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SOME FUNCTIONAL IDENTITIES ARISING FROM DERIVATIONS

Abdellah Mamouni, Lahcen Oukhtite, and Mohammed Zerra

Abstract. This paper considers some functional identities related to derivations of a ring R and their action on the centre of R/P where P is a prime ideal of R. It generalizes some previous results that are in the same spirit. Finally, examples proving that our restrictions cannot be relaxed are given.

1. Introduction

In all that follows, unless stated otherwise, R will be an associative ring with center $Z(R)$. Recall that a proper ideal P of R is said to be prime if whenever $xRy \subseteq P$ implies that $x \in P$ or $y \in P$. The ring R is a prime ring if and only if (0) is a prime ideal of R. A ring R is said to be n-torsion free, where $n \neq 0$ is a positive integer, if whenever $na = 0$, with $a \in R$, then $a = 0$. For any $x, y \in R$, the symbol $[x, y]$ and $x \circ y$ denote the Lie product $xy - yx$ and Jordan product $xy + yx$, respectively. An additive mapping $d : R \longrightarrow R$ is called a *derivation* if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. Let $a \in R$ be a fixed element. A map $d : R \longrightarrow R$ defined by $d(x) = [a, x] = ax - xa$, $x \in R$, is a derivation on R, which is called an *inner derivation* defined by a. Recently, many results in literature indicate how the global structure of a ring R is often tightly connected to the behaviour of additive mappings defined on R (for example, see [2], [3], [5], [6] and [13]). Herstein [14] showed that a 2-torsion free prime ring R must be a commutative integral domain if it admits a nonzero derivation d satisfying $[d(x), d(y)] = 0$ for all $x, y \in R$, and if the characteristic of R equals two, the ring R must be commutative or an order in a simple algebra which is 4-dimensional over its center. Several authors have proved commutativity theorems for prime rings admitting derivations which are centralizing on R. We begin recalling that a mapping $f : R \longrightarrow R$ is called centralizing on R if $[f(x), x] \in Z(R)$ for all $x \in R$. A well known result of Posner [16] states that if d is a derivation of the prime ring R such that

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$[d(x), x] \in Z(R)$ for any $x \in R$, then either $d = 0$ or R is commutative. In [10] Lanski generalizes the result of Posner to a Lie ideal.

More recently several authors considered similar situation in the case the derivation d is replaced by a generalized derivation. More specifically an additive map $F : R \longrightarrow R$ is said to be a generalized derivation if there exists a derivation d of R such that, for all $x, y \in R$, $F(xy) = F(x)y + xd(y)$. Basic examples of generalized derivations are the usual derivations on R and a left R-module mappings from R into itself. An important example is a map of the form $F(x) = ax + xb$ for some $a, b \in R$; such generalized derivations are called *inner*. Generalized derivations have been primarily studied on operator algebras. Therefore any investigation from the algebraic point of view might be interesting (see for example [11] and [15]).

The present paper is motivated by the previous results and we here continue this line of investigation by studying some functional identities related to derivations of a ring R and their action on the centre of R/P where P is a prime ideal of R.

2. Some results inspired by Herstein theorems

In what follows, $\bar{x}$ for x in R denotes $x + P$ in R/P. We begin our discussion with the following lemma which is essential for developing the proof of our main results.

Lemma 2.1. *Let R be a ring and P a prime ideal of R. If d is a derivation of R and $a \in R$ such that $[a, d(x)] \in P$ for all $x \in R$, then:*

1. *If $char(R/P) \neq 2$, then $d(R) \subseteq P$ or $\overline{a} \in Z(R/P)$.*
2. *If $char(R/P) = 2$, then $\overline{a}^2 \in Z(R/P)$. Moreover, if $\overline{a} \notin Z(R/P)$, then d satisfies $\overline{d(x)} = \lambda\overline{[a, x]}$ for all $x \in R$, where λ in the extended centroid of R/P.*

