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TERRACINI LOCI OF CODIMENSION 1 AND A CRITERION FOR PARTIALLY SYMMETRIC TENSORS

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Abstract. The Terracini t-locus of an embedded variety $X \subset \mathbb{P}^r$ is the set of all cardinality t subsets of the smooth part of X at which a certain differential drops rank, i.e., the union of the associated double points is linearly dependent. We give an easy to check criterion to exclude some sets from the Terracini loci. This criterion applies to tensors and partially symmetric tensors. We discuss the non-existence of codimension 1 Terracini t-loci when t is the generic X-rank.

1. Introduction

Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate variety defined over the complex number field $\mathbb{C}$, but the interested reader may use any algebraically closed field of characteristic zero. The case of the complex numbers is essential to get the case over $\mathbb{R}$ and our first result (Theorem 1.1) is motivated by the following problem which arises over $\mathbb{R}$ but not over $\mathbb{C}$: deleting a hypersurface from a connected real manifold M we may disconnect M and this may obstruct the use of the homotopic tools of Numerical Algebraic Geometry ([5, 6]). Over $\mathbb{C}$ deleting a hypersurface we may change the fundamental group and this is again a problem (although a minor one) for some algorithms in Numerical Algebraic Geometry. These two problems do not occur if we delete a union of higher codimension varieties of a connected smooth variety. See [10] for more on tensors, tensor rank, symmetric tensors, secant varieties and some of their applications.

Set $n := \dim X$. Let X_{reg} denote the set of all smooth points of X. For any positive integer t let $S(X_{\mathrm{reg}}, t)$ denote the set of all subsets of X_{reg} with cardinality t. The set $S(X_{\mathrm{reg}}, t)$ is a smooth quasi-projective variety of dimension tn. For any $p \in X_{\mathrm{reg}}$ let $(2p, X)$ denote the closed subscheme of X with $(\mathcal{I}_{p,X})^2$ as its ideal sheaf. The scheme $(2p, X)$ is zero-dimensional, $\deg(2p, X) = n + 1$ and $(2p, X)_{\mathrm{red}} = \{p\}$. For any finite set $S \subset X_{\mathrm{reg}}$ set $(2S, X) := \cup_{p\in S}(2p, X)$

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and $\delta(2S,X) := h^1(\mathbb{P}^r, \mathcal{I}_{(2S,X)}(1))$. The *Terracini locus* or the *open Terracini locus* or the *Terracini t-set* $\mathbb{T}_1(X,t)$ is the set of all $S \in S(X_{\mathrm{reg}},t)$ such that $h^0(\mathbb{P}^r, \mathcal{I}_{(2S,X)}(1)) > 0$ and $h^1(\mathbb{P}^r, \mathcal{I}_{(2S,X)}(1)) > 0$. The *closed Terracini locus* or *Terracini t-locus* is the closure of $\mathbb{T}_1(X,t)$ in the Hilbert scheme of X or (easier) in the t-symmetric power $X^{(t)}$ of X.

For any integer $t > 0$ the t-secant variety $\sigma_t(X) \subseteq X$ is the closure in $\mathbb{P}^r$ of the union of all linear spaces $\langle S\rangle$, where S is a subset of X with cardinality t and $\langle\ \ \rangle$ denotes the linear span. Let $r_{\mathrm{gen}}(X)$ denote the minimal integer t such that $\sigma_t(X) = \mathbb{P}^r$. The embedded variety X is said to be *secant defective* if there is a positive integer t such that $\dim \sigma_t(X) < \min\{r, t(n+1)-1\}$. Obviously, if X is secant defective, then there is an integer $t \le \lceil \frac{r+1}{n+1} \rceil$ such that $\dim \sigma_t(X) < \min\{r, t(n+1)-1\}$. If X is not secant defective, then $r_{\mathrm{gen}}(X) = \lceil (r+1)/(n+1) \rceil$ and we set $r_{\mathrm{crit}}(X) := \lceil (r+1)/(n+1) \rceil$. If X is secant defective, let $r_{\mathrm{def}}(X)$ denote the first integer $t \le \lceil (r+1)/(n+1) \rceil$ such that $\dim \sigma_t(X) \le \min\{r-1, t(n+1)-2\}$. If X is secant defective, then $\mathbb{T}_1(X,t) = S(X_{\mathrm{reg}},t)$ for all $t \ge r_{\mathrm{def}}(X)$ by the semicontinuity theorem for cohomology and the Terracini lemma ([1, Cor. 1.11]). If X is secant defective, then we set $r_{\mathrm{crit}}(X) := r_{\mathrm{def}}(X) - 1$. We have $\delta(2S,X)) > 0$ for all $t > r_{\mathrm{crit}}(X)$ and all $S \in S(X_{\mathrm{reg}},t)$ and for $t = \lceil (r+1)/(n+1) \rceil$ if X is not secant defective and $(r+1)/(n+1) \notin \mathbb{Z}$. Thus the largest t to be considered for a general $S \in S(X_{\mathrm{reg}},t)$ is $r_{\mathrm{crit}}(X) - 1$ if either X is secant defective or $(r+1)/(n+1) \notin \mathbb{Z}$ and $r_{\mathrm{crit}}(X)$ if X is not secant defective and $r+1 \equiv 0 \pmod{n+1}$. The latter is called the perfect case and it is rare. We may have $\mathbb{T}_1(X_{\mathrm{reg}},t) \ne \emptyset$ even if $t > r_{\mathrm{crit}}(X)$, but of course in this range $S \in \mathbb{T}_1(X_{\mathrm{reg}},t)$ if and only if $\langle (2S,X)\rangle \ne \mathbb{P}^r$. It is very easy to give examples of such triples (X,t,S) (Example 2.4).

