Commun. Korean Math. Soc. **38** (2023), No. 1, pp. 89–96
https://doi.org/10.4134/CKMS.c220015
pISSN: 1225-1763 / eISSN: 2234-3024

GENERALIZING SOME FIBONACCI–LUCAS RELATIONS

Junghyun Hong, Jongmin Lee, and Ho Park

Abstract. Edgar obtained an identity between Fibonacci and Lucas numbers which generalizes previous identities of Benjamin–Quinn and Marques. Recently, Dafnis provided an identity similar to Edgar's. In the present article we give some generalizations of Edgar's and Dafnis's identities.

1. Introduction and statement of main results

Let F_n denote the nth Fibonacci number defined by

$$F_0 = 0,\ F_1 = 1,\ F_n \ =\ F_{n-1} + F_{n-2} \quad (n \geq 2).$$

Lucas numbers L_n are defined as $L_0 = 2$, $L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$. In 1999, Benjamin and Quinn [1] obtained an identity

$$\sum_{i=0}^{n} 2^i L_i = 2^{n+1} F_{n+1}$$

between Fibonacci and Lucas numbers. An analogous identity was proven by Marques [5] as

$$\sum_{i=0}^{n} 3^i (L_i + F_{i+1}) = 3^{n+1} F_{n+1}$$

and these two identities were generalized by Edgar [3] as

$$\sum_{i=0}^{n} m^i (L_i + (m-2) F_{i+1}) = m^{n+1} F_{n+1}.$$

Received January 24, 2022; Accepted April 1, 2022.
2020 *Mathematics Subject Classification.* 11B39.
Key words and phrases. Fibonacci numbers, Lucas numbers.
H. Park was supported by the National Research Foundation of Korea (NRF-2019R1C1C1010211). This research was partially supported by KAIST (Global Institute For Talented Education) and Dongguk University Science Education Institute for the Gifted, Korea Foundation for the Advancement of Science and Creativity financed by the Ministry of Science and ICT and the Ministry of Economy and Finance.

Recently, Dafnis [2] showed another identity

$$\sum_{i=0}^{n}(-1)^i m^{n-i}(L_{i+1}+(m-2)F_i)=(-1)^n F_{n+1},$$

which is similar to Edgar's identity. In the same article, he further obtained an identity

$$\sum_{i=0}^{n}(-1)^i m^{n-i}(V_{i+1}+(m-2)T_i-3T_{i-1})=(-1)^n T_{n+1}$$

between Tetranacci numbers T_i and Lucas-Tetranacci numbers V_i.

The main purpose of the present article is to generalize Edgar's and Dafnis's identities.

Theorem 1.1. *Let k and n be nonnegative integers and let m be a real number. Then we have*

(1) $\sum_{i=0}^{n} m^i(L_{ki}F_k+(mL_k-2)F_{ki+k})=m^{n+1}L_kF_{kn+k}$,
(2) $\sum_{i=0}^{n} m^i(5F_{ki}F_k+(mL_k-2)L_{ki+k})=m^{n+1}L_kL_{kn+k}-2L_k$,
(3) $\sum_{i=0}^{n}(-1)^i m^{n-i}(L_{ki+k}F_k+(mL_k+2(-1)^k)F_{ki})=(-1)^nL_kF_{kn+k}$,
(4) $\sum_{i=0}^{n}(-1)^i m^{n-i}(5F_{ki+k}F_k+(mL_k+2(-1)^k)L_{ki})=L_k((-1)^nL_{kn+k}+2m^{n+1})$.

Note that we can derive Edgar's and Dafnis's identities from Theorem 1.1(1) and (3) with $k=1$. In Section 2, we prove Theorem 1.1. In Section 3, we also provide several additional identities similar to Edgar's and Dafnis's identities.

2. Proof of Theorem 1.1

The following is well-known identities of Fibonacci and Lucas numbers which will often be used in the proofs of our theorems.

Lemma 2.1 ([4, p. 91 and p. 97])**.** *For any integers m and n, we have*

(1) $F_{m+n}-(-1)^nF_{m-n}=L_mF_n$,
(2) $F_{m+n}+(-1)^nF_{m-n}=F_mL_n$,
(3) $L_{m+n}-(-1)^nL_{m-n}=5F_mF_n$,
(4) $L_{m+n}+(-1)^nL_{m-n}=L_mL_n$.