Proof. We are given that

$$[a, d(x)] \in P \quad \text{for all } x \in R. \tag{2.1}$$

Substituting xy instead of x in (2.1), we get

$$[a, d(x)]y + +d(x)[a, y] + x[a, d(y)] + [a, x]d(y) \in P \quad \text{for all } x, y \in R \tag{2.2}$$

which, in view of (2.1), the last expression yields

$$d(x)[a, y] + [a, x]d(y) \in P \quad \text{for all } x, y \in R. \tag{2.3}$$

As a special case of (2.3), when we put $y = d(r)$ we may write

$$d(x)[a, d(r)] + [a, x]d^2(r) \in P \quad \text{for all } r, x \in R \tag{2.4}$$

and employing the fact that $[a, d(r)] \in P$ for all $r \in R$, then (2.4) may be restated as

$$[a, x]d^2(r) \in P \quad \text{for all } r, x \in R. \tag{2.5}$$

If we write xy instead of x in (2.5) and using it, we obtain

$$[a,x]Rd^2(r) \subseteq P \quad \text{for all } r,x \in R. \tag{2.6}$$

Invoking the primeness of P, it follows from the above expression that either $[a,x] \in P$ for all $x \in R$ or $d^2(r) \in P$ for all $r \in R$. In the first case we obtain $\overline{a} \in Z(R/P)$. For the later case replacing r by rs, we arrive at

$$d(d(rs)) = d^2(r)s + 2d(r)d(s) + rd^2(s) \in P \quad \text{for all } r,s \in R. \tag{2.7}$$

In such a way that

$$2d(r)d(s) \in P \quad \text{for all } r,s \in R. \tag{2.8}$$

Once again putting rt instead of r in the last relation, we obviously find that

$$2d(r)Rd(s) \subseteq P \quad \text{for all } r,s \in R. \tag{2.9}$$

However, if the characteristic of R/P is not 2, we obtain

$$d(r)Rd(s) \subseteq P \quad \text{for all } r,s \in R. \tag{2.10}$$

Using the primeness of P together with equation (2.10), we conclude that $d(R) \subseteq P$.

Now assuming that the characteristic of the ring R/P is two, and putting ry instead of y in relation (2.3) and applying it, we may write

$$d(x)r[a,y] + [a,x]rd(y) \subseteq P \quad \text{for all } r,x,y \in R. \tag{2.11}$$

This may be restated as

$$\overline{d(x)}\overline{r}\overline{[a,y]} = \overline{[a,x]}\overline{r}\overline{d(y)} \quad \text{for all } r,x,y \in R. \tag{2.12}$$

As a particular case of (2.12), when we put $y = x$, it is obvious to see that

$$\overline{d(x)}\overline{r}\overline{[a,x]} = \overline{[a,x]}\overline{r}\overline{d(x)} \quad \text{for all } r,x \in R. \tag{2.13}$$

If $\overline{[x,a]} = \overline{0}$, then $\overline{a} \in Z(R/P)$.

Now assuming that $\overline{a} \notin Z(R/P)$, then [7, Lemma 1.3.2] proving that $\overline{d(x)} = \lambda\overline{[a,x]}$ where λ in the extended centroid of R/P.

Now the hypothesis $[a,d(x)] \in P$ for all $x \in R$, leads to $\lambda\overline{[a,[a,x]]} = \overline{0}$. So because of $\lambda \neq 0$ we arrive at $\overline{a(ax+xa)} = \overline{(ax+xa)a}$. Accordingly $\overline{a}^2 \in Z(R/P)$. This completes the proof of our result. □

A classical theorem of Herstein [9] states that: if R is a prime ring provided with a nonzero derivation d and $a \in R$ such that $ad(x) - d(x)a = 0$ for all $x \in R$, then; if the characteristic of R is not equal to two, then $a \in Z(R)$, and if the characteristic of R is two, then $a^2 \in Z(R)$.

Our goal in the following theorem is to investigate a more general context of differential identity involving a prime ideal P by omitting the primeness assumption imposed on the considered ring R. This approach allows us to generalize the preceding result, indeed we will study the behaviour of the more general expression $\overline{ad(x) - d(x)a} \in Z(R/P)$ for all $x \in R$, where R is any ring and P is a prime ideal of R rather than $ad(x) - d(x)a = 0$. Moreover, our result is more consistent because we will not get $\overline{a} \in Z(R/P)$ but we will also prove that the derivation d has its range in the prime ideal P. More precisely we will prove the following result.

