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PRÜFER CONDITIONS VS EM CONDITIONS

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Abstract. In this article we relate the six Prüfer conditions with the EM conditions. We use the EM-conditions to prove some cases of equivalence of the six Prüfer conditions. We also use the Prüfer conditions to answer some open problems concerning EM-rings.

1. Introduction

All rings are assumed to be commutative with unity. Let R be a ring such that for each $a, b \in R$, there exist $c, d, f \in R$ with $a = cd, b = df$ and $R = (d, f)$. Kaplansky in [16] named this ring a Hermite ring. Many other different rings were called in the literature Hermite rings, and so it was suggested to call Kaplansky's ring a K-Hermite ring. A generalization to K-Hermite rings was given in [2] to be EM-Hermite rings in which if $a, b \in R$, there exist $c, d, f \in R$ with $a = cd, b = df$ and (c, f) is a regular ideal. It was shown there that R is K-Hermite if and only if R is a Bézout and EM-Hermite ring. Yet another generalization was given in [1], R is called an EM-ring if for each $f(x) \in R[x]$, there exist $a \in R$ and a regular polynomial $g(x) \in R[x]$ such that $f(x) = ag(x)$. The ring R is called a locally EM-ring if for each prime ideal P of R, we have R_P is an EM-ring.

It was shown in [3] that a ring R is an EM-ring if and only if for any finitely generated ideal I of R there exist $a \in R$ and a finitely generated ideal J of R such that $I = aJ$ and $Ann(J) = \{0\}$.

It is clear that an EM-Hermite ring is an EM-ring with property A. An example of an EM-ring that is not an EM-Hermite ring can be found in [2].

In this article we try to relate the EM conditions with the six well known Prüfer conditions:

(P1) R is a Prüfer ring (every finitely generated regular ideal in R is invertible).

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(P2) R is a locally Prüfer ring (R_P is Prüfer for every prime ideal P of R).
(P3) R is a Gaussian ring (for every $f, g \in R[x], c(fg) = c(f)c(g)$).
(P4) R is an arithmetical ring (every finitely generated ideal of R is locally principal).
(P5) $w.\dim(R) \leq 1$ (every finitely generated ideal of R is flat).
(P6) R is semihereditary (every finitely generated ideal of R is projective).

It is known that if R is an integral domain, then (P1) to (P6) are all equivalent, but if R is not an integral domain, then (P6) $\Rightarrow$ (P5) $\Rightarrow$ P(4) $\Rightarrow$ P(3) $\Rightarrow$ P(2) $\Rightarrow$ P(1), while the reverse implications are all false. A lot of work in the literature are done to investigate the cases at which some of these conditions are equivalent. For a survey for the six Prüfer conditions, see [11].

In Section 2, we studied the relations between the EM conditions and the Prüfer conditions. While K-Hermite rings and semihereditary rings are incomparable, there are some other implications. At the end of the section there is a diagram illustrating these interactions.

In Section 3, we used the EM-conditions to prove some cases of equivalence of the six Prüfer conditions. Theorem 3.9 shows that if R is a reduced EM-ring such that $\mathrm{Min}(R)$ is compact, then the Prüfer conditions (P1) to (P6) are equivalent. Weaker conditions are given in Theorem 3.1 to show that (P1) to P(4) are equivalent and Theorem 3.2 to show (P1) to (P5) are equivalent. It was proved in [1] that if R is a PP-ring (every principal ideal is projective), then R is an EM-ring, but the relation with PF-rings (every principal ideal is flat) was not investigated, now and while looking for relations between EM-rings and Prüfer rings, we found an example of a PF-ring that is not an EM-ring. We also manage to find an example of a locally EM-ring that is not an EM-ring, while investigating relations between EM-rings and arithmetical rings.

In Section 4, we used the Prüfer conditions and Theorems 3.1, 3.2 and 3.9, to answer some open problems concerning EM-rings. It is proved in [1] that if R is an EM-ring, then $T(R)$ (the total quotient ring of R) and $R[x]$ are EM-rings. Here we give partial answers to the converses.

