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ON THE STRUCTURE OF A k-ANNIHILATING IDEAL HYPERGRAPH OF COMMUTATIVE RINGS

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Abstract. In this paper we obtain a new structure of a k-annihilating ideal hypergraph of a reduced ring R, by determine the order and size of a hypergraph $\mathcal{AG}_k(R)$. Also we describe and count the degree of every nontrivial ideal of a ring R containing in vertex set $\mathcal{A}(R,k)$ of a hypergraph $\mathcal{AG}_k(R)$. Furthermore, we prove the diameter of $\mathcal{AG}_k(R)$ must be less than or equal to 2. Finally, we determine the minimal dominating set of a k-annihilating ideal hypergraph of a ring R.

1. Introduction

In the last twenty years, the structure of finite commutative rings associated graphs has been an intriguing issue that has been studied by some authors such as [2], [6], [7] and [8]. According to [4], Eslahchi and Rahimi introduced and studied a graph known as the k-zero divisor hypergraph of a commutative ring R, which is defined as: Let R be a commutative ring and $k \geq 2$ a fixed integer, a nonzero nonunit element a_1 in R is said to be a k-zero-divisor in R if there exist $k-1$ distinct nonunit elements a_2, a_3,..., a_k in R different from a_1 such that $a_1 \cdot a_2 \cdot a_3 \cdot \dots \cdot a_k = 0$ and the product of no elements of any proper subset of $A = \{a_1, a_2, \dots, a_k\}$ is zero.

In [8], K. Selvakumar, V. Ramanathan have introduced a k-annihilating ideal hypergraph of a commutative ring and defined as: Let R be a commutative ring and let $\mathcal{A}(R,k)$ be the set of all k-annihilating ideals in R and $k > 2$ an integer. The k-annihilating ideal hypergraph of R, denoted by $\mathcal{AG}_k(R)$ is a hypergraph with vertex set $\mathcal{A}(R,k)$ and for distinct elements $I_1, I_2, \dots, I_k$ in $\mathcal{A}(R,k)$, the set $\{I_1, I_2, \dots, I_k\}$ is an edge of $\mathcal{AG}_k(R)$ if and only if $\prod_{i=1}^{k} I_i = (0)$ and the product of $(k-1)$ element of $\{I_1, I_2, \dots, I_k\}$ is nonzero. Clearly, if R is an integral domain with $\mathcal{A}(R,k) = \phi$ for all $k \geq 2$, then $\mathcal{AG}_k(R) = \phi$.

Throughout this paper we suppose R is a finite commutative ring with identity. One of the most significant structures of a finite commutative ring is that

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a ring R is a reduced ring if and only if $R \cong F_1 \times F_2 \times \cdots \times F_n$, where F_i are finite fields and $1 \leq i \leq n$, we denote $J_{n-s} = (F_1 \times F_2 \times \cdots \times F_s \times 0 \times 0 \times \cdots \times 0)$ be any nontrivial ideal of a ring R with arbitrary s, where $2 \leq s \leq n-1$. We Assume that $k = 3$ is a fixed integer in the definition of a k-annihilating ideal hypergraph of R with the set of a nontrivial ideals J_{n-s} of a ring R; $\mathcal{A}(R,k)$ as a vertex set of a hypergraph $\mathcal{AG}_k(R)$.

A hypergraph $\mathcal{H}$ is a pair $(\mathcal{V}(\mathcal{H}), \mathcal{E}(\mathcal{H}))$ of disjoint sets, where $\mathcal{V}(\mathcal{H})$ is a non-empty finite set whose elements are called vertices, and the number of elements of $\mathcal{V}(\mathcal{H})$, is called order of hypergraph $\mathcal{H}$; denote by $n(\mathcal{H})$. Also the elements of $\mathcal{E}(\mathcal{H})$ are nonempty subsets of $\mathcal{V}(\mathcal{H})$ called hyperedges and the number of elements of hyperedges is called size of hypergraph $\mathcal{H}$; denote by $m(\mathcal{H})$. While graph edges are pair of vertices, hyperedges are arbitrary sets of vertices and can contain an arbitrary number of vertices. Special hypergraphs called uniform hypergraph where all the hyperedges have the same cardinality. The hypergraph $\mathcal{H}$ is called k-uniform if every edge e of $\mathcal{H}$ is of size k. The number of edges containing a vertex $v \in \mathcal{V}(\mathcal{H})$ is its degree $d_{\mathcal{H}}(v)$. A path of length k from a vertex x_1 to another vertex x_2 in a hypergraph $\mathcal{H}$ is a finite sequence of the form $x_1, E_1, y_1, E_2, y_2, \ldots, E_{k-1}, y_{k-1}, E_k, x_2$ such that $x_1 \in E_1$ and $y_i \in E_i \cap E_{i+1}$ for $i = 1, 2, \ldots, k-1$ and $x_2 \in E_k$. Let $\mathcal{H}$ be a connected hypergraph. For $u, v \in \mathcal{H}(v)$, the distance between u and v is the length of a shortest path from u and v in $\mathcal{H}$, denoted by $d_{\mathcal{H}}(u,v)$. In particular, $d_{\mathcal{H}}(u,u) = 0$. The diameter of $\mathcal{H}$ is the maximum distance between all vertex pairs of $\mathcal{H}$ (see [9]).

The hypergraph and algebraic properties have been covered by r-Stirling numbers which were introduced by Broder [3] and were counted as restricted partitions with the restriction being that the first r elements must be in distinct subsets. Although let X be a finite set with n elements, then the partitions of X which contain exactly k blocks are called k-partitions of X. The numbers of k-partitions are denoted by $\left\{ {n \atop k} \right\}$ which are known as the regular Stirling numbers of the second kind with some special values of them as $\left\{ {n \atop 2} \right\} = 2^{n-1} - 1$, and $\left\{ {n \atop 3} \right\} = \frac{1}{2}(3^{n-1} - 2^n + 1)$ see [5].