Theorem 2.2. *Let R be a ring and P be a prime ideal of R. If d is a derivation of R and $a \in R$ such that $\overline{[a, d(x)]} \in Z(R/P)$ for all $x \in R$, then:*

1. *If $char(R/P) \neq 2$, then $d(R) \subseteq P$ or $\overline{a} \in Z(R/P)$.*
2. *If $char(R/P) = 2$, then $\overline{a}^2 \in Z(R/P)$.*

Proof. Suppose that

$$\overline{[a, d(x)]} \in Z(R/P) \quad \text{for all } x \in R. \tag{2.14}$$

Analogously, substituting $[a, x]$ instead of x in the above relation, it follows that

$$\overline{[a, [d(a), x]] + [a, [a, d(x)]]} \in Z(R/P) \quad \text{for all } x \in R. \tag{2.15}$$

This means that

$$\overline{[a, [d(a), x]]} \in Z(R/P) \quad \text{for all } x \in R. \tag{2.16}$$

Once again putting ax instead of x in (2.16), we thereby obtain

$$\overline{[a, a[d(a), x] + [d(a), a]x]} \in Z(R/P) \quad \text{for all } x \in R. \tag{2.17}$$

Keeping in mind that $\overline{[d(a), a]}$ is central in R/P and using the last expression we have

$$\overline{a[a, [d(a), x]] + [d(a), a][a, x]} \in Z(R/P) \quad \text{for all } x \in R. \tag{2.18}$$

Commuting the relation (2.18) with a and invoking (2.16), we arrive at

$$[d(a), a][[a, x], a] + [[d(a), a], a][a, x] \in P \quad \text{for all } x \in R. \tag{2.19}$$

This can be rewritten as

$$[d(a), a]R[[a, x], a] \subseteq P \quad \text{for all } x \in R. \tag{2.20}$$

In light of primeness of P, we get either $[d(a), a] \in P$ or $[[a, x], a] \in P$. In the later case, we can again employ the argument of Lemma 2.1, we obtain the required result. Now suppose that $[d(a), a] \in P$, then the relation (2.18) becomes

$$\overline{a[a, [d(a), x]]} \in Z(R/P) \quad \text{for all } x \in R. \tag{2.21}$$

The fact that $\overline{[a, [d(a), x]]} \in Z(R/P)$ by expression (2.16), forces that either $\overline{a} \in Z(R/P)$ or $[a, [d(a), x]] \in P$ for all $x \in R$, if $D_{d(a)}(x) = [d(a), x]$ denotes the inner derivation induced by $d(a)$, then the preceding relation may be restated as

$$[a, D_{d(a)}(x)] \in P \quad \text{for all } x \in R. \tag{2.22}$$

If the characteristic of R/P is not equal to two, then invoking Lemma 2.1, we get either $\overline{a} \in Z(R/P)$ or $D_{d(a)}(R) \subseteq P$. In the second case we obtain $\overline{d(a)} \in Z(R/P)$. Replacing x by xa in our hypothesis, we may write

$$\overline{[a, d(x)]a + x[a, d(a)] + [a, x]d(a)} \in Z(R/P) \quad \text{for all } x \in R. \tag{2.23}$$

Using the fact that $[d(a), a] \in P$, we arrive at

$$\overline{[a, d(x)]a + [a, x]d(a)} \in Z(R/P) \quad \text{for all } x \in R. \tag{2.24}$$

Commuting the last relation with a and using the hypothesis, we obtain

$$[[a, x], a]d(a) \in P \quad \text{for all } x \in R. \tag{2.25}$$

So that

$$[[a, x], a]Rd(a) \subseteq P \quad \text{for all } x \in R. \tag{2.26}$$

The primeness of P implies easily that either $[[a, x], a] \in P$ for all $x \in R$ or $d(a) \in P$. In the first case applying Lemma 2.1 we get $\overline{a} \in Z(R/P)$. Now if $d(a) \in P$, then (2.23) becomes

$$\overline{[a, d(x)]a} \in Z(R/P) \quad \text{for all } x \in R. \tag{2.27}$$

Then either $\overline{a} \in Z(R/P)$ or $[a, d(x)] \in P$ for all $x \in R$. In light of Lemma 2.1 it follows that $\overline{a} \in Z(R/P)$ or $d(R) \subseteq P$.

