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SOME RESULTS INVOLVING THE REPRESENTATION OF THE CENTRALIZER OF A MATRIX

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Abstract. In this paper, we investigate the dimension and the structure of the centralizer of a square matrix with entries from an arbitrary field.

1. Introduction

Throughout this paper, K denotes an arbitrary field, $\mathcal{M}_n(K)$ all n by n matrices with entries from K. For a given n by n matrix A over K, its centralizer $\mathcal{C}(A)$ is the subalgebra of $\mathcal{M}_n(K)$ consisting of all matrices B that commute with A. For a given n by n matrix A over K and m by m matrix over K, the Sylvester space $\mathcal{C}(A, B)$ of A and B is the set of all n by m T-matrices over K such that $AT = TB$ (see [3]). There are several reasons to devote a study to the centralizer $\mathcal{C}(A)$. Indeed the centralizer $\mathcal{C}(A)$ is the set of solutions of the homogenous equation of

$$AX - XA = C, \tag{1}$$

where $X \in \mathcal{M}_n(K)$. On another side, the representation of the centralizer of a given square matrix, $\mathcal{C}(A)$ of A is related to the problem of representation of the Sylvester space $\mathcal{C}(A, B)$ which is neither else the set of solutions of the homogenous matrix equation

$$AX - XB = C, \tag{2}$$

where $X \in \mathcal{M}_{n\times m}(K)$, that plays a central role in many areas of applied mathematics and, in particular in systems and control theory. The structure of $\mathcal{C}(A)$ is only known for non-derogatory matrices (resp. cyclic endomorphism). In [8] author treats the problem of determining $\mathcal{C}(A)$ over the real and the complex field. We aim to give some results in the characterization of $\mathcal{C}(A)$ over an arbitrary field by using results from module theory.

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2. Preliminaries

In this section, we introduce notations and mathematical objects used in this paper.

2.1. Notations

Throughout this paper, the following notations are used.

- K, an arbitrary field.
- $\mathcal{M}_n(K)$, the set of n by n matrices with entries from K.
- $\mathcal{C}(A)$, the centralize of A.
- $\mathcal{C}(A, B)$ of A and B is the set of all n by m T-matrices over K such that $AT = TB$.
- C_A or $C_A(X)$, the characteristic polynomial of A.
- m_A or $m_A(X)$, the minimal polynomial of A.
- $diag(A_1, A_2, \dots, A_m)$, a block diagonal matrix with matrices A_1, A_2, $\dots, A_m$ on its main diagonal blocks.
- $K[B]$ denotes the subspace of $\mathcal{M}_n(K)$ spanned by all powers of B.
- $\mathcal{M}_r(K[B])$ denote the set of all $r \times r$ matrices $A = (f_{ij})$ with entries $f_{ij} \in K[B]$.
- $Comp(m_A)$, the companion matrix of m_A.
- $\oplus_{i=1}^{r} \oplus_{j=1}^{r} \mathcal{C}(A_i,\ A_j)$, the direct sum of the $\mathcal{C}(A_i,\ A_j)$.
- $\prod_{i=1}^{s} \mathcal{C}(A_i)$, the product of the $\mathcal{C}(A_i)$.
- $\deg f$, the degree of f.
- $\dim_K \mathcal{C}(A)$, dimension of $\mathcal{C}(A)$.

2.2. Notions from module theory

We include some well-known results which are used for developing the proof of our main result. Let $A \in \mathcal{M}_n(K)$ be a non zero matrix, and M_A be the $K[X]$-module induced by A. From theory of finitely generated torsion module over P.I.D (see [5, p. 215], [6, §2, p. 556], and [1, p. 235]), we have the following theorems.

Theorem 2.1. *Let $A \in \mathcal{M}_n(K)$ be a non zero matrix, and M_A be the $K[X]$-module induced by A. Then there exists a unique sequence of polynomials $q_1, \dots, q_r$ such that:*

$$M_A \simeq \frac{K[X]}{(q_1)} \oplus \frac{K[X]}{(q_2)} \oplus \cdots \oplus \frac{K[X]}{(q_r)}$$

and

- *q_i divides q_{i+1} for all $1 \leq i \leq r-1$,*
- *$q_r = m_A$ the minimal polynomial of A and $\prod_{i=1}^{r} q_i = C_A$ the characteristic polynomial of A.*

The ascending sequence of polynomials $q_1, \dots, q_r$ are unique up similarity and called the invariant factors of A.

A matrix A with entries from a field K is said to be in rational canonical form if there exists an ordered set $\{f_1, f_2, \dots, f_r\}$ of polynomials in $K[X]$ such that f_i divides f_{i+1} for all $i \in \{1, \dots, r-1\}$ and

$$A = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \cdots & \cdots & \vdots \\ 0 & 0 & \cdots & A_r \end{pmatrix},$$

where A_i is the companion matrix of f_i, and each 0 is zero matrix of appropriate order.