For any ring R, let $Z(R) = \{a \in R : ab = 0 \text{ for some } b \in R - \{0\}\}$, $Nil(R) = \{a \in R : a^n = 0 \text{ for some } n \in \mathbb{N}\}$ for any subset S of R, let $Ann(S) = \{a \in R : as = 0 \text{ for all } s \in S\}$, $Min(R) = \{P : P \text{ is a minimal prime ideal of } R\}$. Recall that a ring R is said to be reduced if $Nil(R) = \{0\}$. R is said to have property A if every finitely generated ideal $I \subseteq Z(R)$ has a nonzero annihilator. A ring R is an a.c. ring if for every pair of elements $a, b \in R$, there is an element $c \in R$ such that $Ann(a, b) = Ann(c)$. It is shown in [17] that if R is a reduced ring, then R has property A and $Min(R)$ is compact if and only if R is an a.c. ring and $Min(R)$ is compact.

2. Prüfer conditions vs EM conditions

A semihereditary non-Bézout ring is not K-Hermite, and we found an example of a K-Hermite ring that is not semihereditary, see Example 3.3 below, and so semihereditary and K-Hermite are incomparable. Similarly K-Hermite rings and rings R with $w.\dim(R) \leq 1$ are incomparable; as $\mathbb{Z}_4$ is K-Hermite with $w.\dim(\mathbb{Z}_4) > 1$, and for an example of a ring R with $w.\dim(R) \leq 1$ that is not K-Hermite, see Example 3.4 below, while any K-Hermite ring is Bézout, and so it is arithmetical.

If R is semihereditary, then it is EM-Hermite, and so it is an EM-ring, while $\mathbb{Z}[x]$ is an example of an EM-Hermite ring that is not Prüfer.

The ring in Example 3.5 below has weak global dimension ≤ 1 but it is not an EM-ring. Thus the EM-rings and EM-Hermite rings are incomparable with Prüfer conditions (P1) to (P5). It is clear that any arithmetical ring is a locally EM-ring, while $\mathbb{Z}[x]$ is a locally EM-ring that is not arithmetical. Example 3.7 below is an example of a Gaussian ring that is not a locally EM-ring, and so a locally EM-ring is incomparable with Prüfer conditions (P1) to P(3). But yet rings with Prüfer conditions and EM-rings affect each others. The following diagram illustrates the relations between these rings.

$$\begin{array}{ccccc}
 & & & & \text{Prüfer} \\
 & & & & \uparrow \\
 & & & & \text{Locally Prüfer} \\
 & & & & \uparrow \\
 & & & & \text{Gaussian} \\
 & & & & \uparrow \\
\text{Semihereditary} & \longrightarrow & w.dim(R) \leq 1 & \longrightarrow & \text{Arithmetical} \\
\downarrow & & & \nearrow & \downarrow \\
\text{EM-Hermite} & \longleftarrow & \text{K-Hermite} & & \text{Locally EM-ring} \\
\downarrow & & & \nearrow & \\
\text{EM-ring with property A} & \longrightarrow & \text{EM-ring} & &
\end{array}$$

3. When are the Prüfer conditions equivalent?

In this section we will use the EM conditions to prove some cases of equivalence of the six Prüfer conditions.

Theorem 3.1. *Let R be an EM-ring with property A. Then the Prüfer conditions* (P1) *to* (P4) *are equivalent.*

Proof. For the implications P(4) $\Rightarrow$ P(3) $\Rightarrow$ P(2) $\Rightarrow$ P(1), see [11].

P(1) $\Rightarrow$ P(4): Let I be a finitely generated ideal in R. Then $I = aJ$ where $a \in R$ and J is a finitely generated ideal of R with $Ann(J) = \{0\}$, and since R has property A, J is a regular ideal. Since R is a Prüfer ring, J is invertible, and so it follows by Theorem 3.9 in [11] that J is locally principal, and hence $I = aJ$ is locally principal. Thus R is arithmetical. □

The ring $\mathbb{Z}_4$ is a Prüfer EM-Hermite ring, and so all the four conditions in Theorem 3.1 are satisfied, but it is not reduced, and so $w.\dim(\mathbb{Z}_4) > 1$. We now see that if we add the reduced condition, then the four conditions in Theorem 3.1 are equivalent to P(5).