Many papers studied if a general $S \in S(X_{\mathrm{reg}},t)$ is in the open Terracini t-locus of X, because by the Terracini lemma $h^0(\mathcal{I}_{(2S,X)}(1)) = r - \dim \sigma_t(X)$ for a general S ([1, Cor. 1.11], [10]). A few recent papers aim to give a finer description of the sets $\mathbb{T}_1(X_{\mathrm{reg}},t)$ ([2,3]).

By the semicontinuity theorem for cohomology $\mathbb{T}_1(X_{\mathrm{reg}},t) = S(X_{\mathrm{reg}},t)$ if and only if $\mathbb{T}_1(X_{\mathrm{reg}},t)$ contains a general $S \in S(X_{\mathrm{reg}},t)$. Thus the Terracini lemma ([1, Cor. 1.11]) gives that $\mathbb{T}_1(X_{\mathrm{reg}},t) = S(X_{\mathrm{reg}},t)$ if and only if $\sigma_t(X) \ne \mathbb{P}^r$ and $\dim \sigma_t(X) \le t(n+1)-2$. We study the next case, i.e., if $\mathbb{T}_1(X_{\mathrm{reg}},t)$ contains a hypersurface of $S(X_{\mathrm{reg}},t)$, for the first integer t for which the t-secant variety of X may fill the whole $\mathbb{P}^r$. We prove the following result.

Theorem 1.1. *Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate variety. Set $n := \dim X$ and $t := \lceil (r+1)/(n+1) \rceil$. If $n = 1$ assume X smooth.*

(a) *If X is secant defective, then $\mathbb{T}_1(X_{\mathrm{reg}},t) = S(X_{\mathrm{reg}},t)$.*

(b) *Assume X not secant defective and $r+1 \equiv 0 \pmod{n+1}$. Then*
   - (i) *$\mathbb{T}_1(X_{\mathrm{reg}},t) = \emptyset$ if and only if X is a rational normal curve.*
   - (ii) *If X is not a rational normal curve, then $\mathbb{T}_1(X_{\mathrm{reg}},t)$ contains a hypersurface of $S(X_{\mathrm{reg}},t)$.*

For arbitrary r, n and t we give an explicit criteria to check if a given $S \in S(X_{\mathrm{reg}}, t)$ is an element of $\mathbb{T}_1(X_{\mathrm{reg}}, t)$.

Theorem 1.2. *Fix a finite set $S \subset X_{\mathrm{reg}}$ and set $t := \#S$. Assume the existence of line bundles $\mathcal{L}, \mathcal{R}$ on X and linear subspaces $V \subseteq H^0(\mathcal{L})$, $W \subseteq H^0(\mathcal{R})$ such that $\mathcal{L} \otimes \mathcal{R} \cong \mathcal{O}_X(1)$ and this isomorphism sends $V \otimes W$ into the image of the restriction map $H^0(\mathcal{O}_{\mathbb{P}^r}(1)) \to H^0(\mathcal{O}_X(1))$. Assume that $(2a, X) \cup (S \setminus \{a\})$ gives $t + n$ independent conditions to V for all $a \in S$ and that S gives t independent conditions to W. Then $S \notin \mathbb{T}_1(X_{\mathrm{reg}}, t)$.*

Note that in Theorem 1.2 we do not assume that $\mathcal{L}$ or $\mathcal{R}$ is very ample, but $\mathcal{L}$ must induce an embedding at all points of S, otherwise one of the assumptions of the theorem is not satisfied.