Our aim in this paper is to determine some basic graphical properties of a k-annihilating ideal hypergraph of a ring R, such as the order and size of $\mathcal{AG}_k(R)$, and to explain the degree of any nontrivial ideal of a ring R containing in $\mathcal{A}(R,k)$ in Section 2. In Section 3, we find the diameter of $\mathcal{AG}_k(R)$, which differs from [9, Theorem 3.5], that is, $diam(\mathcal{AG}_k(R)) \leq 2$. In addition, we discover the minimal dominating set of $\mathcal{AG}_k(R)$ of a ring R.

2. Basic properties of a k-annihilating ideal hypergraph of a ring R

Recall that $\mathcal{A}(R,k)$ is a set of all k-annihilating ideals in R, where k is an integer; as a vertex set, and $R \cong F_1 \times F_2 \times \cdots \times F_n$, where F_i are finite fields for $1 \leq i \leq n$. In this section, we obtain the properties of a set of vertices $\mathcal{A}(R,k)$, and relate it to a k-annihilating ideal hypergraph of a ring R. Also we get more detailed explanation of the degree of every vertex of $\mathcal{A}(R,k)$. Furthermore, we find the order and size of $\mathcal{AG}_k(R)$.

Firstly, we instigate this section by getting the property of minimal ideals of R.

Lemma 2.1. *Let R be a ring and let I_i be a minimal ideal of R. Then I_i is not contain in $\mathcal{A}(R,k)$.*

Proof. Suppose that I_i is a k-annihilating ideal of R. Then there are k-annihilating ideals as $I_1, I_2, \ldots, I_k$ such that the product of any $(k-1)$ ideals of $\{I_1, I_2, \ldots, I_k\}$ is nonzero and $\prod_{n=1}^{k} I_n = (0)$. Since I_i is a minimal ideal of R, then for some j different from i; $(0) \neq I_i \cdot I_j \subset I_i$ implies that $I_i \cdot I_j = I_i$, that is contradiction; so I_i is not contain in $\mathcal{A}(R,k)$. □

Theorem 2.2. *Let R be a ring such that $R \cong F_1 \times F_2 \times \cdots \times F_n$, where F_i are finite fields for $1 \leq i \leq n$ and let $J_{n-2} = (F_1 \times F_2 \times 0 \times 0 \times \cdots \times 0)$ be any ideal of a ring R with randomly s, where $2 \leq s \leq n-1$. Then* $\deg(J_{n-2}) = \left\{ {n+1 \atop 1+3} \right\}_3$, *where $\left\{ {n+1 \atop 1+3} \right\}_3$ is 3-Stirling number of the second kind.*

Proof. Let R be a ring such that $R \cong F_1 \times F_2 \times \cdots \times F_n$, where F_i are finite fields for $1 \leq i \leq n$ and let J_{n-2} be an ideal containing in $\mathcal{A}(R,k)$ such that F_i's are all zeros except F_1 and F_2, then the combination of zeros F_i are given by $\sum_{k=1}^{n-2} \binom{n-2}{k}$ and every combination in this expansion has some exception, so we can explained it by fixing $F_1 \neq 0$ and $F_2 = 0$ for any ideal in $\mathcal{A}(R,k)$ such L_1 with combination of zeros F_i, and fixing $F_2 \neq 0$ and $F_1 = 0$ for another ideal in $\mathcal{A}(R,k)$, as L_2 with combination of zeros F_i, therefore we conclude the following general form for J_{n-2} which possess randomly s, where $2 \leq s \leq n-1$, as

$$\begin{aligned}
(1) \qquad \deg(J_{n-2}) = &\binom{n-2}{1}\left(\sum_{k=1}^{n-2}\binom{n-2}{k} - \sum_{k=1}^{n-3}\binom{n-3}{k}\right) \\
&+ \binom{n-2}{2}\left(\sum_{k=1}^{n-2}\binom{n-2}{k} - \sum_{k=1}^{n-4}\binom{n-4}{k}\right) + \cdots \\
&+ \binom{n-2}{n-2}\left(\sum_{k=1}^{n-2}\binom{n-2}{k} - \sum_{k=1}^{n-(n-1)}\binom{n-(n-1)}{k}\right).
\end{aligned}$$

Observe the terms of $\sum_{k=1}^{n-2} \binom{n-2}{k}$ and simplified expansion (1) by using binomial coefficients

$$\begin{aligned}
\deg(J_{n-2}) = &\binom{n-2}{1}\Big((2^{n-2}-1) - (2^{n-3}-1)\Big) + \binom{n-2}{2} \\
&\Big((2^{n-2}-1) - (2^{n-4}-1)\Big) + \cdots
\end{aligned}$$

$$+\binom{n-2}{n-2}\left((2^{n-2}-1)-(2^{2-1}-1)\right),$$

$$\deg(J_{n-2}) = \binom{n-2}{1}\left(\left\{{n-1 \atop 2}\right\}-\left\{{n-2 \atop 2}\right\}\right)+\binom{n-2}{2}$$

$$\left(\left\{{n-1 \atop 2}\right\}-\left\{{n-3 \atop 2}\right\}\right)+\cdots+\binom{n-2}{n-2}\left(\left\{{n-1 \atop 2}\right\}-\left\{{2 \atop 2}\right\}\right),$$