Therefore we assume henceforth that the characteristic of the ring R/P is equal to two, then invoking Lemma 2.1 from relation (2.22), we find that $\overline{a}^2 \in Z(R/P)$. Moreover if $\overline{a} \notin Z(R/P)$, then $\overline{D_{d(a)}(x)} = \lambda\overline{[a, x]}$ where λ is in the extended centroid of R/P. However the relation (2.22), becomes $\lambda\overline{[a, [a, x]]} = \overline{0}$, and thus $\overline{a(ax + xa)} = \overline{(ax + xa)a}$. Accordingly $\overline{a}^2 \in Z(R/P)$. This completes the proof of our theorem. □

If we consider R is a prime ring in Theorem 2.2, then $P = (0)$ is a prime ideal of R, in this case we get a generalization of Herstein's result [9].

Corollary 2.3. *Let R be a prime ring. If d is a nonzero derivation of R and $a \in R$ such that $[a, d(x)] \in Z(R)$ for all $x \in R$, then*:

1. *If char$(R) \neq 2$, then $a \in Z(R)$.*
2. *If char$(R) = 2$, then $a^2 \in Z(R)$.*

In 1969 Herstein [8, Theorem 2] proved that if a prime ring R of characteristic different from two admits a nonzero derivation d such that $[d(x), d(y)] = 0$ holds for all $x, y \in R$, then R is commutative. Motivated by the above result we investigate a more general context of differential identity involving a prime ideal by omitting the primeness assumption imposed on the ring. Especially, we will investigate in the following proposition the behavior of the more general expression $\overline{[d_1(x), d_2(y)]} \in Z(R/P)$ for all $x, y \in R$ where R is any ring and P is a prime ideal of R.

Proposition 2.4. *Let R be a ring and P is a prime ideal of R such that char$(R/P) \neq 2$. If d_1, d_2 are derivations of R satisfying $\overline{[d_1(x), d_2(y)]} \in Z(R/P)$ for all $x, y \in R$, then one of the following assertions holds*:

(1) $d_1(R) \subseteq P$.
(2) $d_2(R) \subseteq P$.
(3) *R/P is a commutative integral domain.*

Proof. Of course we have $\overline{[d_1(x), d_2(y)]} \in Z(R/P)$ for all $x, y \in R$, and the characteristic of the ring R/P is not 2, then according to Theorem 2.2, we have $\overline{d_1(x)} \in Z(R/P)$ for all $x \in R$ or $d_2(R) \subseteq P$. The relation of the first case reduces to $[d_1(x), x] \in P$ and applying [1, Lemma 1], we conclude that $d_1(R) \subseteq P$ or R/P is a commutative integral domain and we are done. □

Now, we get a similar result of P. H. Lee et al. [12, Theorem 2], which is a generalization of Herstein's result [9, Theorem 2].

Corollary 2.5 ([12, Theorem 2])**.** *Let R be a 2-torsion free prime ring. If d_1, d_2 are nonzero derivations of R, then the following assertions are equivalent*:

(1) $[d_1(x), d_2(y)] \in Z(R)$ *for all* $x, y \in R$;
(2) *R is a commutative integral domain.*

In [12, Theorem 4], it is showed that if R is a 2-torsion free prime ring and d_1, d_2 are two nonzero derivations of R such that $d_1d_2(R) \subseteq Z(R)$, then R must be commutative. Motivated by this result the author in [4, Theorem 2], established that: if R is any ring and P is a prime ideal of R such that the characteristic of R/P is not 2 and d_1, d_2 are two derivations of R such that $d_1d_2(R) \subseteq P$ for all $x, y \in R$, then $d_1(R) \subseteq P$ or $d_2(R) \subseteq P$.

A trivial question that now appears: Is that conclusion remains satisfied if we consider the identity $\overline{d_1d_2(x)} \in Z(R/P)$ for all $x \in R$ where R is any ring and P is a prime ideal of R? The following proposition gives an affirmative answer to this question.