Theorem 3.2. *Let R be a reduced EM-ring with property A. Then the Prüfer conditions* (P1) *to* (P5) *are equivalent.*

Proof. For the implications P(5) $\Rightarrow$ P(4) $\Rightarrow$ P(3) $\Rightarrow$ P(2) $\Rightarrow$ P(1), see [11].

P(1) $\Rightarrow$ P(5): It follows by Theorem 3.1 that R is Gaussian. But if R is a Gaussian reduced ring, then $w.\dim(R) \leq 1$, see [11, Theorem 5.6]. □

The question now is: if the conditions in Theorem 3.2 are sufficient to imply that R is semihereditary. Unfortunately the answer is no, as seen in the following examples.

Example 3.3. Let $X = \beta\mathbb{R}^+ - \mathbb{R}^+$, where $\beta\mathbb{R}^+$ is the Stone-Čech compactification of $\mathbb{R}^+$. It was proved in [8, Example 3.3] that X is a compact connected F-space, and its ring of real valued continuous functions $C(X)$ is Bézout, K-Hermite ring, and so it is a reduced EM-Hermite ring with $w.\dim(C(X)) \leq 1$, but it is not semihereditary, since X is connected.

Example 3.4. Let $X = [0,\infty) \times [-1,1]$. It was proved in [8, Example 3.4] that $R = C(\beta X - X)$ is a Bézout ring that is not K-Hermite and as noted in [2] that this implies that R is not EM-Hermite. Then R is an EM-ring with $w.\dim(R) \leq 1$, but R is not semihereditary.

Example 3.5. Let $T = \prod_{i\in\mathbb{N}} \mathbb{Q}[x]$, $f = (x, 0, x^2, 0, x^3, 0, \ldots)$, $I = fT$, D the set of all sequences in T that are eventually constant and let $R = I + D$. It was noted in [11, Example 4.1] that $w.\dim(R) \leq 1$, but R is not semihereditary. We now show that R is not an EM-ring. Let

$$\overline{a} = (xf_1, 0, x^2f_2, 0, x^3f_3, 0, \ldots) \text{ with } f_i = x + i,$$
$$\overline{b} = (xg_1, 0, x^2g_2, 0, x^3g_3, 0, \ldots) \text{ with } g_i = x^2 + i.$$

Assume $k(y) = \overline{a} + \overline{b}y = \overline{\alpha}\sum_{i=0}^{k} \overline{\beta}_i y^i$ with $\bigcap Ann(\overline{\beta}_i) = \{\overline{0}\}$.

Note that the tail of any element in R is of the form $(x^n f_n(x) + c(x), c(x), x^{n+1}f_{n+1}(x) + c(x), c(x), \ldots)$. We can find $n \in \mathbb{N}$ such that

$$\overline{a} = (\ldots, x^n f_n(x), 0, x^{n+1}f_{n+1}(x), \ldots),$$
$$\overline{b} = (\ldots, x^n g_n(x), 0, x^{n+1}g_{n+1}(x), 0, \ldots),$$
$$\overline{\alpha} = (\overline{\alpha}_1, x^n h_n(x) + d(x), d(x), x^{n+1}h_{n+1}(x) + d(x), d(x), \ldots),$$
$$\overline{\beta}_i = (\overline{\beta}_{i,1}, x^n r_{i,n}(x) + c_i(x), c_i(x), x^{n+1}r_{i,n+1}(x) + c_i(x), c_i(x), \ldots),$$

where $\overline{\alpha}_1, \overline{\beta}_{i,1} \in \prod_{i=1}^{n-1} \mathbb{Q}[x]$ for $i = 0, 1, \ldots, k$. Since $dc_i = 0$ for all i and $\sum_{i=0}^{k} c_i^2 \neq 0$, we must have $d = 0$. Thus we have

$$x^n h_n(x) \mid \gcd(x^n f_n(x), x^n g_n(x)),$$

and so we get $h_n \in \mathbb{Q}$. Hence we have for each $n > 2$

$$x + n = h_n c_0(x),$$
$$x^2 + n = h_n c_1(x),$$

a contradiction. Hence R is not an EM-ring.