Proof of Theorem 1.1. (1) Applying the identities in Lemma 2.1(1) and (2), we obtain

$$\begin{aligned}&\sum_{i=0}^{n} m^i\Big(L_{ki}F_k+(mL_k-2)F_{ki+k}\Big)\\ =&\sum_{i=0}^{n} m^i\Big((F_{ki+k}-(-1)^kF_{ki-k})+(mL_k-2)F_{ki+k}\Big)\\ =&\sum_{i=0}^{n} m^i\Big(-(F_{ki+k}+(-1)^kF_{ki-k})+mL_kF_{ki+k}\Big)\end{aligned}$$

$$\begin{aligned}
&= \sum_{i=0}^{n} m^i(-F_{ki}L_k + mL_kF_{ki+k}) \\
&= L_k\Big(-\sum_{i=0}^{n} m^iF_{ki} + \sum_{i=1}^{n+1} m^iF_{ki}\Big) \\
&= m^{n+1}L_kF_{kn+k}.
\end{aligned}$$

(2) Using Lemma 2.1(3) and (4), we compute

$$\begin{aligned}
&\sum_{i=0}^{n} m^i\Big(5F_{ki}F_k + (mL_k-2)L_{ki+k}\Big) \\
&= \sum_{i=0}^{n} m^i\Big((L_{ki+k} - (-1)^nL_{ki-k}) + (mL_k-2)L_{ki+k}\Big) \\
&= \sum_{i=0}^{n} m^i\Big(-(L_{ki+k} + (-1)^nL_{ki-k}) + mL_kL_{ki+k}\Big) \\
&= \sum_{i=0}^{n} m^i(-L_{ki}L_k + mL_kL_{ki+k}) \\
&= L_k\Big(-\sum_{i=0}^{n} m^iL_{ki} + \sum_{i=1}^{n} m^iL_{ki}\Big) \\
&= m^{n+1}L_kL_{kn+k} - 2L_k.
\end{aligned}$$

(3) By Lemma 2.1(1) and (2), we have

$$\begin{aligned}
&\sum_{i=0}^{n}(-1)^im^{n-i}\Big(L_{ki+k}F_k + (mL_k+2(-1)^k)F_{ki}\Big) \\
&= \sum_{i=0}^{n}(-1)^im^{n-i}\Big(F_{ki+2k} - (-1)^kF_{ki} + (mL_k+2(-1)^k)F_{ki}\Big) \\
&= \sum_{i=0}^{n}(-1)^im^{n-i}\Big(F_{ki+2k} + (-1)^kF_{ki} + mL_kF_{ki}\Big) \\
&= \sum_{i=0}^{n}(-1)^im^{n-i}(F_{ki+k}L_k + mL_kF_{ki}) \\
&= L_k\Big(\sum_{i=1}^{n+1}(-1)^{i-1}m^{n-i+1}F_{ki} + \sum_{i=0}^{n}(-1)^im^{n-i+1}F_{ki}\Big) \\
&= L_k\Big((-1)^nF_{k(n+1)} + m^{n+1}F_0\Big) \\
&= (-1)^nL_kF_{kn+k}.
\end{aligned}$$

(4) From Lemma 2.1(1) and (2), we derive

$$\begin{aligned}
&\sum_{i=0}^{n}(-1)^i m^{n-i}\Big(5F_{ki+k}F_k+(mL_k+2(-1)^k)L_{ki}\Big)\\
&=\sum_{i=0}^{n}(-1)^i m^{n-i}\Big(L_{ki+2k}-(-1)^kL_{ki}+(mL_k+2(-1)^k)L_{ki}\Big)\\
&=\sum_{i=0}^{n}(-1)^i m^{n-i}\Big(L_{ki+2k}+(-1)^kL_{ki}+mL_kL_{ki}\Big)\\
&=\sum_{i=0}^{n}(-1)^i m^{n-i}(L_{ki+k}L_k+mL_kL_{ki})\\
&=L_k\Big(\sum_{i=1}^{n+1}(-1)^{i-1}m^{n-i+1}L_{ki}+\sum_{i=0}^{n}(-1)^i m^{n-i+1}L_{ki}\Big)\\
&=L_k\Big((-1)^nL_{k(n+1)}+m^{n+1}L_0\Big)\\
&=L_k\Big((-1)^nL_{kn+k}+2m^{n+1}\Big).
\end{aligned}$$