Proposition 2.6. *Let R be a ring and P is a prime ideal of R such that char$(R/P) \neq 2$. If d_1, d_2 are derivations of R satisfying $\overline{d_1d_2(x)} \in Z(R/P)$ for all $x \in R$, then one of the following assertions holds*:

(1) $d_1(R) \subseteq P$.
(2) $d_2(R) \subseteq P$.
(3) *R/P is a commutative integral domain.*

Proof. We are given that

$$\overline{d_1d_2(x)} \in Z(R/P) \quad \text{for all } x \in R. \tag{2.28}$$

Replacing x by $[x, y]$ in this relation and applying the hypothesis, we obviously obtain

$$\overline{[d_1(x), d_2(y)] + [d_2(x), d_1(y)]} \in Z(R/P) \quad \text{for all } x, y \in R. \tag{2.29}$$

Once again putting $x = d_2(r)$ in (2.29), it follows that

$$\overline{[d_2^2(r), d_1(y)]} \in Z(R/P) \quad \text{for all } r, y \in R. \tag{2.30}$$

Then according to Theorem 2.2, we have either $\overline{d_2^2(r)} \in Z(R/P)$ for all $r \in R$ or $d_1(R) \subseteq P$. Suppose that

$$\overline{d_2^2(x)} \in Z(R/P) \quad \text{for all } x \in R. \tag{2.31}$$

Writing $[x, y]$ instead of x in this expression, we find that

$$\overline{[d_2^2(x), y] + 2[d_2(x), d_2(y)] + [x, d_2^2(y)]} \in Z(R/P) \quad \text{for all } x, y \in R. \tag{2.32}$$

Using the fact that $\overline{d_2^2(x)} \in Z(R/P)$ for all $x \in R$, we arrive at

$$2\overline{[d_2(x), d_2(y)]} \in Z(R/P) \quad \text{for all } x, y \in R. \tag{2.33}$$

Because of the characteristic of R/P is not 2, leads to $\overline{[d_2(x), d_2(y)]} \in Z(R/P)$ for all $x, y \in R$. Hence Proposition 2.4 forces that $d_2(R) \subseteq P$ or R/P is a commutative integral domain. □

Now, the following corollary deduces the result of P. H. Lee et al. [12, Theorem 4].

Corollary 2.7 ([12, Theorem 4])**.** *Let R be a 2-torsion free prime ring. If d_1, d_2 are nonzero derivations of R, then the following assertions are equivalent*:

(1) $d_1 d_2(x) \in Z(R)$ *for all* $x \in R$;
(2) *R is a commutative integral domain.*

The following example demonstrates that the primeness condition imposed on the ideal P in Theorem 2.2 can not be omitted.

Example 2.8. Let us consider the ring $R = \left\{ \left(\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}\right) \,\middle|\, x, y, z \in \mathbb{Z} \right\}$ and $d\left(\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & y \\ 0 & 0 \end{smallmatrix}\right)$. It straightforward to check that d is a derivation of R and $P = (0)$ is a non-prime ideal of R. Moreover, for $a = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ we have

$$[a, d(X)] \in Z(R) \quad \text{for all } X \in R.$$

However $a \notin Z(R)$.

To close this circle of ideas, it's natural to ask whether Herstein's theorem in [9] is true for semi-prime rings. Since all our proof attempts have failed, we are forced to consider the following conjecture:

Conjecture. Hestein's theorem in [9] cannot be extended to semi-prime rings.

The next example gives an affirmative answer to the above conjecture.

Example 2.9. Let us consider the 2-torsion free semi-prime ring $\mathcal{R} = \mathbb{Q}[X] \times R$ where R is a non-commutative prime ring and define $d(P, M) = (P', 0)$ for all

$(P, M) \in \mathcal{R}$ where P' denotes the usually derivation. If we set $a = (X, \alpha) \in \mathcal{R}$, then d is a nonzero derivation of $\mathcal{R}$ such that

$$\begin{aligned}[a, d(P, M)] &= (X, \alpha)\,(P', 0) - (P', 0)\,(X, \alpha) \\ &= (0, 0) \in Z(\mathcal{R}).\end{aligned}$$

However $a \notin Z(\mathcal{R})$.

