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ON THE 2-ABSORBING SUBMODULES AND ZERO-DIVISOR GRAPH OF EQUIVALENCE CLASSES OF ZERO DIVISORS

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Abstract. Let R be a commutative ring, M be a Noetherian R-module, and N a 2-absorbing submodule of M such that $r(N :_R M) = \mathfrak{p}$ is a prime ideal of R. The main result of the paper states that if $N = Q_1 \cap \cdots \cap Q_n$ with $r(Q_i :_R M) = \mathfrak{p}_i$, for $i = 1, \ldots, n$, is a minimal primary decomposition of N, then the following statements are true.

(i) $\mathfrak{p} = \mathfrak{p}_k$ for some $1 \leq k \leq n$.

(ii) For each $j = 1, \ldots, n$ there exists $m_j \in M$ such that $\mathfrak{p}_j = (N :_R m_j)$.

(iii) For each $i, j = 1, \ldots, n$ either $\mathfrak{p}_i \subseteq \mathfrak{p}_j$ or $\mathfrak{p}_j \subseteq \mathfrak{p}_i$.

Let $\Gamma_E(M)$ denote the zero-divisor graph of equivalence classes of zero divisors of M. It is shown that $\{Q_1 \cap \cdots \cap Q_{n-1}, Q_1 \cap \cdots \cap Q_{n-2}, \ldots, Q_1\}$ is an independent subset of $V(\Gamma_E(M))$, whenever the zero submodule of M is a 2-absorbing submodule and $Q_1 \cap \cdots \cap Q_n = 0$ is its minimal primary decomposition. Furthermore, it is proved that $\Gamma_E(M)[(0 :_R M)]$, the induced subgraph of $\Gamma_E(M)$ by $(0 :_R M)$, is complete.

1. Introduction

Let R be a commutative ring. A proper ideal I of R is called a 2-absorbing ideal if whenever $abc \in I$ for $a, b, c \in R$, then $ab \in I$ or $bc \in I$ or $ac \in I$. The concept of 2-absorbing ideals was introduced and studied in [3]. The basic properties of the set $\mathcal{A} = \{\mathrm{Ann}_R(x + I) : I$ is a 2-absorbing ideal of R and $x \in R\}$ have been studied in [11], and in that paper it is shown $\mathrm{Ann}_R(x + I)$ is a prime or is a 2-absorbing ideal of R, and $\mathcal{A}$ is a totally ordered set or is union of two totally ordered sets. After that, the concept of 2-absorbing submodule was introduced in [10]. A proper submodule N of an R-module M is called a 2-absorbing submodule if whenever $abm \in N$ for $a, b \in R$ and $m \in M$, then $am \in N$ or $bm \in N$ or $ab \in (N :_R M)$.

The zero-divisor graph of equivalence classes of zero divisors in a commutative ring was introduced and investigated in [7, 14]. This kind of graph has some advantages comparing to the zero-divisor graph discussed in [2, 4]. In many cases, the zero-divisor graph of equivalence classes of zero divisors in

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a commutative ring is finite when the zero-divisor graph is infinite. Another important aspect of zero-divisor graph of equivalence classes of zero divisors is the connection to associated primes of the ring.

In Section 2, for a 2-absorbing submodule N of M with a primary decomposition $N = Q_1 \cap \cdots \cap Q_n$ with $r(Q_i :_R M) = \mathfrak{p}_i$ for $i = 1, \ldots, n$ it is shown that the set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ is a totally ordered set or is union of two totally ordered sets. Furthermore, it is shown that if N is a 2-absorbing submodule of M such that $r(N :_R M) = \mathfrak{p}$ is a prime ideal of R, then $\{(N :_M a) : a \in R \setminus \mathfrak{p}\} = \{N = \cap_{i=1}^{n} Q_i, \cap_{i=1}^{n-1} Q_i, \ldots, Q_1\}$ is a totally ordered set. Let the zero submodule of M be a 2-absorbing submodule and $Q_1 \cap \cdots \cap Q_n = 0$ with $r(Q_i :_R M) = \mathfrak{p}_i$, for $i = 1, \ldots, n$, be its minimal primary decomposition. In Section 3, we define the zero-divisor graph of equivalence classes of zero divisors of M, $\Gamma_E(M)$, and we show that $\{Q_1 \cap \cdots \cap Q_{n-1}, Q_1 \cap \cdots \cap Q_{n-2}, \ldots, Q_1\}$ is an independent subset of $V(\Gamma_E(M))$.