Theorem 1.2 may be applied in specific cases. For instance the next theorem is an easy application of Theorem 1.2 to the Segre-Veronese varieties. We recall that the Segre-Veronese varieties determine the partially symmetric rank of partially symmetric tensors.

Let $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ be a multiprojective space. For any $E \subseteq \{1, \dots, k\}$, $E \neq \emptyset$, set $Y(E) := \prod_{i \in E} \mathbb{P}^{n_i}$ and let $\pi_E : Y \to Y(E)$ denote the projection.

Theorem 1.3. *Let $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ be a multiprojective space. Fix $(d_1, \dots, d_k) \in \mathbb{N}^k$ such that $d_i > 0$ for all i. Let $\nu_{d_1,\dots,d_k} : Y \to \mathbb{P}^r$, $r = -1 + \prod_{i=1}^k \binom{d_i + n_i}{n_i}$, be the Segre-Veronese embedding of Y with multidegree $(d_1, \dots, d_k)$. Set $X := \nu_{d_1,\dots,d_k}(Y)$. Fix a finite set $S \subset X$, say $S = \nu_{d_1,\dots,d_k}(A)$ with $A \subset Y$. Assume the existence of $(a_1, \dots, a_k) \in \mathbb{N}^k$ such that $0 < a_i \le d_i$ for all i, $h^1(Y, \mathcal{I}_{(2a,Y) \cup A}(a_1, \dots, a_k)) = 0$ for all $a \in A$ and $h^1(Y, \mathcal{I}_A(d_1 - a_1, \dots, d_k - a_k)) = 0$. Then $S \notin \mathbb{T}_1(X, \#A)$.*

Since in Theorem 1.3 we assume $a_i > 0$ for all i, if $\#A \ge 2$ we cannot apply it to the Segre embedding of Y. For the Segre embedding we prove a similar result (Theorem 1.4). To state it we use the following notation.

Theorem 1.4. *Let $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ be a multiprojective space and let $\nu : Y \to \mathbb{P}^r$, $r = -1 + \prod_{i=1}^k (n_i + 1)$, be the Segre embedding of Y. Set $X := \nu(Y)$. Fix a finite set $A \subset Y$. Assume the existence of $E \subseteq \{1, \dots, k\}$ with the following properties*:

(1) *$h^1(Y, \mathcal{I}_A(\{1, \dots, k\} \setminus E)) = 0$,*
(2) *$\pi_{E|A}$ is injective,*
(3) *for each $o \in A$ we have $h^1(Y(E), \mathcal{I}_{\pi_E(A) \cup (2o, Y(E))}(1, \dots, 1)) = 0$.*

Then $\nu(A) \notin \mathbb{T}_1(X, \#A)$.

2. Curves

Let $X \subset \mathbb{P}^r$, $r > 1$, be an integral and non-degenerate curve. It is well-known that X is not defective ([1, Remark 1.6]). Thus $r_{\mathrm{gen}}(X) = \lceil (r+1)/2 \rceil$ and $\mathbb{T}_1(X_{\mathrm{reg}}, 1 + r/2) = S(X_{\mathrm{reg}}, 1 + r/2)$ if r is even.

Remark 2.1. Let $X \subset \mathbb{P}^r$ be a rational normal curve. Since each line bundle on X with degree ≥ -1 has no higher cohomology, every zero dimensional scheme $Z \subset X$ is linearly independent if $\deg(Z) \leq r+1$ and spans $\mathbb{P}^r$ if $\deg(Z) \geq r+1$. Thus $\mathbb{T}_1(X,t) = 0$ if $2t \leq r+1$ and $h^0(\mathcal{I}_{(2S,X)}(1)) > 0$ for all $S \in S(X,t)$ it $2t \geq r+2$.