Theorem 2.2 (Rational canonical form)**.** *Every square matrix with entries in a field K is similar to a unique matrix in rational canonical form.*

A partition $\lambda = (\lambda_1, \dots, \lambda_r)$ of a positive integer n is a finite nonincreasing sequence of positive integers $\lambda_1, \lambda_2, \dots, \lambda_r$ such that $\sum_{i=1}^{r} \lambda_i = n$. The λ_i are called the parts of the partition λ.

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ be a partition of the number n and let λ'_i be the number of parts of the partition λ that are $\geq i$, or equivalently the largest j such that $\lambda_j \geq i$. Then $\lambda^* = (\lambda'_1, \lambda'_2, \dots, \lambda'_s)$ is a partition of the number n called the conjugate partition of the partition λ.

The partition $\lambda = (\lambda_1, \dots, \lambda_r)$ of n can be graphically visualized with Young's diagram: the i^{th} row of the graphical representation of $(\lambda_1, \dots, \lambda_r)$ contains λ_i unit square (see [2, pp. 1–7]).

3. Main results

In this part, we aim to give some characterizations of $\mathcal{C}(A)$.

Theorem 3.1. *Let $A \in \mathcal{M}_n(K)$ be a non zero matrix and r the number of its invariant factors. If $B = Comp(m_A)$, then*

$\mathcal{C}(A)$ is isomorph to $\mathcal{M}_r(K[B])$ as K algebras if and only if $C_A = (m_A)^r$.

The proof of Theorem 3.1 involves the two following lemmas.

Lemma 3.2 ([1, Theorem 5.15, p. 336])**.** *Let $A \in \mathcal{M}_n(K)$ be a non zero matrix. Then*

$$\dim_K \mathcal{C}(A) = \sum_{i=1}^{r} (2r - 2i + 1) d_i \qquad (\textit{Frobenius formula}),$$

where q_i are the invariant factors of A and $d_i = \deg q_i$.

Lemma 3.3. *Let $A \in \mathcal{M}_n(K)$. If A is a block diagonal matrix, i.e., $A = diag(A_1, A_2, \dots, A_m)$, where A_i are $n_i \times n_i$ square matrices such that $n_1 + \cdots + n_r = n$. Then the centralizer of the matrix A is*

$$\mathcal{C}(A) \simeq \oplus_{i=1}^{r} \oplus_{j=1}^{r} \mathcal{C}(A_i,\ A_j).$$

Proof. Let $T \in \mathcal{M}_n(K)$. Write $T = (T_{ij})$ as an $r \times r$ block matrix with the same block structure as A. If $T \in \mathcal{C}(A)$, then $A_i T_{ij} = T_{ij} A_j$ hence $T_{ij} \in \mathcal{C}(A_i, A_j)$. Conversely if $T = (T_{ij})$ with $T_{ij} \in \mathcal{C}(A_i, A_j)$, then $AT = (A_i T_{ij})$ and $TA = (A_j T_{ij})$, since $T_{ij} \in \mathcal{C}(A_i, A_j)$. Then $T \in \mathcal{C}(A)$. $\square$

Proof of Theorem 3.1. Let $A \in \mathcal{M}_n(K)$ and $q_1, \ldots, q_r$ be the invariant factors of A. Assume that $\mathcal{C}(A)$ is isomorphic to $\mathcal{M}_r(K[B])$. Then

$$\dim_K \mathcal{C}(A) = r^2 \deg(m_A). \tag{3}$$

By virtue of Lemma 3.2 we have

$$\dim_K \mathcal{C}(A) = \sum_{i=1}^{r} (2r - 2i + 1) d_i, \tag{4}$$

where $d_i = \deg q_i$ for all $1 \leq i \leq r$, it follows from (3) that

$$\sum_{i=1}^{r} (2r - 2i + 1) d_i = r^2 d_r. \tag{5}$$

According to (5) and by remarking that $r^2 = \sum_{i=1}^{r} (2r - 2i + 1)$, we have

$$\sum_{i=1}^{r} (2r - 2i + 1)(d_r - d_i) = 0. \tag{6}$$

The equality (6) gives $d_r = d_i$ for all $1 \leq i \leq r$ since $[2(r - i) + 1] > 0$ and $d_r \geq d_i$ for each $1 \leq i \leq r$. Consequently $q_i = q_r$ for each $1 \leq i \leq r$. Therefore $C_A = (m_A)^r$. Conversely, suppose that $C_A = (m_A)^r$. According to the following decomposition $C_A = \prod_{i=1}^{r} q_i$, where $q_1, \ldots, q_r$ are the invariant factors of A such that $q_i \mid q_{i+1}$ for all $1 \leq i \leq r - 1$, so we have $q_1 = \cdots = q_{r-1} = q_r$. By applying Theorem 2.1 we get, the following rational canonical form of A,