$$\begin{aligned}
\deg(J_{n-2}) &= \sum_{k=1}^{n-2}\binom{n-2}{1}\left(\left\{{n-1 \atop 2}\right\}-\left\{{n-k-1 \atop 2}\right\}\right)\\
&= \sum_{k=1}^{n-2}\binom{n-2}{1}\left\{{n-1 \atop 2}\right\}-\sum_{k=1}^{n-2}\binom{n-2}{1}\left\{{n-k-1 \atop 2}\right\}\\
&= (2^{n-2}-1)(2^{n-2}-1)-\sum_{k=1}^{n-2}\binom{n-2}{n-k-2}(2^{n-k-2}-1)\\
&= (2^{n-2}-1)^2-\left(\sum_{k=0}^{n-2}\binom{n-2}{n-k-2}(2^{n-k-2}-1)-(2^{n-2}-1)\right)\\
&= (2^{n-2}-1)^2-\left(\sum_{k=0}^{n-2}\binom{n-2}{n-k-2}2^{n-k-2}-\sum_{k=0}^{n-2}\binom{n-2}{n-k-2}\right)\\
&\quad + (2^{n-2}-1)\\
&= 2^{2(n-2)}-2\cdot 2^{n-2}+1-3^{n-2}+2^{n-2}+2^{n-2}-1\\
&= 2^{2(n-2)}-3^{n-2},
\end{aligned}$$

$$\deg(J_{n-2}) = 4^{n-2}-3^{n-2} = (1+3)^{(n+1)-3}-3^{(n+1)-3}. \tag{2}$$

That is, $\deg(J_{n-2}) = \left\{{n+1 \atop 1+3}\right\}_3$, where $\left\{{n+1 \atop 1+3}\right\}_3$ is 3-Stirling number of the second kind. $\square$

Theorem 2.3. *Let R be a ring such that $R \cong F_1 \times F_2 \times \cdots \times F_n$, where F_i are finite fields for $1 \le i \le n$ and let $J_s = (F_1 \times F_2 \times \cdots \times F_s \times 0)$ be any ideal of a ring R with randomly s, where $2 \le s \le n-1$. Then $\deg(J_s) = \left\{{s+1 \atop 3}\right\}$, where $\left\{{s+1 \atop 3}\right\}$ is regular Stirling number of the second kind.*

Proof. Let R be a ring such that $R \cong F_1 \times F_2 \times \cdots \times F_n$, where F_i are finite fields for $1 \le i \le n$ and let $J_s = (F_1 \times F_2 \times \cdots \times F_s \times 0)$ be any ideal containing in $\mathcal{A}(R,k)$ such that F_1 and F_2 and $\ldots$ and F_s are nonzeros for $2 \le s \le n-1$. Suppose J_s, L_1 and L_2 are ideals of a ring R containing in $\mathcal{A}(R,k)$ such that $J_s \cdot L_1 \ne (0)$, $J_s \cdot L_2 \ne (0)$ and $L_1 \cdot L_2 \ne (0)$, so $F_1 \times F_2 \times \cdots \times F_s$ is not equal to zero in L_1 and L_2. Now, to obtain $J_s \cdot L_1 \cdot L_2 \ne (0)$, let $F_s \ne 0$ in L_1 for $2 \le s \le n-1$, then F_s must be zeros in L_2. For getting $F_1 \ne 0$ in L_1, then

$F_1 = 0$ in L_2 so, we can explain the number of cases of J_s as

$$\begin{aligned}\sum_{j=0}^{s-2}\binom{s-1}{j}\left(\sum_{k=1}^{s-j-1}\binom{s-j-1}{k}\right) &= \sum_{j=0}^{s-2}\binom{s-1}{j}(2^{s-j-1}-1)\\ &= \sum_{j=0}^{s-2}\binom{s-1}{s-j-1}(2^{s-j-1}-1)\\ &= \sum_{j=1}^{s-1}\binom{s-1}{j}(2^{j}-1)\\ &= \sum_{j=0}^{s-1}\binom{s-1}{j}(2^{j}-1)\\ &= \sum_{j=0}^{s-1}\binom{s-1}{j}2^{j}-\sum_{j=0}^{s-1}\binom{s-1}{j},\end{aligned}$$

$$\sum_{j=0}^{s-2}\binom{s-1}{j}\left(\sum_{k=1}^{s-j-1}\binom{s-j-1}{k}\right) = 3^{s-1}-2^{s-1}. \tag{3}$$

Also, fixed $F_1 = 0$ and $F_2 \neq 0$ in L_1, then $F_1 = F_2 = 0$ in L_2, we get

$$\begin{aligned}\sum_{j=0}^{s-3}\binom{s-2}{j}\left(\sum_{k=1}^{s-j-2}\binom{s-j-2}{k}\right) &= \sum_{j=0}^{s-3}\binom{s-2}{j}(2^{s-j-2}-1)\\ &= \sum_{j=0}^{s-3}\binom{s-2}{s-j-2}(2^{s-j-2}-1)\\ &= \sum_{j=1}^{s-3}\binom{s-2}{j}(2^{j}-1)\\ &= \sum_{j=0}^{s-2}\binom{s-2}{j}(2^{j}-1)\\ &= \sum_{j=0}^{s-2}\binom{s-2}{j}2^{j}-\sum_{j=0}^{s-2}\binom{s-2}{j},\end{aligned}$$

$$\sum_{j=0}^{s-3}\binom{s-2}{j}\left(\sum_{k=1}^{s-j-2}\binom{s-j-2}{k}\right) = 3^{s-2}-2^{s-2}. \tag{4}$$

By continuing this process and using (3) and (4), for $1 \leq t \leq s-1$, we have the degree of J_s as

$$\begin{aligned}
\deg(J_s) = & \sum_{j=0}^{s-2} \binom{s-1}{j}(2^{s-j-1}-1) + \sum_{j=0}^{s-3} \binom{s-2}{j}(2^{s-j-2}-1) \\
& + \cdots + \sum_{j=0}^{s-i} \binom{s-i}{j}(2^{s-j-i}-1) \\
= & \, 3^{s-1} - 2^{s-1} + 3^{s-2} - 2^{s-2} + \cdots + 3^1 - 2^1,
\end{aligned}$$