Remark 2.2. Let $X \subset \mathbb{P}^r$, $r \geq 4$, be an integral and non-degenerate curve. Set $d := \deg(X)$. The algebraic set $\mathbb{T}_1(X_{\mathrm{reg}},2)$ has codimension at least 2 in $S(X_{\mathrm{reg}},2)$, i.e., $\mathbb{T}_1(X_{\mathrm{reg}},2)$ is finite or empty, if and only if for a general $p \in X_{\mathrm{reg}}$ the line T_pX meets X_{reg} only at p and the linear projection from T_pX induces an unramified morphism $\gamma : X_{\mathrm{reg}} \setminus \{p\} \to \mathbb{P}^2$. The first condition is always satisfied (and even $T_pX \cap (X \setminus \{p\}) = \emptyset$) if X is smooth or at least the normalization map of X is unramified by a theorem due to H. Kaji ([8, Theorem 3.1 and Remark 3.8]) and in a few other cases, but it is not known in general ([4, 9]). In positive characteristic the first condition fails even for some smooth curves ([8, Example 4.1]). Assume $T_pX \cap X = \{p\}$ set-theoretically. Since p is general, T_pX has order of contact 2 with X at p. Since $p \in X_{\mathrm{reg}}$, the linear projection from p induces a morphism $\ell : X \to \mathbb{P}^{r-2}$ such that $\deg(\ell) \circ \deg(\ell(X)) = d-2$.

Lemma 2.3. *Assume that r is odd, that X is smooth and that X is not a rational normal curve. Fix a general $A \in S(X,(r-1)/2)$. Then*

(a) *$\langle 2p \rangle \cap \langle 2A \rangle = \emptyset$ for a general $p \in X$.*
(b) *There is $p \in X_{\mathrm{reg}} \setminus A$ such that $\langle 2p \rangle \cap \langle 2A \rangle \neq \emptyset$.*

Proof. Since $\sigma_{(r+1)/2}(X) = \mathbb{P}^r$ ([1, Remark 1.6]) part (a) follows from the Terracini lemma ([1, Corollary 1.11]) and the fact that for a general p the set $A \cup \{p\}$ is a general element of $S(X_{\mathrm{reg}},(r+1)/2)$. Now we prove part (b). Set $d := \deg(X)$. By assumption $d > r$. By Remark 2.2 the scheme $\langle A \rangle \cap X$ is equal to $(2A,X)$ and it has degree $r-1$. Let $\ell_{\langle A \rangle} : \mathbb{P}^r \setminus \langle A \rangle \to \mathbb{P}^1$ denote the linear projection from the linear space $\langle A \rangle$. Since X is a smooth curve and $\langle A \rangle \cap X \setminus A = \emptyset$, $\ell_{\langle A \rangle | X \setminus A}$ extends to a morphism $\ell : X \to \mathbb{P}^1$. Since $\langle A \rangle \cap X = (2A,X)$, $\deg(\ell) = d-r+1 \geq 2$. Since X is smooth, ℓ must be ramified. Fix a ramification point $p \in X$ of ℓ. If $p \notin A$, then $T_pX \cap \langle (2A,X) \rangle \neq \emptyset$, proving the theorem in this case by Remark 2.2. Now assume $p \in A$. We get that the osculating plane $O(X,2,p)$ of X at p is contained in $\langle (2A,X) \rangle$, contradicting [7] and the generality of A. □

Proof of Theorem 1.1 for smooth curves. No non-degenerate curve is secant defective ([1, Remark 1.6]). Remark 2.1 describes the rational normal curve.

Now assume that X is not a rational normal curve. Since $\sigma_{(r+1)/2}(X) = \mathbb{P}^r$, $\langle 2S \rangle = \mathbb{P}^r$ for a general $S \in S(X_{\mathrm{reg}},(r+1)/2)$, $\mathbb{T}_1(X,(r+1)/2)$ is a proper closed subset of $S(X,(r+1)/2)$. Lemma 2.3 says that for a general $A \subset X_{\mathrm{reg}}$ with cardinality $(r-1)/2$ there is $p \in X_{\mathrm{reg}} \setminus A$ such that $\langle 2p \rangle \cap \langle 2A \rangle \neq \emptyset$. Hence $\mathbb{T}_1(X,(r+1)/2) \neq \emptyset$ and $\mathbb{T}_1(X,(r+1)/2)$ has codimension 1 in $S(X_{\mathrm{reg}},(r+1)/2)$. □