$$\begin{aligned} A &= diag(B, B, \ldots, B) \\ &= \begin{pmatrix} B & 0 & \cdots & 0 \\ 0 & B & \cdots & 0 \\ \vdots & \cdots & \cdots & \vdots \\ 0 & 0 & \cdots & B \end{pmatrix}, \end{aligned}$$

where B is the companion matrix of $m_A = q_1 = \cdots = q_r$. Then we can invoke Lemma 3.3 to conclude that the centralizer $\mathcal{C}(A)$ of the matrix A is isomorphic to $\mathcal{M}_r(K[B])$. $\square$

3.1. Case of a primary matrix

Let $A \in \mathcal{M}_n(K)$ and $B \in \mathcal{M}_m(K)$. Recall that the matrices A and B are said to be coprime if there exist two coprime polynomials P and Q in $K[X]$ such that $P(A) = 0$ and $Q(B) = 0$. If the matrices A and B are coprime, then $\mathcal{C}(A, B) = 0$ (see [3]).

Let $A \in \mathcal{M}_n(K)$ and P be an irreducible monic polynomial of $K[X]$. We will say that A is a P-primary matrix if the characteristic polynomial of A is a power of P.

Lemma 3.4. *Let $A \in \mathcal{M}_n(K)$ be a non zero matrix. Then A is similar to a block diagonal P_i-primary matrices $diag(A_1, A_2, \dots, A_s)$.*

Proof. Let $m_A = \prod_{i=1}^{s} P_i^{\alpha_i}$ be the prime decomposition of m_A. By the primary decomposition theorem (see Theorem 1.5.1, p. 29 in [7]), A is similar to a block diagonal P_i-primary matrices $diag(A_1, A_2, \dots, A_s)$. □

Lemma 3.5. *Let $A \in \mathcal{M}_n(K)$ be a non zero matrix. Then the centralizer $\mathcal{C}(A)$ of the matrix A is isomorph to the ring $\prod_{i=1}^{s} \mathcal{C}(A_i)$, where $A_1, A_2, \dots, A_s$ are pairwise coprime matrices.*

Proof. By Lemma 3.4, A is similar to a block diagonal P_i-primary matrices $diag(A_1, A_2, \dots, A_s)$ furthermore the A_i's are pairwise coprime matrices. Hence, by Lemma 3.3, the centralizer of the matrix A is isomorph to $\prod_{i=1}^{s} \mathcal{C}(A_i)$. □

Remark that by Lemma 3.5 the study of the centralizer $\mathcal{C}(A)$ reduces to the study of the centralizer of a P-primary matrix.

Lemma 3.6. *If $A \in \mathcal{M}_n(K)$ is a P-primary matrix such that $m_A = P^\alpha$ and $C_A = P^\beta$ with $\alpha \le \beta$, then there exists a partition $\lambda = (\lambda_1, \dots, \lambda_r)$ of β such that $\lambda_1 = \alpha$ and $q_i = P^{\lambda_{r-i+1}}$ for any $1 \le i \le r$ are the invariant factors of A.*

Proof. Since $m_A = P^\alpha$ and $C_A = P^\beta$, the invariant factors of A are of the form P^{α_i}, $1 \le i \le r$ with $\alpha_1 \le \alpha_2 \le \cdots \le \alpha_r = \alpha$ and $\Sigma_{i=1}^{r} \alpha_i = \beta$. Set $\lambda_i = \alpha_{r-i+1}$ and $q_i = P^{\lambda_{r-i+1}}$ for each $1 \le i \le r$, we can verify easily that

- $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_r = \alpha$ and $\Sigma_{i=1}^{r} \lambda_i = \beta$.
- $q_i \mid q_{i+1}$ for $1 \le i \le r-1$, $q_r = m_A$ and $\prod_{i=1}^{r} q_i = C_A$. □

Theorem 3.7. *Let $A \in \mathcal{M}_n(K)$ be a P-primary matrix. If r is the number of invariant factors of A and B is the companion matrix of m_A, then the following assertions are equivalent*:

(1) $m_A = P$.
(2) $\dim_K \mathcal{C}(A) = r^2 \deg(P)$.
(3) $\mathcal{C}(A) \simeq \mathcal{M}_r(K[B])$.

Proof. Let $m_A = P^\alpha$ and $C_A = P^\beta$.