(5) $$\deg(J_s) = \sum_{t=1}^{s-1} \left(\sum_{j=0}^{s-t-1} \binom{s-t}{j}(2^{s-j-t}-1) \right),$$

$$\begin{aligned}
\deg(J_s) &= \sum_{t=1}^{s-1}(3^{s-t} - 2^{s-t}) \\
&= \sum_{t=1}^{s-1} 3^{s-t} - \sum_{t=1}^{s-1} 2^{s-t} \\
&= \frac{1}{2}(3^s - 3) - (2^s - 2) \\
&= \frac{1}{2}(3^s - 2^{s+1} + 1) \\
&= \left\{ {s+1 \atop 3} \right\},
\end{aligned}$$

where $\left\{ {s+1 \atop 3} \right\}$ is regular Stirling number of second kind. □

Theorem 2.4. *Let R be a ring such that $R \cong F_1 \times F_2 \times \cdots \times F_n$, where F_i are finite fields for $1 \leq i \leq n$ and let $J_{n-s} = (F_1 \times F_2 \times \cdots \times F_s \times 0 \times 0 \times \cdots \times 0)$ be any ideal of a ring R with randomly s, where $2 \leq s \leq n-1$. Then* $\deg(J_{n-s}) = \left\{ {s+1 \atop 3} \right\} \left\{ {n-s+3 \atop 1+3} \right\}_3$.

Proof. Let R be a ring such that $R \cong F_1 \times F_2 \times \cdots \times F_n$, where F_i are finite fields for $1 \leq i \leq n$ and let $J_{n-s} = (F_1 \times F_2 \times \cdots \times F_s \times 0 \times 0 \times \cdots \times 0)$ be any ideal of R containing in $\mathcal{A}(R,k)$ for $2 \leq s \leq n-1$, by depending on expansion (1) and (5), we can conclude the general formula for any ideal as a form, for randomly s; $J_{n-s} = (F_1 \times F_2 \times \cdots \times F_s \times 0 \times 0 \times \cdots \times 0)$ without loss the generality. So the proof is complete from Theorem 2.2 and Theorem 2.3. □

Theorem 2.5. *Let R be a ring such that $R \cong F_1 \times F_2 \times \cdots \times F_n$, where F_i are finite fields for $1 \leq i \leq n$. Then $n(\mathcal{AG}_k(R)) = \sum_{i=2}^{n-1} \binom{n}{i} = \left\{ {n+1 \atop 2} \right\} - (n+1)$, where $\left\{ {n+1 \atop 2} \right\}$ is regular Stirling number of the second kind.*

Proof. Let R be a ring such that $R \cong F_1 \times F_2 \times \cdots \times F_n$, where F_i are finite fields for $1 \leq i \leq n$. Let $J_1, J_2, \ldots, J_{n-1}$ are nontrivial distinct ideals in $\mathcal{A}(R,k)$. Since $R \cong F_1 \times F_2 \times \cdots \times F_n$ for F_i are fields and for $1 \leq i \leq n$, then $n(J_{n-1}) = \binom{n}{1}$ and $n(J_{n-2}) = \binom{n}{2}$ and $\ldots$ and $n(J_s) = \binom{n}{n-1}$. Since the set of ideals of the form J_1 are minimal by Lemma 2.1, then the order of $\mathcal{AG}_k(R)$ is expansion of $\binom{n}{s}$, where $2 \leq s \leq n-1$. That is,

$$n(\mathcal{AG}_k(R)) = \binom{n}{2} + \cdots + \binom{n}{n-1} = \sum_{s=2}^{n-1} \binom{n}{s}. \tag{6}$$

By using regular Stirling number of the second kind, we obtain

$$\begin{aligned} n(\mathcal{AG}_k(R)) &= 2^n - n - 2 \\ &= (2^n - 1) - (n+1) \\ &= \left\{ {n+1 \atop 2} \right\} - (n+1), \end{aligned}$$

where $\left\{ {n+1 \atop 2} \right\}$ is regular Stirling number of the second kind. $\square$

Theorem 2.6. *Let R be a ring such that $R \cong F_1 \times F_2 \times \cdots \times F_n$, where F_i are finite fields for $1 \leq i \leq n$. Then the size of hypergraph $\mathcal{AG}_k(R)$ is equal to $m(\mathcal{AG}_k(R)) = \frac{1}{3}(\sum_{s=2}^{n-1} \binom{n}{s} \left\{ {s+1 \atop 3} \right\} \left\{ {n-s+3 \atop 1+3} \right\}_3)$.*

Proof. Let $\mathcal{A}(R,k)$ be the set of all nontrivial k-annihilating ideals of R and not minimal ideals, since the order of hypergraph of $\mathcal{AG}_k(R) = \sum_{s=2}^{n-1} \binom{n}{s}$, since $k = 3$, then there are three nontrivial ideals in $\mathcal{A}(R,k)$ contained in every edges of $\mathcal{AG}_k(R)$, that is,

$$\begin{aligned} &m(\mathcal{E}(\mathcal{AG}_k(R))) \\ &= \frac{1}{3}\left(\binom{n}{2} \deg(J_{n-2}) + \binom{n}{3} \deg(J_{n-3}) + \cdots + \binom{n}{n-1} \deg(J_s) \right). \end{aligned}$$

So, we can conclude the following

$$m(\mathcal{E}(\mathcal{AG}_k(R))) = \frac{1}{3}\left(\sum_{s=2}^{n-1} \binom{n}{s} \deg(J_{n-s}) \right). \tag{7}$$