□

3. Some additional identities

Theorem 3.1. *Let k and n be nonnegative integers and let m be a real number. Then we obtain*

(1) *If $m \neq 1$, then*

$$\begin{aligned}
&\sum_{i=0}^{n}(-1)^{ki}m^i(L_{k(i-t)}-(-1)^kmL_{k(i-t+2)t})F_{ki+k}\\
&=(-1)^{kt}\left(m^{n+1}L_{k(t-n-2)}F_{k(n+1)}-(-1)^kL_{ki}F_k\frac{m^{n+2}-1}{m-1}\right),
\end{aligned}$$

(2)
$$\begin{aligned}
&\sum_{i=0}^{n}(-1)^{ki}F_{k(i-t+1)}F_{k(i+1)}\\
&=\frac{(-1)^{kt}}{5F_k}\Big(L_{k(t-n-2)}F_{k(n+1)}-(-1)^k(n+1)L_{ki}F_k\Big).
\end{aligned}$$

Proof. Using Lemma 2.1(1) and (2), we obtain

$$\begin{aligned}
&L_{kt}F_k-L_{k(t-i)}F_{ki+k}\\
&=(F_{kt+k}-(-1)^kF_{kt-k})-(F_{kt+k}-(-1)^{ki+k}F_{kt-2ki-k})\\
&=(-1)^{k+1}(F_{kt-k}-(-1)^{ki}F_{kt-2ki-k})\\
&=(-1)^{k+1}L_{kt-ki-k}F_{ki}.
\end{aligned}$$

Now we compute

$$\begin{aligned}
&\sum_{i=0}^{n} m^i\Big(L_{kt}F_k + \big((-1)^k mL_{k(t-i-2)} - L_{k(t-i)}\big)F_{ki+k}\Big)\\
&= \sum_{i=0}^{n} m^i(-1)^{k+1}(L_{kt-ki-k}F_{ki} - mL_{k(t-i-2)}F_{ki+k})\\
&= (-1)^{k+1}\left(\sum_{i=0}^{n} m^i L_{kt-ki-k}F_{ki} - \sum_{i=0}^{n} m^{i+1}L_{kt-ki-2k}F_{ki+k}\right)\\
&= (-1)^{k+1}\left(\sum_{i=0}^{n} m^i L_{kt-ki-k}F_{ki} - \sum_{i=1}^{n+1} m^{i}L_{kt-ki-k}F_{ki}\right)\\
&= (-1)^k m^{n+1}L_{k(t-n-2)}F_{k(n+1)}.
\end{aligned}$$

Since $L_{-k} = (-1)^k L_k$ for any integer k, it follows that

$$\begin{aligned}
&\sum_{i=0}^{n} m^i\Big((-1)^{kt-ki-k}mL_{k(i-t+2)} - (-1)^{k(t-i)}L_{k(i-t)}\Big)F_{ki+k}\\
&= (-1)^k m^{n+1}L_{k(t-n-2)}F_{k(n+1)} - \sum_{i=0}^{n} m^i L_{kt}F_k
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{i=0}^{n}(-1)^{ki}m^i(mL_{k(i-t+2)} - (-1)^k L_{k(i-t)})F_{ki+k}\\
&= (-1)^{kt}m^{n+1}L_{k(t-n-2)}F_{k(n+1)} - (-1)^{kt+k}\sum_{i=0}^{n} m^i L_{kt}F_k.
\end{aligned}$$

