix}$$

is a *-isomorphism of C^*-algebras and therefore the linking C^*-algebra of $C_0(X,V)$ is $C_0(X,A(V))$. Since $C_0(X,A(V))$ is exact for exact TRO V thus from ([14], Theorem 4.4) it follows that if V is exact, then $C_0(X,V)$ is also an exact TRO. For each $x \in X$, let I_x be an ideal of V. Then the set of $f \in C_0(X,V)$ satisfying $f(x) \in I_x$ is an ideal of $C_0(X,V)$. By Proposition 2.4, it follows that every ideal of $C_0(X,M)$ has this form.

Proposition 3.4. *Let X be a locally compact Hausdorff space and V a TRO. Then $C_0(X) \otimes^{\mathrm{tmin}} V$ is a TRO isomorphic to $C_0(X,V)$. In addition, if V is separable, then* $\mathrm{Prim}(C_0(X,V))$ *is homeomorphic to* $X \times \mathrm{Prim}(V)$.

Proof. By ([14], Proposition 2.2), it is enough to show that $A(C_0(X) \otimes^{\mathrm{tmin}} V) = C_0(X) \otimes^{\min} A(V)$ is C^*-isomorphic to $A(C_0(X,V)) = C_0(X,A(V))$ which follows from ([12], Proposition 1.5.6). If in addition V is separable, using Theorem 2.6 and Proposition 2.4, we observe that

$$\mathrm{Prim}(C_0(X) \otimes^{\mathrm{tmin}} V) \cong \mathrm{Prim}(C_0(X)) \times \mathrm{Prim}(A(V)) \cong X \times \mathrm{Prim}(V),$$

where the first homeomorphism is obtained using ([3], Propositions 2.16, 2.17). ☐

Proposition 3.5. *Let V be a TRO and B a C^*-algebra.*

(1) *If V and B are simple, then $V \otimes^{\mathrm{tmin}} B$ is a simple TRO.*
(2) *Every nonzero ideal of $V \otimes^{\mathrm{tmin}} B$ contains a nonzero elementary tensor.*

Proof. (1) In view of ([14], Proposition 3.1) and Theorem 2.8, it suffices to show that $A(V) \otimes^{\min} B$ is simple which follows from ([21], Corollary 4.21).

(2) Let I be an ideal of $V \otimes^{\mathrm{tmin}} B$ so by ([14], Proposition 3.1) $A(I)$ is an ideal of $A(V) \otimes^{\min} B$. By ([3], Lemma 2.12), $A(I)$ has an elementary tensor so assume $\left[\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right] \otimes b \in A(I)$ and therefore, we may assume $0 \neq q \otimes b \in I$. ☐

Let Id'(V) denote the set of proper closed ideals of V. For an ideal I of V, we denote by q_I the quotient TRO-morphism of V onto V/I. Let V be a TRO and B be a C^*-algebra. Consider the map $\Phi : \mathrm{Id}(V) \times \mathrm{Id}(B) \to \mathrm{Id}(V \otimes^{\mathrm{tmin}} B)$ defined as

$$\Phi(I,J) := \mathrm{Ker}(q_I \otimes q_2).$$

Proposition 3.6. *The map Φ is a homeomorphism of* Id'$(V) \times$ Id'(B) *onto its image which is dense in* Id'$(V \otimes^{\mathrm{tmin}} B)$.

Proof. By Proposition 2.4, $\theta \times id : \mathrm{Id'}(V) \times \mathrm{Id'}(B) \to \mathrm{Id'}(A(V)) \times \mathrm{Id'}(B)$ is a homeomorphism. By ([15], Theorem 6), Φ is a homeomorphism of $\mathrm{Id'}(A(V)) \times \mathrm{Id'}(B)$ onto its image which is dense in $\mathrm{Id'}(A(V) \otimes^{\min} B)$. Rest of the proof is clear by considering the following diagram:

$$\begin{array}{ccc}
\mathrm{Id}'(V) \times \mathrm{Id'}(B) & \longrightarrow & \mathrm{Id}'(V \otimes^{\mathrm{tmin}} B) \\
{\scriptstyle \theta \times id}\Big\downarrow & & {\scriptstyle \theta}\Big\downarrow \\
\mathrm{Id}'(A(V)) \times \mathrm{Id'}(B) & \xrightarrow{\Phi} & \mathrm{Id}'(A(V) \otimes^{\min} B)
\end{array}$$

□

Theorem 3.7. *Let V be a topologically simple TRO and let B be a C^*-algebra. If either V is exact or B is nuclear, then every closed TRO ideal of the TRO $V \otimes^{\mathrm{tmin}} B$ is a product ideal of the form $V \otimes^{\mathrm{tmin}} J$ for some closed ideal J in B.*

Proof. Let I be a closed TRO ideal of $V \otimes^{\min} B$ so $A(I)$ is a closed ideal of $A(V) \otimes^{\min} B$. If V is exact, then $A(V)$ must be exact by [14] and therefore $A(I)$ must be of the form $A(V) \otimes^{\min} J$ for some closed ideal J in B. But by ([14], Proposition 3.1), $A(V \otimes^{\mathrm{tmin}} J) = A(V) \otimes^{\min} J$ so by Proposition 2.4, $I = V \otimes^{\mathrm{tmin}} J$ that is I is a product ideal. In case B in nuclear, the result follows on the similar lines. □