Throughout, R denotes a commutative ring with a nonzero identity, M is a unitary Noetherian R-module, and $Z(M)$ the set of its zero divisors. Let $\mathrm{Ass}_R(M) = \{\mathfrak{p} \in \mathrm{Spec}(R) : \mathfrak{p} = \mathrm{Ann}_R(m) \text{ for some } 0 \neq m \in M\}$ denote the set of associated primes of M. Set $(0 :_M a) = \mathrm{Ann}_M(a) := \{m \in M : am = 0\}$ for all $a \in R$. For notations and terminologies not given in this article, the reader is referred to [13].

2. Primary decomposition of a 2-absorbing submodule

In this section, R is a commutative ring and M is a Noetherian R-module. We study the properties of a minimal primary decomposition of a 2-absorbing submodule of M. A proper submodule Q of M is said to be primary if $rm \in Q$ for some $r \in R$ and $m \in M$, then $m \in Q$ or $r \in r(Q :_R M) = \{a \in R : a^t M \subseteq Q \text{ for some } t \in \mathbb{N}\}$.

Lemma 2.1. *Let $\mathfrak{p}$ be a prime ideal of R and Q be a $\mathfrak{p}$-primary submodule of M. Then the following statements are true.*

(i) *If $m \in M \setminus Q$, then $(Q :_R m)$ is a $\mathfrak{p}$-primary ideal of R.*
(ii) *If $a \in R \setminus \mathfrak{p}$, then $(Q :_M a) = Q$.*

Recall that a proper submodule N of M is called 2-absorbing if whenever $abm \in N$ for $a, b \in R$ and $m \in M$, then $am \in N$ or $bm \in N$ or $ab \in (N :_R M)$. In the sequel, we suppose that $N = Q_1 \cap \cdots \cap Q_n$ with $r(Q_i :_R M) = \mathfrak{p}_i$, for $i = 1, \ldots, n$, is a minimal primary decomposition of N.

Theorem 2.2. *Let N be a 2-absorbing submodule of M such that $r(N :_R M) = \mathfrak{p}$ is a prime ideal of R. Then the following statements are true.*

(i) *$\mathfrak{p} = \mathfrak{p}_j$ for some j with $1 \leq j \leq n$.*
(ii) *For each $j = 1, \ldots, n$ there exists $m_j \in M$ such that $\mathfrak{p}_j = (N :_R m_j)$.*
(iii) *For each $i, j = 1, \ldots, n$ either $\mathfrak{p}_i \subseteq \mathfrak{p}_j$ or $\mathfrak{p}_j \subseteq \mathfrak{p}_i$.*

Proof. (i) By the assumption

$$\mathfrak{p} = r(N :_R M) = r(\cap_{i=1}^n Q_i :_R M) = r(\cap_{i=1}^n (Q_i :_R M)) = \cap_{i=1}^n \mathfrak{p}_i.$$

Thus there exists j with $1 \leq j \leq n$ such that $\mathfrak{p} = \mathfrak{p}_j$, see [13, Corollary 3.57].

(ii) By the assumption there is $m_j \in \cap_{i=1, i\neq j}^n Q_i \setminus Q_j$ thus $(N :_R m_j) = (Q_j :_R m_j)$ so by Lemma 2.1(i), $r(N :_R m_j) = r(Q_j :_R m_j) = \mathfrak{p}_j$. In view of [10, Theorem 2.5], either $(N :_R m_j)$ is a prime ideal of R or there exists $a \in R$ such that $(N :_R am_j)$ is prime. If $(N :_R m_j)$ is prime, then $(N :_R m_j) = \mathfrak{p}_j$ and we are done. Now, suppose that $(N :_R m_j) \subset \mathfrak{p}_j$ and $a \in \mathfrak{p}_j \setminus (N :_R m_j)$. Thus $am_j \in \cap_{i=1, i\neq j}^n Q_i \setminus Q_j$ as above $(N :_R am_j)$ is a $\mathfrak{p}_j$-primary ideal of R. By [10, Theorem 2.4] and [3, Theorem 2.4] it follows that $\mathfrak{p}_j^2 \subseteq (N :_R m_j)$. Hence, $\mathfrak{p}_j \subseteq (N :_R am_j) \subseteq \mathfrak{p}_j$ and $(N :_R am_j) = \mathfrak{p}_j$.