If $m = 1$, then the left-hand side becomes

$$\begin{aligned}
&\sum_{i=0}^{n}(-1)^{ki}(L_{k(i-t)} - (-1)^k L_{k(i-t+2)})F_{ki+k}\\
&= \sum_{i=0}^{n}(-1)^{ki}5F_kF_{ki-kt+k}F_{ki+k}
\end{aligned}$$

and the right-hand side becomes

$$(-1)^{kt}L_{k(t-n-2)}F_{k(n+1)} - (-1)^{kt+k}(n+1)L_{ki}F_k.$$

If $m \neq 1$, then we have

$$\begin{aligned}
&\sum_{i=0}^{n}(-1)^{ki}m^i(L_{k(i-t)} - (-1)^k mL_{k(i-t+2)})F_{ki+k}\\
&= (-1)^{kt}\left(m^{n+1}L_{k(t-n-2)}F_{k(n+1)} - (-1)^k L_{ki}F_k\frac{m^{n+2}-1}{m-1}\right).
\end{aligned}$$

□

Theorem 3.2. *Let $i_1, i_2, \ldots, i_r, k, n$ be nonnegative integers and let m be a real number. Then*

$$\sum_{i_r=0}^{n}\Big(m^{i_r}(L_{ki_r}F_k - 2F_{ki_r+k}) + \sum_{i_{r-1}=0}^{i_r}\Big(m^{i_{r-1}}(L_{ki_{r-1}}F_k - 2F_{ki_{r-1}+k}) + \cdots + \sum_{i_2=0}^{i_3}\Big(m^{i_2}(L_{ki_2}F_k - 2F_{ki_2+k}) + \sum_{i_1=0}^{i_2} m^{i_1}(L_{ki_1}F_k + (mL_k-2)F_{ki_1+k})\Big)\cdots\Big)\Big)$$
$$= m^{n+1}L_kF_{kn+k}.$$

Proof. By Theorem 1.1(1), we obtain

$$\begin{aligned}
&\sum_{i_r=0}^{n}\Big(m^{i_r}(L_{ki_r}F_k - 2F_{ki_r+k}) + \sum_{i_{r-1}=0}^{i_r}\Big(m^{i_{r-1}}(L_{ki_{r-1}}F_k - 2F_{ki_{r-1}+k}) + \cdots \\
&\quad + \sum_{i_2=0}^{i_3}\Big(m^{i_2}(L_{ki_2}F_k - 2F_{ki_2+k}) + \sum_{i_1=0}^{i_2} m^{i_1}(L_{ki_1}F_k + (mL_k-2)F_{ki_1+k})\Big)\cdots\Big)\Big) \\
&= \sum_{i_r=0}^{n}\Big(m^{i_r}(L_{ki_r}F_k - 2F_{ki_r+k}) + \sum_{i_{r-1}=0}^{i_r}\Big(m^{i_{r-1}}(L_{ki_{r-1}}F_k - 2F_{ki_{r-1}+k}) + \cdots \\
&\quad + \sum_{i_2=0}^{i_3}\Big(m^{i_2}(L_{ki_2}F_k - 2F_{ki_2+k}) + m^{i_2+1}L_kF_{ki_1+k}\Big)\cdots\Big)\Big) \\
&= \sum_{i_r=0}^{n}\Big(m^{i_r}(L_{ki_r}F_k - 2F_{ki_r+k}) + \sum_{i_{r-1}=0}^{i_r}\Big(m^{i_{r-1}}(L_{ki_{r-1}}F_k - 2F_{ki_{r-1}+k}) + \cdots \\
&\quad + \sum_{i_2=0}^{i_3} m^{i_2}(L_{ki_2}F_k + (mL_k-2)F_{ki_2+k})\cdots\Big)\Big) \\
&\ \vdots \\
&= \sum_{i_r=0}^{n}\Big(m^{i_r}(L_{ki_r}F_k - 2F_{ki_r+k}) + m^{i_r+1}L_kF_{ki_r+k}\Big) \\
&= \sum_{i_r=0}^{n} m^{i_r}(L_{ki_r}F_k + (mL_k-2)F_{ki_r+k}) \\
&= m^{n+1}L_kF_{kn+k}.
\end{aligned}$$
□

Taking $m = 2$ and $k = 1$ in Theorem 1.1(2), we easily obtain a result similar to the identity of Benjamin and Quinn.

Lemma 3.3.

$$\sum_{i=0}^{n} 2^iF_i = \frac{1}{5}(2^{n+1}L_{n+1} - 2).$$

Using the above lemma and Benjamin–Quinn's identity, we can prove the following.