Remark 3.6. The above examples show that the Prüfer conditions (P1) to (P5) are incomparable with EM, and EM-Hermite rings. Moreover the last example shows that if R is a PF-ring or a locally EM-ring (since clearly an arithmetical ring is a locally EM-ring), then R needs not be an EM-ring.

Example 3.7. Let K be a field and let $R = K[x, y] / (x, y)^2$, where x and y are indeterminants over K. Then R is a Gaussian local ring, see [6, Example 3.9], but as was noted in [3] that R is not an EM-ring, and so it is not locally EM.

The question now is: what an extra condition must be added to a reduced EM-ring to ensure that all the six Prüfer conditions are equivalent? Before we give a partial answer to this question, we give the following example.

Example 3.8. Let $X = \beta\mathbb{N} - \mathbb{N}$. As noted in [12, Example 5.9] that $C(X)$ is a Bézout ring, but $\mathrm{Min}(C(X))$ is not compact and $C(X)$ is not semihereditary, and so it is an EM-ring with $w.\dim(C(X)) \leq 1$.

Theorem 3.9. *Let R be a reduced EM-ring such that* $\mathrm{Min}(R)$ *is compact. Then the Prüfer conditions* (P1) *to* (P6) *are equivalent.*

Proof. For the implications P(6) ⇒ P(5) ⇒ P(4) ⇒ P(3) ⇒ P(2) ⇒ P(1), see [11].

P(1) ⇒ P(6): Since R is an EM-ring, it is an a.c. ring, and so $T(R)$ is a von Neumann regular ring and $w.\dim(R) \leq 1$. Hence it follows by Theorem 2.3 in [10] that R is semihereditary. □

Recall that a ring R is coherent if each finitely generated ideal of R is finitely presented. It is known that a ring R is coherent if and only if the intersection of any finitely generated ideals of R is finitely generated, and for each $a \in R$, $Ann(a)$ is finitely generated. So if R is a reduced coherent EM-ring, then $\mathrm{Min}(R)$ is compact, since R is an a.c. ring. Thus we have the following corollary.

Corollary 3.10. *Let R be a reduced coherent EM-ring. Then the Prüfer conditions* (P1) *to* (P6) *are equivalent.*

4. Solving some problems on EM-rings using Prüfer conditions

In this section, we will use results in the previous section to solve some open problems concerning EM-rings: what extra conditions on R to ensure if $T(R)$ or $R[x]$ is an EM-ring, then so is R. But first we give an example of an EM-ring that does not have property A.

Example 4.1. Let K be an algebraic closed field and $D = K[x, y]$. Let $R = A + B$ be the ring defined as in Example 2.4 in [17]. The author showed that this ring is a.c. but it does not have property A. We now show that R is an EM-ring. For each $n \in \mathbb{N}$, let $\psi_n : D \to \prod_{j \in I - \{1,2,\dots,n\}} D / P_j$ such that

$(\psi_n(z))_i = z + P_i$. Let $(\overline{f}_1, \overline{f}_2, \ldots, \overline{f}_n)$ be a finitely generated ideal in R. We can find an $m \in \mathbb{N}$ such that $\overline{f}_i = [\overline{\beta}_i, \psi_m(z_i)]$, with $\overline{\beta}_i \in \sum_{j=1}^{m} D/P_j$ for each i. Since $\sum_{j=1}^{m} D/P_j$ is an EM-Hermite ring, there exist $\overline{\beta}, \overline{\gamma}_1, \ldots, \overline{\gamma}_n \in \sum_{j=1}^{m} D/P_j$ such that $\overline{\beta}_i = \overline{\beta}\overline{\gamma}_i$ for each i and $Ann(\overline{\gamma}_1, \ldots, \overline{\gamma}_n) = \{\overline{0}\}$. Let $w = GCD(z_1, \ldots, z_n)$, and so we have $z_i = wk_i$ for each i with $GCD(k_1, \ldots, k_n) = 1$. Then $\psi_m(z_i) = \psi_m(w)\psi_m(k_i)$ with $Ann(\psi_m(k_1), \ldots, \psi_m(k_n)) = Ann(\psi_m(1)) = \{\overline{0}\}$. Thus we have