(iii) In view of [10, Theorem 2.6(ii)], the assertion follows. □

Corollary 2.3. *Let N be a 2-absorbing submodule of M such that $r(N :_R M) = \mathfrak{p}$ is a prime ideal of R. Then $\mathrm{Ass}_R(M/N) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ is a totally ordered set.*

Proof. This is an immediate consequence of Theorem 2.2(iii). □

Remark 2.4. Let N be a 2-absorbing submodule of M such that $r(N :_R M) = \mathfrak{p}$ is a prime ideal of R. Suppose that $N = Q_1 \cap \cdots \cap Q_n$ with $r(Q_i :_R M) = \mathfrak{p}_i$, for $i = 1, \ldots, n$, is a minimal primary decomposition of N. In the rest of this section, we suppose that $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ have been numbered (renumbered if necessary) such that $\mathfrak{p} = \mathfrak{p}_1$ and $\mathfrak{p}_1 \subset \mathfrak{p}_2 \subset \cdots \subset \mathfrak{p}_n$.

A proper submodule P of M is said to be prime if $rm \in P$ for some $r \in R$ and $m \in M$, then $m \in P$ or $r \in (P :_R M) = \{a \in R : aM \subseteq P\}$.

Theorem 2.5. *Let N be a 2-absorbing submodule of M such that $r(N :_R M) = \mathfrak{p}$ is a prime ideal of R. Then the following statements are true.*

(i) $Q_1 = (N :_M a)$ *for some* $a \in R$.
(ii) Q_1 *is a prime submodule of* M *whenever* $(N :_R M) = \mathfrak{p}$.
(iii) *Assume that* $(N :_R M) = \mathfrak{p}$ *and* $b \in R$. *If* $P = (N :_M b)$ *is a prime submodule of* M *with* $\mathfrak{p}' = (P :_R M)$, *then* $\mathfrak{p}'$ *is a minimal element of* $\mathrm{Ass}_R(M/N)$ *and* $Q_1 = P$.

Proof. (i) By Remark 2.4, $\mathfrak{p}_1 = r(Q_1 :_R M)$ is a minimal element of $\mathrm{Ass}_R(M/N)$. Then $\cap_{i=2}^n (Q_i :_R M) \not\subseteq \mathfrak{p}_1$. Suppose that $a \in \cap_{i=2}^n (Q_i :_R M) \setminus \mathfrak{p}_1$. We show that $Q_1 = (N :_M a)$. By the assumption $(N :_M a) = (\cap_{i=1}^n Q_i :_M a) = \cap_{i=1}^n (Q_i :_M a) = (Q_1 :_M a)$. It is clear that $Q_1 \subseteq (Q_1 :_M a)$. If there exists $m \in (Q_1 :_M a) \setminus Q_1$, then there is $t \in \mathbb{N}$ such that $a^t M \subseteq Q_1$ which implies that $a \in \mathfrak{p}_1$ that is a contradiction. Hence, $Q_1 = (N :_M a)$.

(ii) From (i) it follows that $Q_1 = N :_M a$ for some $a \in \cap_{i=2}^n \mathfrak{p}_i \setminus \mathfrak{p}_1$. We show that Q_1 is a prime submodule. Suppose that $b \in R, m \in M \setminus Q_1$ and $bm \in Q_1$. Thus there is $t \in \mathbb{N}$ such that $b^t M \subseteq Q_1$. So $(ab)^t M \subseteq Q_1$. On the other hand,

$ab \in \cap_{i=2}^{n}\mathfrak{p}_i$ thus $(ab)^t M \subseteq \cap_{i=2}^{n} Q_i$. Hence, $(ab)^t M \subseteq \cap_{i=1}^{n} Q_i = N$. Therefore, by the hypothesis $abM \subseteq N$ so $bM \subseteq Q_1$ and Q_1 is prime.