For more interpretation, we can rewrite (7) by using Theorem 2.4, as

$$\begin{aligned} m(\mathcal{E}(\mathcal{AG}_k(R))) = \frac{1}{3}\Bigg(&\binom{n}{2} \left\{ {2+1 \atop 3} \right\} \left\{ {n-2+3 \atop 1+3} \right\}_3 + \binom{n}{3} \left\{ {3+1 \atop 3} \right\} \left\{ {n-3+3 \atop 1+3} \right\}_3 \\ &+ \cdots + \binom{n}{n-1} \left\{ {n \atop 3} \right\} \left\{ {n-(n-1)+3 \atop 1+3} \right\}_3 \Bigg) \end{aligned}$$

$$= \frac{1}{3}\left(\sum_{s=2}^{n-1}\binom{n}{s}\left\{\begin{matrix} s+1 \\ 3 \end{matrix}\right\}\left\{\begin{matrix} n-s+3 \\ 4 \end{matrix}\right\}_3\right).$$

□

3. Adjacency property of a *k*-annihilating ideal hypergraph of ring *R*

In this section, we study the idea of adjacency between the ideals of R containing in $\mathcal{A}(R,k)$, where $k=3$, that based on its, which found the diameter and minimal dominating set of a k-annihilating ideal hypergraph of a ring R.

Theorem 3.1. *Let R be a ring such that $R \cong F_1 \times F_2 \times \cdots \times F_n$, where F_i are finite fields for $1 \leq i \leq n$ and let $J_{n-s} = (F_1 \times F_2 \times \cdots \times F_s \times 0 \times 0 \times \cdots \times 0)$ be any ideal of a ring R with randomly s, where $2 \leq s \leq n-1$. Then $diam(\mathcal{AG}_k(R)) \leq 2$.*

Proof. Let R be a ring such that $R \cong F_1 \times F_2 \times \cdots \times F_n$, where F_i are finite fields for $1 \leq i \leq n$ and let $I = (F_{i_1} \times F_{i_2} \times \cdots \times F_{i_s} \times 0 \times 0 \times \cdots \times 0)$ be any ideal of a ring R containing in $\mathcal{A}(R,k)$ and F_{i_j} be a field in position i_j. Our purpose is to prove that $diam(R) \leq 2$. It is enough to find a path between any two ideals of R in $\mathcal{AG}_k(R)$. So we must discuss these cases:

Case 1. Let I and J be any two ideals of R such that $I \subset J$ or $J \subset I$, without lost generality, let $I \subset J$, then there are two fields F_{i_1} and F_{i_2} in i_j-th position are nonzero in I and J also F_{i_3} is nonzero in J, but the i_3-th position is equal to zero in I. Furthermore; there are i_4-th position are equal to zero in I and J. Now let $K_1 = T_1 \times T_2 \times \cdots \times T_n$ be an ideal of R, where $T_i \in \{0, F_i\}$ such that

$$T_i = \begin{cases} F_i & \text{if } i = i_1, i_3 \text{ or } i_4, \\ 0 & \text{if otherwise} \end{cases}$$

and $K_2 = T_1 \times T_2 \times \cdots \times T_n$ be an ideal of R such that

$$T_i = \begin{cases} F_i & \text{if } i = i_2 \text{ or } i_4, \\ 0 & \text{if otherwise.} \end{cases}$$

That is, $I \cdot K_1 \cdot K_2 = (0)$ with $I \cdot K_1 \neq (0)$, $I \cdot K_2 \neq (0)$, $K_1 \cdot K_2 \neq (0)$ and $J \cdot K_1 \cdot K_2 = (0)$ with $J \cdot K_1 \neq (0)$, $J \cdot K_2 \neq (0)$, $K_1 \cdot K_2 \neq (0)$. So we conclude that; there are two hyperedges in $\mathcal{AG}_k(R)$ known as $e_1 = \{I, K_1, K_2\}$ and $e_2 = \{J, K_1, K_2\}$. Then $diam(I,J) = 2$.

Case 2. Again, let I and J be any two ideals of R such that $I \cdot J = (0)$. Then F_{i_1} and F_{i_2} be two fields in i_j-th position are nonzero in I, and F_{i_3} and F_{i_4} be two fields in i_j-th position are nonzero in J. Also the i_3-th position and i_4-th position are equal to zero in I, but i_1-th position and i_2-th position are equal to zero in J.

Now let $L_1 = T_1 \times T_2 \times \cdots \times T_n$ be an ideal of R, where $T_i \in \{0, F_i\}$ such that

$$T_i = \begin{cases} F_i & \text{if } i = i_1, i_3 \text{ or } i_4, \\ 0 & \text{if otherwise} \end{cases}$$

and $L_2 = T_1 \times T_2 \times \cdots \times T_n$ be an ideal of R such that

$$T_i = \begin{cases} F_i & \text{if } i = i_2, i = i_3 \text{ or } i_4, \\ 0 & \text{if otherwise,} \end{cases}$$

also let $L_3 = T_1 \times T_2 \times \cdots \times T_n$ be an ideal of R, such that

$$T_i = \begin{cases} F_i & \text{if } i = i_1, i_2 \text{ or } i_4, \\ 0 & \text{if otherwise.} \end{cases}$$

Therefore, $I \cdot L_1 \cdot L_2 = (0)$ with $I \cdot L_1 \neq (0)$, $I \cdot L_2 \neq (0)$, $L_1 \cdot L_2 \neq (0)$ and $J \cdot L_1 \cdot L_3 = (0)$ with $J \cdot L_1 \neq (0)$, $J \cdot L_3 \neq (0)$, $L_1 \cdot L_3 \neq (0)$, that is, there are two hyperedges in $\mathcal{E}(\mathcal{AG}_k(R))$ known as $e_1 = \{I, L_1, L_2\}$ and $e_2 = \{J, L_1, L_3\}$, which are explained that $diam(I, J) = 2$.