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References

[1] F. Abadie and D. Ferraro, *Applications of ternary rings to C^*-algebras*, Adv. Oper. Theory **2** (2017), no. 3, 293–317. https://doi.org/10.22034/aot.1612-1085
[2] L. Arambašić, *Irreducible representations of Hilbert C^*-modules*, Math. Proc. R. Ir. Acad. **105A** (2005), no. 2, 11–24. https://doi.org/10.3318/PRIA.2005.105.2.11
[3] E. Blanchard and E. Kirchberg, *Non-simple purely infinite C^*-algebras: the Hausdorff case*, J. Funct. Anal. **207** (2004), no. 2, 461–513. https://doi.org/10.1016/j.jfa.2003.06.008
[4] K. R. Davidson, *C^*-algebras by example*, Fields Institute Monographs, 6, American Mathematical Society, Providence, RI, 1996. https://doi.org/10.1090/fim/006
[5] E. G. Effros, N. Ozawa, and Z.-J. Ruan, *On injectivity and nuclearity for operator spaces*, Duke Math. J. **110** (2001), no. 3, 489–521. https://doi.org/10.1215/S0012-7094-01-11032-6
[6] E. G. Effros and Z.-J. Ruan, *Operator spaces*, London Mathematical Society Monographs. New Series, 23, The Clarendon Press, Oxford University Press, New York, 2000.
[7] M. Hamana, *Injective envelopes of dynamical systems*, in Operator algebras and operator theory (Craiova, 1989), 69–77, Pitman Res. Notes Math. Ser., 271, Longman Sci. Tech., Harlow, 1992.
[8] M. Hamana, *Triple envelopes and Šilov boundaries of operator spaces*, Math. J. Toyama Univ. **22** (1999), 77–93.
[9] L. A. Harris, *A generalization of C^*-algebras*, Proc. London Math. Soc. (3) **42** (1981), no. 2, 331–361. https://doi.org/10.1112/plms/s3-42.2.331

[10] M. R. Hestenes, *A ternary algebra with applications to matrices and linear transformations*, Arch. Rational Mech. Anal. **11** (1962), 138–194. `https://doi.org/10.1007/BF00253936`

[11] Y. F. Jin, C. Park, and M. Th. Rassias, *Hom-derivations in C^*-ternary algebras*, Acta Math. Sin. (Engl. Ser.) **36** (2020), no. 9, 1025–1038. `https://doi.org/10.1007/s10114-020-9323-3`

[12] E. Kaniuth, *A course in commutative Banach algebras*, Graduate Texts in Mathematics, 246, Springer, New York, 2009. `https://doi.org/10.1007/978-0-387-72476-8`

[13] A. Kansal, A. Kumar, and V. Rajpal, *Inductive limit in the category of TRO*, Ann. Funct. Anal. **11** (2020), no. 3, 748–760. `https://doi.org/10.1007/s43034-020-00052-2`

[14] M. Kaur and Z.-J. Ruan, *Local properties of ternary rings of operators and their linking C^*-algebras*, J. Funct. Anal. **195** (2002), no. 2, 262–305. `https://doi.org/10.1006/jfan.2002.3951`

[15] A. J. Lazar, *The space of ideals in the minimal tensor product of C^*-algebras*, Math. Proc. Cambridge Philos. Soc. **148** (2010), no. 2, 243–252. `https://doi.org/10.1017/S0305004109990351`

[16] M. S. Moslehian, *Almost derivations on C^*-ternary rings*, Bull. Belg. Math. Soc. Simon Stevin **14** (2007), no. 1, 135–142. `http://projecteuclid.org/euclid.bbms/1172852249`

[17] G. J. Murphy, *C^*-Algebras and Operator Theory*, Academic Press, Inc., Boston, MA, 1990.

[18] C. Park and M. Th. Rassias, *Additive functional equations and partial multipliers in C^*-algebras*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **113** (2019), no. 3, 2261–2275. `https://doi.org/10.1007/s13398-018-0612-y`

[19] R. Pluta and B. Russo, *Ternary operator categories*, J. Math. Anal. Appl. **505** (2021), no. 2. `http://doi.org/10.1016/j.jmaa.2021.125590`

[20] M. Skeide, *Ideal submodules versus ternary ideals versus linking ideals*, Algebr. Represent. Theory **25** (2022), no. 2, 359–386. `https://doi.org/10.1007/s10468-020-10025-7`

[21] M. Takesaki, *Theory of Operator Algebras. I*, Springer-Verlag, New York, 1979.

[22] D. P. Williams, *Crossed products of C^*-algebras*, Mathematical Surveys and Monographs, 134, American Mathematical Society, Providence, RI, 2007. `https://doi.org/10.1090/surv/134`

[23] H. Zettl, *A characterization of ternary rings of operators*, Adv. in Math. **48** (1983), no. 2, 117–143. `https://doi.org/10.1016/0001-8708(83)90083-X`

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