(iii) Let $b \in R$ and $P = (N :_M b)$ be a prime submodule of M. Then one can see that $((N :_M b) :_R M) = \mathfrak{p}'$ is a prime ideal of R. It is easy to see that $\mathfrak{p}' = (N :_R bM)$. Let $m \in M$ and $bm \notin N$. We show that $\mathfrak{p}' = (N :_R bm)$. It is obvious that $\mathfrak{p}' \subseteq (N :_R bm)$. Assume that $r \in R$ and $rbm \in N$. Thus $rm \in P = (N :_M b)$ and $m \notin P = (N :_M b)$ so $rbM \subseteq N$ and $r \in (N :_R bM) = \mathfrak{p}'$. Hence, $\mathfrak{p}' = (N :_R bm) \in \mathrm{Ass}_R(M/N)$. If $b \in \cap_{i=1}^{n}\mathfrak{p}_i$, then there is $t \in \mathbb{N}$ such that $b^t \in \cap_{i=1}^{n}(Q_i :_R M)$ so $b^t M \subseteq \cap_{i=1}^{n} Q_i = N$. Therefore, $b^t M \subseteq N$ and $bM \subseteq N$ which is a contradiction. Thus $b \notin \mathfrak{p}_1$. Assume that $r \in \mathfrak{p}'$. Thus $rbM \subseteq N$ and so $rbM \subseteq \cap_{i=1}^{n} Q_i$. Hence, $rb \in (\cap_{i=1}^{n} Q_i :_R M) \subseteq \cap_{i=1}^{n}(Q_i :_R M) \subseteq \cap_{i=1}^{n}\mathfrak{p}_i$. Therefore, from $rb \in \mathfrak{p}_1$ and $b \notin \mathfrak{p}_1$ it follows that $\mathfrak{p}' \subseteq \mathfrak{p}_1$ so $\mathfrak{p} = \mathfrak{p}_1$. Now, we show that $P = Q_1$. Assume that $m \in Q_1$. Thus $am \in N \subseteq P$. If $m \notin P$, then $a \in \mathfrak{p}' = \mathfrak{p}_1$ which is a contradiction so $Q_1 \subseteq P$. Assume that $m \in P$ so $bm \in N \subseteq Q_1$. If $m \notin Q_1$, then there is $s \in \mathbb{N}$ such that $b^s M \subseteq Q_1 = (N :_M a)$. Hence, $ab^{s-1}(bM) \subseteq N$ so $ab^{s-1} \in \mathfrak{p}' = \mathfrak{p}_1$ which implies that $b \in \mathfrak{p}' = \mathfrak{p}_1$ since $a \notin \mathfrak{p}_1$. This means that $b^2 M \subseteq N$ and $b \in r(N :_R M) = (N :_R M)$ which is a contradiction. Therefore, $m \in Q_1$ and so $P \subseteq Q_1$. □

Theorem 2.6. *Let N be a 2-absorbing submodule of M such that $r(N :_R M) = \mathfrak{p}$ is a prime ideal of R. Then $\{(N :_M a) : a \in R \setminus \mathfrak{p}\} = \{N = \cap_{i=1}^{n} Q_i, \cap_{i=1}^{n-1} Q_i, \ldots, Q_1\}$ is a totally ordered set.*

Proof. By [12, Corollary 2.4], $(N :_M a)$ is a 2-absorbing submodule of M for all $a \in R$. If $a \notin \mathfrak{p}_n$, then in view of Lemma 2.1(ii), $(N :_M a) = \cap_{i=1}^{n}(Q_i :_M a) = \cap_{i=1}^{n} Q_i = N$. Suppose that there is j with $1 \leq j < n$ such that $a \in \mathfrak{p}_{j+1} \setminus \bigcup_{i=1}^{j} \mathfrak{p}_i$. Thus there is $t \in \mathbb{N}$ such that $a^t M \subseteq \cap_{i=j+1}^{n} Q_i$ and $a^t \notin \mathfrak{p}_j$. By Lemma 2.1(ii), we have $(N :_M a^t) = (\cap_{i=1}^{n} Q_i :_M a^t) = \cap_{i=1}^{n}(Q_i :_M a^t) = \cap_{i=1}^{j} Q_i$. Now, it is enough to show that $(N :_M a^t) = (N :_M a)$. It is clear that $(N :_M a) \subseteq (N :_M a^t)$. For the reverse inclusion assume that $m \in (N :_M a^t)$. Thus $a^t m \in N$ since N is a 2-absorbing submodule $am \in N$ or $a^{t-1} m \in N$ or $a^t \in (N :_R M) \subseteq \mathfrak{p}_j$. If $am \in N$ the assertion follows. The third case is impossible. So assume that $a^{t-1} m \in N$. Now, by an easy induction one can show that $am \in N$ as desired. Hence, $(N :_M a) = (N :_M a^t) = \cap_{i=1}^{j} Q_i$. □