Case 3. At least, let I and J be any two ideals of R such that $I \cdot J \neq (0)$ and $I \not\subset J$, and $J \not\subset I$. Then there are F_{i_1} and F_{i_2} are two fields in i_j-th position are nonzero in I and F_{i_1}, F_{i_3} are two fields in i_j-th position are nonzero in J. Also the i_3-th position is equal to zero in I, with i_2-th position is equal to zero in J.

Now let $N = T_1 \times T_2 \times \cdots \times T_n$ be an ideal of R, where $T_i \in \{0, F_i\}$ such that

$$T_i = \begin{cases} F_i & \text{if } i = i_2 \text{ or } i_3, \\ 0 & \text{if otherwise} \end{cases}$$

since $I \cdot J \neq (0)$, by assumption we get $I \cdot N \neq (0)$, $J \cdot N \neq (0)$, so $I \cdot N \cdot J = (0)$. That is, there is exactly one hyper edge containing I and J. So $diam(I, J) = 2$. □

Corollary 3.2. *Let R be a ring and $R \cong F_1 \times F_2 \times \cdots \times F_n$, where F_i are finite fields for $1 \leq i \leq n$ and let I and J be two nontrivial ideals containing in $\mathcal{A}(R, k)$ such that $I \cdot J \neq (0)$ and $I \not\subset J$ ($J \not\subset I$). Then there is another nontrivial ideal as K in $\mathcal{A}(R, k)$, different from I and J, that is, $\{I, J, K\}$ is contained in $\mathcal{E}(\mathcal{AG}_k(R))$.*

According to [1], Acharya introduced the dominating set and minimal dominating set in hypergraphs as an extension of basic results from the theory of domination in graphs, which defined as:

Definition. Let $(\mathcal{V}(\mathcal{H}), \mathcal{E}(\mathcal{H}))$ be any hypergraph. Then, $\mathcal{D}(\mathcal{H}) \subset (\mathcal{V}(\mathcal{H})$ is an adominating set of $\mathcal{H}$ if for every $v \in \mathcal{V}(\mathcal{H}) - \mathcal{D}(\mathcal{H})$, there exists $u \in \mathcal{D}(\mathcal{H})$ such that u and v are adjacent in $\mathcal{H}$; that is, if there exists $E \in \mathcal{E}$ such that $u, v \in E$. Furthermore, $\mathcal{D}(\mathcal{H}) \subset (\mathcal{V}(\mathcal{H})$ is a minimal dominating set of $\mathcal{H}$ if $\mathcal{D}(\mathcal{H})$ does not contain proper dominating set.

Theorem 3.3. *Let R be a ring and $R \cong F_1 \times F_2 \times \cdots \times F_n$, where F_i are finite fields for $1 \leq i \leq n$. Then*

(i) *If n is even, then the dominating set of a k-annihilating hypergraph of $\mathcal{AG}_k(R)$ is defined as $\mathcal{D}(\mathcal{AG}_k(R)) = \{D_1, D_2, \ldots, D_{\frac{n}{2}}, D_{\frac{n}{2}+1}, \}$, where $D_1 = (F_1 \times F_2 \times 0 \times \cdots \times 0), D_2 = (0 \times 0 \times F_3 \times F_4 \times 0 \times \cdots \times 0), D_{\frac{n}{2}} = (0 \times \cdots \times 0 \times F_{n-1} \times F_n), D_{\frac{n}{2}+1} = (F_1 \times 0 \times F_3 \times 0 \times F_5 \times \cdots \times F_{n-1} \times 0)$.*

(ii) *If n is odd, then the dominating set of a k-annihilating hypergraph of $\mathcal{AG}_k(R)$ is defined as $\mathcal{D}(\mathcal{AG}_k(R)) = \{D_1, D_2, \ldots, D_{\frac{n-1}{2}}, D_{\frac{n+1}{2}}, D_{\frac{n+3}{2}}\}$, where $D_1 = (F_1 \times F_2 \times 0 \times \cdots \times 0), D_2 = (0 \times 0 \times F_3 \times F_4 \times 0 \times \cdots \times 0), D_{\frac{n-1}{2}} = (0 \times \cdots \times 0 \times F_{n-2} \times F_{n-1} \times 0), D_{\frac{n+1}{2}} = (0 \times \cdots \times 0 \times F_{n-2} \times F_{n-1} \times F_n), D_{\frac{n+3}{2}} = (F_1 \times 0 \times F_3 \times 0 \times F_5 \times 0 \times \cdots \times 0 \times F_n)$.*

Proof. Let R be a ring and $R \cong F_1 \times F_2 \times \cdots \times F_n$, where F_i are finite fields for $1 \leq i \leq n$ and let I be an ideal of a ring R containing in $\mathcal{A}(R, k)$ such that $I \in \mathcal{A}(R, k) - \mathcal{D}(\mathcal{AG}_k(R))$.

(i) If n is even, we instigate to show that $\mathcal{D}(\mathcal{AG}_k(R))$ is a dominating set of $\mathcal{AG}_k(R)$.

Firstly, we observe an axiom, if $D_j \subset I$ for all $1 \leq j \leq \frac{n}{2}$, then we have $I = R$, which contradicts the assumption. Also if $I \subset D_j$, then we get I is a minimal ideal of R, so by Lemma 2.1, that any minimal ideal is not a vertex in $\mathcal{A}(R, k)$. Hence there exists D_j in $\mathcal{D}(\mathcal{AG}_k(R))$ such that $D_j \not\subset I (I \not\subset D_j)$ for some $1 \leq j \leq \frac{n}{2}$. Furthermore, if $D_{\frac{n}{2}+1} \cdot I = 0$, then there exist two nonzero even positions as F_{i_1} and F_{i_2} in I, so there is D_j in $\mathcal{D}(\mathcal{AG}_k(R))$ such that satisfied the conditions of Corollary 3.2.