References

[1] F. A. A. Almahdi, A. Mamouni, and M. Tamekkante, *A generalization of Posner's theorem on derivations in rings*, Indian J. Pure Appl. Math. **51** (2020), no. 1, 187–194. https://doi.org/10.1007/s13226-020-0394-8

[2] H. E. Bell and M. N. Daif, *On derivations and commutativity in prime rings*, Acta Math. Hungar. **66** (1995), no. 4, 337–343. https://doi.org/10.1007/BF01876049

[3] K. Bouchannafa, M. A. Idrissi, and L. Oukhtite, *Relationship between the structure of a quotient ring and the behavior of certain additive mappings*, Commun. Korean Math. Soc. **37** (2022), no. 2, 359–370. https://doi.org/10.4134/CKMS.c210126

[4] T. Creedon, *Derivations and prime ideals*, Math. Proc. R. Ir. Acad. **98A** (1998), no. 2, 223–225.

[5] H. El Mir, A. Mamouni, and L. Oukhtite, *Commutativity with algebraic identities involving prime ideals*, Commun. Korean Math. Soc. **35** (2020), no. 3, 723–731. https://doi.org/10.4134/CKMS.c190338

[6] I. N. Herstein, *Topics in Ring Theory*, University of Chicago Press, Chicago-London, 1969.

[7] I. N. Herstein, *Rings with Involution*, University of Chicago Press, Chicago-London, 1976.

[8] I. N. Herstein, *A note on derivations*, Canad. Math. Bull. **21** (1978), no. 3, 369–370. https://doi.org/10.4153/CMB-1978-065-x

[9] I. N. Herstein, *A note on derivations. II*, Canad. Math. Bull. **22** (1979), no. 4, 509–511. https://doi.org/10.4153/CMB-1979-066-2

[10] C. Lanski, *Differential identities, Lie ideals, and Posner's theorems*, Pacific J. Math. **134** (1988), no. 2, 275–297. http://projecteuclid.org/euclid.pjm/1102689262

[11] T.-K. Lee, *Generalized derivations of left faithful rings*, Comm. Algebra **27** (1999), no. 8, 4057–4073. https://doi.org/10.1080/00927879908826682

[12] P. H. Lee and T. K. Lee, *On derivations of prime rings*, Chinese J. Math. **9** (1981), no. 2, 107–110.

[13] A. Mamouni, L. Oukhtite, and H. Elmir, *New classes of endomorphisms and some classification theorems*, Comm. Algebra **48** (2020), no. 1, 71–82. https://doi.org/10.1080/00927872.2019.1632330

[14] A. Mamouni, L. Oukhtite, and M. Zerra, *On derivations involving prime ideals and commutativity in rings*, São Paulo J. Math. Sci. **14** (2020), no. 2, 675–688. https://doi.org/10.1007/s40863-020-00187-z

[15] L. Oukhtite and A. Mamouni, *Generalized derivations centralizing on Jordan ideals of rings with involution*, Turkish J. Math. **38** (2014), no. 2, 225–232. https://doi.org/10.3906/mat-1203-14

[16] E. C. Posner, *Derivations in prime rings*, Proc. Amer. Math. Soc. **8** (1957), 1093–1100. https://doi.org/10.2307/2032686

Abdellah Mamouni
Department of Mathematics
Faculty of Sciences
University Moulay Ismaïl
Meknes, Morocco
Email address: a.mamouni.fste@gmail.com

Lahcen Oukhtite
Department of Mathematics
Laboratory of Modeling and Mathematical Structures
Faculty of Sciences and Technology
University Sidi Mohamed Ben Abdellah
Fez, Morocco
Email address: lahcen.oukhtite@usmba.ac.ma

Mohammed Zerra
Department of Mathematics
Laboratory of Modeling and Mathematical Structures
Faculty of Sciences and Technology
University Sidi Mohamed Ben Abdellah
Fez, Morocco
Email address: mohamed.zerra@gmail.com