Corollary 2.7. *Let N be a 2-absorbing submodule of M such that $r(N :_R M) = \mathfrak{p}$ is a prime ideal of R. Then $(N :_R m)$ is a decomposable ideal of R for each $m \in M \setminus N$. Moreover, its primary decomposition is $(N :_R m) = \cap_{i=j+1}^{n}(Q_i :_R m)$ for some j with $0 \leq j \leq n$.*

Proof. (i) Let $m \in M \setminus N$ and let $m \in \cap_{i=1}^{j} Q_i \setminus \cup_{i=j+1}^{n} Q_i$ for some j with $1 \leq j < n$. Then by Lemma 2.1(i), $(N :_R m) = (\cap_{i=1}^{n} Q_i :_R m) = \cap_{i=j+1}^{n}(Q_i :_R m)$. □

Theorem 2.8. *Let N be a 2-absorbing submodule of M such that $r(N :_R M) = \mathfrak{p} \cap \mathfrak{q}$, where $\mathfrak{p}, \mathfrak{q}$ are the only distinct prime ideals of R that are minimal over $(N :_R M)$. Then the following statements are true.*

(i) $\mathfrak{p} = \mathfrak{p}_k$, $\mathfrak{q} = \mathfrak{p}_s$ *for some* k, s *with* $1 \leq k, s \leq n$ *and* $k \neq s$.

(ii) *For each* $j = 1, \dots, n$ *there exists* $m_j \in M$ *such that* $(N :_R m_j) = \mathfrak{p}_j$.

Proof. (i) By the assumption $r(N :_R M) = r(\cap_{i=1}^n Q_i :_R M) = r(\cap_{i=1}^n (Q_i :_R M)) = \cap_{i=1}^n \mathfrak{p}_i = \mathfrak{p} \cap \mathfrak{q}$. Since $\mathfrak{p}$ is a minimal prime ideal of $(N :_R M)$, there exists $1 \leq k \leq n$ such that $\mathfrak{p} = \mathfrak{p}_k$. Also, there exists $1 \leq s \leq n$ with $k \neq s$ such that $\mathfrak{p} = \mathfrak{p}_s$.

(ii) Let $m_j \in \cap_{i=1, i\neq j}^n Q_i \setminus Q_j$. Then $(N :_R m_j) = (\cap_{i=1}^n Q_i :_R m_j) = (Q_j :_R m_j)$. Moreover, $r(N :_R m_j) = r(Q_j :_R m_j) = r(Q_j :_R M) = \mathfrak{p}_j$. By [10, Theorem 2.5] either $(N :_R m_j)$ is a prime ideal of R or there exists $a \in R$ such that $(N :_R am_j)$ is a prime ideal of R. If $(N :_R m_j)$ is a prime ideal, then $(Q_j :_R m_j) = (N :_R m_j) = r(N :_R m_j) = \mathfrak{p}_j$. Now, suppose that $(N :_R m_j) \subset \mathfrak{p}_j$ and $a \in \mathfrak{p}_j \setminus (N :_R m_j)$. Thus $am_j \in \cap_{i=1, i\neq j}^n Q_i \setminus Q_j$ as above $(N :_R am_j)$ is a $\mathfrak{p}_j$-primary ideal of R. By [10, Theorem 2.4] and [3, Theorem 2.4] it follows that $\mathfrak{p}_j^2 \subseteq (N :_R m_j)$. Hence, $\mathfrak{p}_j \subseteq (N :_R am_j) \subseteq \mathfrak{p}_j$. Therefore, $(N :_R am_j) = \mathfrak{p}_j$. □