$$(\overline{f}_1, \overline{f}_2, \ldots, \overline{f}_n) = [\overline{\beta}, \psi_m(w)]\ ([\overline{\gamma}_1, \psi_m(k_1)], \ldots, [\overline{\gamma}_n, \psi_m(k_n)])$$

with $Ann([\overline{\gamma}_1, \psi_m(k_1)], \ldots, [\overline{\gamma}_n, \psi_m(k_n)]) = \{\overline{0}\}$. Thus R is a reduced EM-ring, but clearly $\mathrm{Min}(R)$ can not be compact.

4.1. If $T(R)$ is an EM-ring

If R is an EM-ring, then so is $T(R)$, the total ring of quotients of R. It was shown in [1] that if $R = \mathbb{Z}_6[x, y]/(xy)$, then $T(R)$ is an EM-ring, but R is not. In fact, since R is a Noetherian ring, we have also $T(R)$ is an EM-Hermite ring, while R is not. The question now is: what extra conditions may be added to get the converse. In the following we have partial answers using the Prüfer conditions.

Theorem 4.2. *Let R be a reduced Prüfer ring with* $\mathrm{Min}(R)$ *is compact. Then R is an EM-ring if and only if $T(R)$ is an EM-ring.*

Proof. ($\Rightarrow$) See [1].

($\Leftarrow$) $T(R)$ is an EM-ring, and so R has a.c. condition, and hence $T(R)$ is von Neumann regular. Now using Theorem 3.12(i) in [6], we get that R is semihereditary, and hence it is an EM-ring. ☐

Since a Noetherian ring has only finitely many minimal primes, we see that the above result is true for reduced Noetherian Prüfer rings.

Example 4.3 (Prüfer, reduced with $\mathrm{Min}(R)$ is compact $\not\Rightarrow$ EM-ring)**.** Let K be a countable, algebraically closed field, let J be an infinite set, and denote by K^J the set of all maps from J to K. Let $\mathbb{N}$ denote the set of natural numbers, and let $L = J \times \mathbb{N}^{\mathbb{N}}$. The author in [9, p. 120] constructed an algebra $R \subseteq K^L$, which satisfies the following properties:

(1) R is a reduced ring.
(2) $R = T(R)$.
(3) $\mathrm{Min}(R)$ is compact.
(4) R is not a von Neumann regular ring.
(5) R is Prüfer which is not Gaussian.

Using Theorem 3.9, R cannot be an EM-ring.

If R is a reduced coherent ring, then R is an a.c. ring if and only if $T(R)$ is a von Neumann regular ring. Thus we have the following.

Corollary 4.4. *Let R be a reduced coherent Prüfer ring. Then R is an EM-ring if and only if $T(R)$ is an EM-ring.*

Example 4.5 (Coherent, Prüfer ring $\not\Rightarrow$ EM-ring)**.** Let K be a field, and let T and U be indeterminants over K. Denote by t and u the images of T and U in $K[T,U]/(T,U)^2$. Let $R = K[t,u]_{(t,u)}$. Then R is a local, Noetherian, non-reduced Gaussian ring with $w.\dim R = \infty$, see [10, Example 3.4]. Since R is Noetherian, R is a coherent ring. But R is not an EM-ring, since if we consider the polynomial $f(x) = t + ux$, then $Ann(f(x)) = (t,u)$ is not principal.

Theorem 4.6. *Let R be a semilocal Prüfer ring with property A. Then R is an EM-ring if and only if $T(R)$ is an EM-ring.*

Proof. ($\Rightarrow$) See [1].