Example 2.4. Fix integers $e \geq 1$, $r \geq \max\{3, e+1\}$ and t such that $2t \geq e+2$. Set $d := r + 2t - e - 1$. Let $C \subset \mathbb{P}^d$ be a degree d rational normal curve. Fix $E \subset C$ such that $\#E = t$. Since C is a rational normal curve and $2t \leq d - 1$, the scheme $(2E, C)$ is linearly independent, i.e., $U := \langle (2E, C) \rangle$ has dimension $2t - 1$. Since C is a rational normal curve, $(2E, C)$ is the scheme-theoretic intersection of C and U. Let $V \subset U$ be a general linear subspace of dimension $d - r - 1 = 2t - e - 2$. Since V is general in U and it has codimension at least 2 in U, V meets no line tangent to C at one of the points of E and no line spanned by 2 of the points of E. Let $\ell_V : \mathbb{P}^d \setminus V \to \mathbb{P}^r$ denote the linear projection from V. Since $V \cap C = \emptyset$, $\ell_{V|C} : C \to \mathbb{P}^r$ is a morphism and $\ell_V(C)$ is a non-degenerate curve. We claim that V is an embedding. It is sufficient to prove that $V \cap \sigma_2(X) = \emptyset$. Since V is a general subspace of U of codimension at least 2, it is sufficient to prove that $\dim U \cap \sigma_2(X) \leq 1$. Indeed, for any $o \in C \setminus E$ the scheme $2E \cup \{o\}$ is linearly independent because it has degree $2t + 1 \leq d - 1$. Set $W := \ell_V(U \setminus V)$. W is a dimension e linear subspace spanned by t tangent lines of the smooth curve $X := \ell_V(C)$. If $t > (r+1)/2$, then $t > r_{\mathrm{gen}}(X)$.

3. Proof of Theorem 1.1

Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate n-dimensional variety. Let $G(n+1, r+1)$ denote the Grassmanniann of all n-dimensional linear subspaces of $\mathbb{P}^r$. Let $\Phi \subset G(n+1, r+1)$ be the set of all tangent spaces T_pX, $p \in X_{\mathrm{reg}}$. Let Ψ be the closure of Φ in $G(n+1, r+1)$. The algebraic sets Φ and Ψ are irreducible and $\dim \Phi = \dim \Psi$.

Remark 3.1. We have $\dim \Phi < n$ if and only if a general T_pX is tangent to X_{reg} at infinitely many points. Since we are in characteristic zero, the contact locus of a general T_pX is a linear space containing p. If X is smooth, then $\dim \Phi = n$ because the Gaussian map of a smooth $X \subset \mathbb{P}^r$ is finite ([11, Corollary I.2.8, Theorem 2.3]).

Proof of Theorem 1.1. Since we proved the case $n = 1$, we may assume $n \geq 2$. Take a general $A \in S(X_{\mathrm{reg}}, t-1)$ and set $L := \langle 2A \rangle$. Since X is not secant defective, L has codimension $n + 1$. Let $W \subset G(n+1, r+1)$ denote the set of all n-dimensional linear subspaces $M \subset \mathbb{P}^r$ such that $W \cap L \neq \emptyset$. The Schubert cycle W is a hypersurface of $G(n+1, r+1)$ and $\mathrm{Pic}(G(n+1, r+1)) = \mathbb{Z}\mathcal{O}_{G(n+1,r+1)}(W)$. Since X is not secant defective, Φ is not contained in W and hence Ψ is not contained in W. Since $\mathcal{O}_{G(n+1,r+1)}$ is very ample and Ψ is not contained in W, $W \cap \Psi \neq \emptyset$ and $\dim(W \cap \Psi) = \dim W - 1$. In particular if X is smooth and hence $\dim \Psi \geq 2$ ([11, Corollary I.2.8]), then $A \cup \{p\} \in \mathbb{T}_1(X_{\mathrm{reg}}, t)$ for an $(n-1)$-dimensional family of $p \in X_{\mathrm{reg}} \setminus A$. Thus varying A we get $\dim \mathbb{T}_1(X_{\mathrm{reg}}, t) = nt - 1$ if $n \geq 2$, X is smooth (and not secant defective).

With no smoothness assumption on X we have $\dim \Psi = 1$ only for very particular varieties X (e.g. for $n = 2$ only cones and developable surfaces) and all of them are ruled by $(n-1)$-dimensional linear subspaces. Call $\gamma : X_{\mathrm{reg}} \to$

$G(n+1,r+1)$ the Gaussian map. Fix $a \in A$. Since the closure of $\gamma^{-1}(\gamma(a))$ is an $(n-1)$-dimensional linear space $\delta(2(A \cup \{p\})) > 0$ for any $p \in \gamma_X^{-1}(\gamma(a))$, concluding the proof in this case.