Theorem 3.4. *Let n and k be nonnegative integers. Then*

(1) $\sum_{i=0}^{n} 2^i\Big(F_kL_i + (2^kL_k - 2)F_{i+k}\Big) = \frac{L_k}{5}(2^{n+k+1}L_{n+k+1} - 2^{n+1}L_{n+1} - 2^kL_k + 2)$,

(2) $\sum_{i=0}^{n} 2^i\Big(5F_kF_i + (2^kL_k - 2)L_{i+k}\Big) = L_k(2^{n+k+1}F_{n+k+1} - 2^{n+1}F_{n+1} - 2^kF_k)$.

Proof. (1) By Lemma 2.1(1) and (2), we have

$$\begin{aligned} F_kL_i - 2F_{i+k} &= F_{i+k} - (-1)^kF_{i-k} - 2F_{i+k} \\ &= -(F_{i+k} + (-1)^kF_{i-k}) \\ &= -L_kF_i. \end{aligned}$$

Using Lemma 3.3, we compute

$$\begin{aligned} &\sum_{i=0}^{n} 2^i\Big(F_kL_i + (2^kL_k - 2)F_{i+k}\Big) \\ =& \sum_{i=0}^{n} 2^i(-L_kF_i + 2^kL_kF_{i+k}) \\ =& L_k\Big(\sum_{i=0}^{n} 2^{i+k}F_{i+k} - \sum_{i=0}^{n} 2^iF_i\Big) \\ =& L_k\Big(\sum_{i=0}^{n+k} 2^iF_i - \sum_{i=0}^{k-1} 2^iF_i - \sum_{i=0}^{n} 2^iF_i\Big) \\ =& \frac{L_k}{5}(2^{n+k+1}L_{n+k+1} - 2^{n+1}L_{n+1} - 2^kL_k + 2). \end{aligned}$$

(2) Similarly, we obtain

$$\begin{aligned} 5F_kF_i - 2L_{k+i} &= L_{k+i} - (-1)^iL_{k-i} - 2L_{k+i} \\ &= -(L_{k+i} + (-1)^iL_{k-i}) \\ &= -L_kL_i \end{aligned}$$

and hence we get

$$\begin{aligned} &\sum_{i=0}^{n} 2^i\Big(5F_kF_i + (2^kL_k - 2)L_{i+k}\Big) \\ =& \sum_{i=0}^{n} 2^i(-L_kL_i + 2^kL_kL_{k+i}) \\ =& L_k\Big(\sum_{i=0}^{n} 2^{k+i}L_{k+i} - \sum_{i=0}^{n} 2^iL_i\Big) \end{aligned}$$

$$= L_k\Big(\sum_{i=0}^{n+k} 2^i L_i - \sum_{i=0}^{k-1} 2^i L_i - \sum_{i=0}^{n} 2^i L_i\Big)$$
$$= L_k(2^{n+k+1}F_{n+k+1} - 2^k F_k - 2^{n+1}F_{n+1}). \qquad \square$$

References

[1] A. T. Benjamin and J. J. Quinn, *Fibonacci and Lucas identities through colored tilings*, Util. Math. **56** (1999), 137–142.

[2] S. D. Dafnis, *On the relation between Fibonacci and Lucas numbers*, Fibonacci Quart. **58** (2020), no. 5, 111–114.

[3] T. Edgar, *Extending some Fibonacci-Lucas relations*, Fibonacci Quart. **54** (2016), no. 1, 79.

[4] T. Koshy, *Fibonacci and Lucas Numbers with Applications. Vol. 2*, Pure and Applied Mathematics (Hoboken), John Wiley & Sons, Inc., Hoboken, NJ, 2019.

[5] D. Marques, *A new Fibonacci-Lucas relation*, Amer. Math. Monthly **122** (2015), no. 7, 683. https://doi.org/10.4169/amer.math.monthly.122.7.683

Junghyun Hong
Gochon Middle School
Gimpo-si 10124, Korea

Jongmin Lee
Hansu Middle School
Goyang-si 10384, Korea

Ho Park
Department of Mathematics
Dongguk University
Seoul 04620, Korea
Email address: ph1240@dongguk.edu