Secondly, we investigate $D_{\frac{n}{2}+1}$, when $D_{\frac{n}{2}+1} \subset I$ or $I \subset D_{\frac{n}{2}+1}$.

If $D_{\frac{n}{2}+1} \subset I$, since there exists $D_j \in \mathcal{D}(\mathcal{AG}_k(R))$ such that $D_j \not\subset I (I \not\subset D_j)$ for some $1 \leq j \leq \frac{n}{2}$ and $D_j = (0 \times \cdots \times 0 \times F_{i_1} \times F_{i_1+1} \times 0)$ for a nonzero odd position as F_{i_1} and a nonzero even position as F_{i_1+1} with $D_{\frac{n}{2}+1} \subset I$, then we get $D_j \cdot I \neq 0$. Therefore, Corollary 3.2, involves that; there is another nontrivial ideal as K in $\mathcal{A}(R, k)$, different from I and D_j, that is, $\{I, D_j, K\}$ is contained in $\mathcal{E}(\mathcal{AG}_k(R))$.

If $I \subset D_{\frac{n}{2}+1}$, there are two nonzero odd positions as F_{i_1} and F_{i_3} in I. Therefore if F_{i_1} in D_j for some $1 \leq j \leq \frac{n}{2}$, then F_{i_3} is not in D_j, also F_{i_3} is in D_j but not in I, which implies that $D_j \cdot I \neq 0$ and $D_j \not\subset I (I \not\subset D_j)$, thus the conditions of Corollary 3.2, satisfied and there is another nontrivial ideal as K in $\mathcal{A}(R, k)$, different from I and D_j, that is, $\{I, D_j, K\}$ is contained in $\mathcal{E}(\mathcal{AG}_k(R))$.

Moreover, we discuss, if $D_j \subset I$ for some $1 \leq j \leq \frac{n}{2}$. Then either $D_j \cdot I = 0$ or $D_j \subset I$ for all $1 \leq j \leq \frac{n}{2}$. Thus we obtain $D_{\frac{n}{2}+1} \cdot I \neq 0$ and $D_{\frac{n}{2}+1} \not\subset I (I \not\subset D_{\frac{n}{2}+1})$. That is, there is another nontrivial ideal as K in $\mathcal{A}(R, k)$, different from I and D_j, that is, $\{I, D_{\frac{n}{2}+1}, K\}$ is contained in $\mathcal{E}(\mathcal{AG}_k(R))$.

Finally, we discuss the instances of $D_j \cdot I = 0$ for some $1 \leq j \leq \frac{n}{2} + 1$.

If $D_{\frac{n}{2}+1} \cdot I = 0$, then there are two nonzero even positions as F_{i_2} and F_{i_4} in I such that $D_j \cdot I \neq 0$ for some $1 \leq j \leq \frac{n}{2}$, and $D_j I(ID_j)$. Thus by Corollary 3.2, that is, there is another nontrivial ideal as K in $\mathcal{A}(R,k)$, different from I and D_j, that is, $\{I, D_j, K\}$ is contained in $\mathcal{E}(\mathcal{AG}_k(R))$.

If $D_j \cdot I = 0$ for some $1 \leq j \leq \frac{n}{2}$, then obviously, $D_{\frac{n}{2}+1} \not\subset I$ and so $D_{\frac{n}{2}+1} \cdot I = 0$, therefore, by last confirmation, there is D_i with $i \neq j$ such that $D_i \cdot I \neq 0$ and $D_i \not\subset I(I \not\subset D_i)$, hence by Corollary 3.2, there is another nontrivial ideal as K in $\mathcal{A}(R,k)$, different from I and D_j, that is, $\{I, D_i, K\}$ is contained in $\mathcal{E}(\mathcal{AG}_k(R))$.

Similarly, if $D_j \cdot I = 0$ for some $1 \leq j \leq \frac{n}{2}$, then, may be; $I \subset D_{\frac{n}{2}+1}$, which is explained in secondly instance, so $\{I, D_{\frac{n}{2}+1}, K\}$ is contained in $\mathcal{E}(\mathcal{AG}_k(R))$ for some K in $\mathcal{A}(R,k)$.

Again, if $D_{\frac{n}{2}+1} \cdot I \neq 0$ with $I \not\subset D_{\frac{n}{2}+1}(D_{\frac{n}{2}+1} \not\subset I)$, so there is a hyperedge as $\{I, D_{\frac{n}{2}+1}, K\}$ is contained in $\mathcal{E}(\mathcal{AG}_k(R))$. Therefore for all cases there is an ideal in $\mathcal{D}(\mathcal{AG}_k(R))$ adjacent with I.

(ii) If n is odd, we begin to show that $\mathcal{D}(\mathcal{AG}_k(R))$ is a dominating set of $\mathcal{AG}_k(R)$, in the same way that (i), then D_j is discussed for all $1 \leq j \leq \frac{n-1}{2}$, as well as $D_{\frac{n+3}{2}}$. It is sufficient to show that $D_{\frac{n+1}{2}}$ is contained in $\mathcal{D}(\mathcal{AG}_k(R))$. Now, we suppose, if $D_{\frac{n+1}{2}} \subset I$, then $I = (T_1 \times T_2 \times \cdots \times T_{n-3} \times F_{n-2} \times F_{n-1} \times F_n)$ is an ideal containing in $\mathcal{A}(R,k)$, where T_i are not all zeros such that $T_i \in \{0, F_i\}$ for $i = 1, 2, \ldots, n-3$, then there is $D_{\frac{n+3}{2}}$ in $\mathcal{D}(\mathcal{AG}_k(R))$ such that $D_{\frac{n+3}{2}} \cdot I \neq 0$ and $D_{\frac{n+3}{2}} \not\subset I(I \not\subset D_{\frac{n+3}{2}})$, so by Corollary 3.2, $\{I, D_{\frac{n+1}{2}}, K\}$ is contained in $\mathcal{E}(\mathcal{AG}_k(R))$ for some K in $\mathcal{A}(R,k)$.