Thus we may assume $\dim \Psi \geq 2$. Since $A \subseteq \Phi \cap L$, $\Phi \cap L \neq \emptyset$. Varying A we see that to conclude the proof it is sufficient to use that $\Phi \cap L \neq \emptyset$, because $\dim \Phi \cap L = n-1$ and hence (by the assumption $n \geq 2$) $\Phi \cap L$ is strictly contained in A. □

4. Proofs of Theorems 1.2, 1.3 and 1.4

For any projective variety W, any effective divisor $D \subset X$ and any zero-dimensional scheme $Z \subset W$ the *residual scheme* $\mathrm{Res}_D(Z)$ of Z with respect to D is the closed subsheme of W with $\mathcal{I}_Z : \mathcal{I}_D$ as its ideal sheaf. We have $\mathrm{Res}_D(Z) \subseteq Z$ (and hence $\mathrm{Res}_D(Z)$ is a zero-dimensional scheme) and $\deg(Z) = \deg(Z \cap D) + \deg(\mathrm{Res}_D(Z))$. For any line bundle $\mathcal{L}$ on W the following sequence (often called the *residual exact sequence of* D)

$$0 \to \mathcal{I}_{\mathrm{Res}_D(Z)} \otimes \mathcal{L}(-D) \to \mathcal{I}_Z \otimes \mathcal{L} \to \mathcal{I}_{Z \cap D, D} \otimes \mathcal{L}_{|D} \to 0$$

is exact. If A, B are zero-dimensional subschemes of W and $A \cap B = \emptyset$, then $\mathrm{Res}_D(A \cup B) = \mathrm{Res}_D(A) \cup \mathrm{Res}_D(B)$.

Remark 4.1. Fix $o \in W_{\mathrm{reg}} \cap D$. If $o \in D_{\mathrm{reg}}$, then $(2o, W) \cap D = (2o, D)$ and $\mathrm{Res}_D((2o, W)) = \{o\}$. If $o \in \mathrm{Sing}(D)$, then $(2o, W) \subset D$ and $\mathrm{Res}_D((2o, W)) = \emptyset$.

Proof of Theorem 1.2. It is sufficient to prove that $(2S, X)$ is linearly independent. Since $(S \setminus \{a\}) \cup (2a, X)$ gives $t+n$ independent conditions to V, $\dim V \geq t+n$. Take a general $f \in V$ such that $f(a) = 0$ for all $a \in S$ and set $G := \{f = 0\} \in |V|$. The set G is an effective Cartier divisor of X containing S. Fix $a \in S$. Since $(2a, X) \cup S$ gives independent conditions to V and f is general, G is smooth at all points of S. Thus $\mathrm{Res}_G((2S, X)) = S$ (Remark 4.1). Since S gives t independent conditions to W and $V \otimes W$ goes into the image of the restriction map $H^0(\mathcal{O}_{\mathbb{P}^r}(1)) \to H^0(\mathcal{O}_X(1))$, the residual exact sequence of G gives that $(2S, X)$ is linearly independent. □

Proof of Theorem 1.3. This result is a very particular case of Theorem 1.2 with as X the image of Y and, up to the identification of X and Y, $\mathcal{L} = \mathcal{O}_Y(a_1, \dots, a_k)$, $V = H^0(\mathcal{L})$, $\mathcal{R} = \mathcal{O}_Y(d_1 - a_1, \dots, d_k - a_k)$ and $W = H^0(\mathcal{R})$. □

Proof of Theorem 1.4. We follow the proofs of Theorems 1.2 and 1.3. Since $E \neq \{1, \dots, k\}$, $h^1(Y, \mathcal{I}_{S \cup (2o, Y)}(E)) > 0$. To carry over the proof of Theorem 1.2 we only need to prove that a general $G \in |\mathcal{I}_A(E)|$ is smooth at all points of A. Since A is finite and the projective space $|\mathcal{I}_A(E)|$ is an irreducible variety, it is sufficient to prove that for every $o \in A$ the hypersurface G is smooth at o. Note that $G \cong G' \times Y(F)$ with G' a general element of $\mathbb{P}H^0(Y(E), \mathcal{I}_{\pi_E(A)\cup(2o,Y(E))}(1,\dots,1))$. Since G' is smooth at $\pi_E(o)$, G is smooth at o. $\square$

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