($\Leftarrow$) $T(R)$ is a Prüfer EM-ring, and so it is arithmetical. Now using Theorem 3.12(iii) in [6], we get that R is arithmetical. Using Theorem 5 in [15], we get that R is Bézout, and hence an EM-ring. □

4.2. If $R[x]$ is an EM-ring

It was proved in [1] that if R is an EM-ring, then so is $R[x]$. The converse was proved true in [7], if R is reduced. Also the converse was proved true in [3], if R is Noetherian. Unfortunately, it is unknown yet if the converse is always true or not.

Let $V = \{f \in R[x] : c(f) = R\}$, and let $U = \{f \in R[x] : f \text{ is monic}\}$. Then V and U are multiplicative closed sets. Let $R(x) = V^{-1}R[x]$, and $R\langle x\rangle = U^{-1}R[x]$. A ring R is called a strongly Prüfer ring if every finitely generated ideal I of R with $Ann(I) = 0$ is locally principal.

Theorem 4.7. *Let R be a semilocal strongly Prüfer ring. Then the following are equivalent*:

(1) *R is an EM-ring.*
(2) *$R[x]$ is an EM-ring.*
(3) *$R(x)$ is an EM-ring.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3): If R is an EM-ring, then $R[x]$ is an EM-ring, and so $R(x)$ is an EM-ring.

(3) $\Rightarrow$ (1): Since R is a strongly Prüfer ring, $R(x)$ is a Prüfer ring, see [4, Theorem 3.2]. Since $R(x)$ is an EM-ring with property A, see [13, Theorem 14.2] and using Theorem 3.1, $R(x)$ is arithmetical, and so R is arithmetical, see [4, Theorem 3.1]. Thus R is Bézout, being a semilocal arithmetical ring, see [15, Theorem 5], and so R is an EM-ring. □

Theorem 4.8. *Let R be a semilocal, Gaussian ring and assume that R_P is a field for every non-maximal prime ideal P of R. Then the following are equivalent*:

(1) *R is an EM-ring.*
(2) *$R[x]$ is an EM-ring.*

(3) *$R\langle x\rangle$ is an EM-ring.*

Proof. We only need (3) $\Rightarrow$ (1). Since R is a Gaussian ring and R_P is a field for every non-maximal prime ideal P of R, it follows that $R\langle x\rangle$ is a Gaussian ring, see [14, Theorem 2.2]. Since $R\langle x\rangle$ is a Gaussian EM-ring with property A, and using Theorem 3.1, $R\langle x\rangle$ is arithmetical, and so R is arithmetical, see [4, Theorem 3.1]. Thus R is Bézout, being a semilocal arithmetical ring, see [15, Theorem 5], and so R is an EM-ring. □

It was proved in [13, Theorem 18.12] that $R\langle x\rangle$ is Prüfer if and only if R is strongly Prüfer, $\dim(R) \leq 1$ and R_P is a field for every non-maximal prime ideal P of R. Thus we can obtain the following result.

Theorem 4.9. *Let R be a semilocal, strongly Prüfer ring,* $\dim(R) \leq 1$ *and R_P is a field for every non-maximal prime ideal P of R. Then the following are equivalent*:
(1) *R is an EM-ring.*
(2) *$R[x]$ is an EM-ring.*
(3) *$R\langle x\rangle$ is an EM-ring.*

In the previous three theorems, strong conditions were added to get the equivalence of the statements. Unfortunately, we don't know yet if they are equivalence in the general case or with weaker conditions. The following example shows that although these conditions are strong, they are not essential to give EM-ring.

Example 4.10 (Local, Gaussian ring with property A $\not\Longrightarrow$ EM-ring)**.** Let $K \subsetneq L$ be a field extension. Then the idealization $R = K(+)L$ is a Gaussian ring which is not arithmetical, see [5, Example 2.6]. Since K is a field, then K is von Neumann regular, and so $R = K(+)L$ has property A, see [17, p. 570]. Clearly R is 0-dimensional with only maximal ideal $0(+)L$. If R is an EM-ring, then by Theorem 3.2, R is arithmetical, which is a contradiction. So, R is not an EM-ring.

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