(1) $\Rightarrow$ (2) If $\alpha = 1$, then $q_1 = \cdots = q_r = P$ are the invariant factors of A and $r = \beta$. Then by Lemma 3.2

$$\begin{aligned}\dim_K \mathcal{C}(A) &= \sum_{i=1}^{r} (2r - 2i + 1) \deg(q_i) \\ &= \left[(2r+1)r - 2\sum_{i=1}^{r} i \right] \deg(P)\end{aligned}$$

$$= \left[(2r+1)r - \frac{2r(r+1)}{2}\right] \deg P$$
$$= r^2 \deg P.$$

(2) $\Rightarrow$ (1) If $\dim_K \mathcal{C}(A) = r^2 \deg(P)$, let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be the partition of β such that $\lambda_1 = \alpha$ and $q_i = P^{\lambda_{r-i+1}}$, $1 \leq i \leq r$ are the invariant factors of A, it's enough to prove that $\lambda_1 = \cdots = \lambda_r = 1$. Indeed by hypothesis and by virtue of Lemma 3.2 we have $\dim_K \mathcal{C}(A) = \sum_{i=1}^{r}(2r-2i+1)\lambda_{r-i+1} \deg P = r^2 \deg P$. Then by remarking that $\sum_{i=1}^{r}(2r-2i+1) = r^2$ we get $\sum_{i=1}^{r}[(2r-2i+1)(\lambda_{r-i+1}-1)] = 0$. Therefore $(\lambda_{r-i+1}-1) = 0$ for all $1 \leq i \leq r$ because $[2(r-i)+1] > 0$ for each $1 \leq i \leq r$. Thus $\lambda_1 = \cdots = \lambda_r = 1$.

For the equilibrium (1) $\Leftrightarrow$ (3), it suffice to apply Theorem 3.1. $\square$

Theorem 3.8. *Let $A \in \mathcal{M}_n(K)$ be a P-primary matrix such that $C_A = P^{\beta}$. Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be the partition of β such that $q_1, \ldots, q_r$ are the invariant factors of A where, $q_i = P^{\lambda_{r-i+1}}$ for any $1 \leq i \leq r$. Then*

$$\dim_K \mathcal{C}(A) = \left[\sum_{k=1}^{s} {\lambda'}_k^2\right] \deg(P),$$

where $\lambda^* = (\lambda'_1, \ldots, \lambda'_s)$ is the conjugate partition of the partition $\lambda = (\lambda_1, \ldots, \lambda_r)$.

To prove the theorem we need the following lemma:

Lemma 3.9 ([7, Proposition 3.3.1, p. 106])**.** *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partition of the number n and $\lambda^* = (\lambda'_1, \lambda'_2, \ldots, \lambda'_s)$ be its conjugate partition. Then*

$$\sum_{i=1}^{k}(2i-1)\lambda_i = \sum_{i=1}^{s} {\lambda'}_i^2.$$

Proof. By Lemma 3.2 and Lemma 3.9, we have

$$\begin{aligned}
\dim_K \mathcal{C}(A) &= \sum_{i=1}^{r}(2r-2i+1)\deg(q_i) \\
&= \sum_{i=1}^{r}(2(r-i+1)-1)\deg(q_i) \\
&= \sum_{j=1}^{r}(2j-1)\deg(q_{r-j+1}) \\
&= \sum_{j=1}^{r}(2j-1)\deg(P^{\lambda_j}) \\
&= \left[\sum_{j=1}^{r}(2j-1)\lambda_j\right]\deg(P)
\end{aligned}$$

$$= \left[\sum_{i=1}^{s} {\lambda'}_i^2\right] \deg(P).$$

Thus the desired result. $\square$

Corollary 3.10. *Let $A \in \mathcal{M}_n(K)$ be a non zero matrix. If $m_A = \prod_{i=1}^r P_i$ and $C_A = \prod_{i=1}^r P_i^{\beta_i}$, where P_i is an irreducible monic polynomial of $K[X]$ for all $1 \le i \le r$. Then*

$$\dim_K \mathcal{C}(A) = \sum_{i=1}^{r} \beta_i^2 \deg(P_i).$$

Proof. By Theorem 3.7 we have $\dim_K \mathcal{C}(A_i) = \beta_i^2 \deg(P)$ for all $1 \le i \le r$. Then we invoke Lemma 3.5, to obtain

$$\begin{aligned}\dim_K \mathcal{C}(A) &= \dim_K \left(\prod_{i=1}^{r} \mathcal{C}(A_i)\right)\\ &= \sum_{i=1}^{r} \dim_K \mathcal{C}(A_i)\\ &= \sum_{i=1}^{r} \beta_i^2 \deg(P_i).\end{aligned}$$

$\square$

We deduce the well known result (see [4], or [7]).

Corollary 3.11. *Let $A \in \mathcal{M}_n(K)$. If $m_A = \prod_{i=1}^r (X - \lambda_i)$ and $C_A(X) = \prod_{i=1}^r (X - \lambda_i)^{\beta_i}$, then $\dim_K \mathcal{C}(A) = \sum_{i=1}^r \beta_i^2$.*

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