Moreover, if $I \subset D_{\frac{n+1}{2}}$, then $I = (0 \times \cdots \times 0 \times F_{n-1} \times F_n)$ or $I = (0 \times \cdots \times 0 \times F_{n-2} \times 0 \times F_n)$ is contained in $\mathcal{A}(R,k)$, implying that; I is adjacent to $D_{\frac{n-1}{2}}$ that is contained in a hyperedge of $\mathcal{E}(\mathcal{AG}_k(R))$.

At last, we suppose, if $D_{\frac{n-1}{2}} \cdot I = 0$, then $I = (T_1 \times T_2 \times \cdots \times T_{n-3} \times 0 \times 0 \times 0)$, where T_i are not all zeros such that $T_i \in \{0, F_i\}$ for $i = 1, 2, \ldots, n-3$, contained in $\mathcal{A}(R,k)$, so there is D_j in $\mathcal{D}(\mathcal{AG}_k(R))$ for $1 \leq j \leq \frac{n-1}{2}$ such that $D_j \cdot I \neq 0$ and $D_j \not\subset I(I \not\subset D_j)$, so by Corollary 3.2, $\{I, D_j, K\}$ is contained in $\mathcal{E}(\mathcal{AG}_k(R))$ for some K in $\mathcal{A}(R,k)$. $\square$

Theorem 3.4. *Let R be a ring and $R \cong F_1 \times F_2 \times \cdots \times F_n$, where F_i are finite fields for $1 \leq i \leq n$. Then the dominating set $\mathcal{D}(\mathcal{AG}_k(R))$, which is defined in Theorem 3.3,* (i) *and* (ii) *is a minimal.*

Proof. Let R be a ring and let $R \cong F_1 \times F_2 \times \cdots \times F_n$, where F_i are finite fields for $1 \leq i \leq n$.

(i) Assume that, n is even, we instigate to show that

$$\mathcal{D}(\mathcal{AG}_k(R)) = \{D_1, D_2, \ldots, D_{\frac{n}{2}}, D_{\frac{n}{2}+1}\}$$

is a minimal dominating set of $\mathcal{AG}_k(R)$. Now let $\mathcal{D}(\mathcal{AG}_k(R)) - D_1$ be a dominating set of $\mathcal{AG}_k(R)$, and suppose that $I = (F_1 \times 0 \times F_3 \times F_4 \times \cdots \times F_n)$ be an ideal containing in $\mathcal{A}(R,k)$, then there is no D_j in $\mathcal{D}(\mathcal{AG}_k(R)) - D_1$ for $2 \leq j \leq \frac{n}{2}+1$, adjacent to I in a hyperedge of $\mathcal{AG}_k(R)$; such that $I \not\subset D_j (D_j \not\subset I)$ and $D_j \cdot I \neq 0$. Thus D_1 must be contained in $\mathcal{D}(\mathcal{AG}_k(R))$. To continuing for all $\mathcal{D}(\mathcal{AG}_k(R)) - D_j$, where $2 \leq j \leq \frac{n}{2}$, there is an ideal I not adjacent with every ideals of $\mathcal{D}(\mathcal{AG}_k(R)) - D_j$.

Also, if we get $\mathcal{D}(\mathcal{AG}_k(R)) - D_{\frac{n}{2}+1}$ a dominating set of $\mathcal{AG}_k(R)$, and for $I = (F_1 \times F_2 \times F_3 \times F_4 \times 0 \times \cdots \times 0)$ containing in $\mathcal{A}(R,k)$, then there is no D_j in $\mathcal{D}(\mathcal{AG}_k(R)) - D_{\frac{n}{2}+1}$ for $1 \leq j \leq \frac{n}{2}$, adjacent to I in a hyperedge of $\mathcal{AG}_k(R)$. Thus $D_{\frac{n}{2}+1}$ must be contained in $\mathcal{D}(\mathcal{AG}_k(R))$. Therefore, $\mathcal{D}(\mathcal{AG}_k(R))$ must be a minimal dominating set.

(ii) Suppose that, n is odd, according to the same method and a same assumption as in (i), we show that $\mathcal{D}(\mathcal{AG}_k(R)) = \{D_1, D_2, \ldots, D_{\frac{n-1}{2}}, D_{\frac{n+1}{2}}, D_{\frac{n+3}{2}}\}$ is minimal, so we only need to prove that $D_{\frac{n+1}{2}}$ is a minimal dominating set containing in $\mathcal{D}(\mathcal{AG}_k(R))$. Let us now continue $\mathcal{D}(\mathcal{AG}_k(R)) - D_{\frac{n+1}{2}}$ is a dominating set of $\mathcal{AG}_k(R)$, and assume that $I = (F_1 \times 0 \times F_3 \times F_4 \times \cdots \times F_n)$ be an ideal containing in $\mathcal{A}(R,k)$. Then there is no D_j in $\mathcal{D}(\mathcal{AG}_k(R)) - D_{\frac{n+1}{2}}$ for $1 \leq j \leq \frac{n+3}{2}$, adjacent to I in a hyperedge of $\mathcal{AG}_k(R)$. Thus $D_{\frac{n+1}{2}}$ must be contain in $\mathcal{D}(\mathcal{AG}_k(R))$. $\square$

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