Corollary 2.9. *Let N be a 2-absorbing submodule of M such that $r(N :_R M) = \mathfrak{p} \cap \mathfrak{q}$, where $\mathfrak{p}, \mathfrak{q}$ are the only distinct prime ideals of R that are minimal over $(N :_R M)$. Then $\mathrm{Ass}_R(M/N)$ is union of two totally ordered sets such as $\{\mathfrak{p}_k\} \cup \{\mathfrak{p}_1, \dots, \mathfrak{p}_{k-1}, \mathfrak{p}_{k+1}, \dots, \mathfrak{p}_n\}$ or $\{\mathfrak{p}_s\} \cup \{\mathfrak{p}_1, \dots, \mathfrak{p}_{s-1}, \mathfrak{p}_{s+1}, \dots, \mathfrak{p}_n\}$.*

Proof. Let $N = \cap_{i=1}^n Q_i$ be a minimal primary decomposition of N with $r(Q_i :_R M) = \mathfrak{p}_i$ for each $1 \leq i \leq n$. Then by Theorem 2.8, $\mathfrak{p} = \mathfrak{p}_k$, $\mathfrak{q} = \mathfrak{p}_s$ for some k, s with $1 \leq k, s \leq n$ and $k \neq s$. Without loss of generality we may assume that $\mathfrak{p} = \mathfrak{p}_1$ and $\mathfrak{q} = \mathfrak{p}_2$. Suppose that $3 \leq l, t \leq n$ and $l \neq t$. By the assumption there exist $m_l \in \cap_{i=1, i\neq l}^n Q_i \setminus Q_l$ and $m_t \in \cap_{i=1, i\neq t}^n Q_i \setminus Q_t$. Thus $r(N :_R m_l) = r(\cap_{i=1}^n Q_i :_R m_l) = \cap_{i=1}^n r(Q_i :_R m_l) = r(Q_l :_R m_l) = r(Q_l :_R M) = \mathfrak{p}_l$ and $r(N :_R m_t) = \mathfrak{p}_t$. Let $\mathfrak{p}_t \not\subseteq \mathfrak{p}_l$; we show that $\mathfrak{p}_l \subseteq \mathfrak{p}_t$. By the hypotheses we may assume that $\mathfrak{p}_1 \subseteq \mathfrak{p}_l$ moreover $\mathfrak{p}_t \not\subseteq \mathfrak{p}_l \cup \mathfrak{p}_2$. Suppose that $a \in \mathfrak{p}_l$ and $b \in \mathfrak{p}_t \setminus \mathfrak{p}_l \cup \mathfrak{p}_2$. So there exists $s \in \mathbb{N}$ such that $a^s m_l \in N, b^s m_t \in N$ and $b^s m_l \notin N$. If $a^s(m_l + m_t) \in N$, then $a \in \mathfrak{p}_t$ and the proof is completed. Now, let $a^s(m_l + m_t) \notin N$. Then $a^s b^s \in (N :_R M)$ since $b^s(m_l + m_t) \notin N$ and $a^s b^s(m_l + m_t) \in N$. So $ab \in \mathfrak{p}_1 \cap \mathfrak{p}_2$. Since $b \notin \mathfrak{p}_1 \cup \mathfrak{p}_2$, we have $a \in \mathfrak{p}_1 \cap \mathfrak{p}_2$. So $a^s M \subseteq N$ and $a^s m_t \in N$ which implies that $a \in \mathfrak{p}_t$. Hence, $\mathrm{Ass}_R(M/N)$ is the union of two totally ordered sets such as $\mathrm{Ass}_R(M/N) = \{\mathfrak{p} = \mathfrak{p}_1\} \cup \{\mathfrak{p}_2, \mathfrak{p}_3, \dots, \mathfrak{p}_n\}$ or $\mathrm{Ass}_R(M/N) = \{\mathfrak{q} = \mathfrak{p}_2\} \cup \{\mathfrak{p}_1, \mathfrak{p}_3, \dots, \mathfrak{p}_n\}$. □

Remark 2.10. Let N be a 2-absorbing submodule of M such that $r(N :_R M) = \mathfrak{p} \cap \mathfrak{q}$, where $\mathfrak{p}, \mathfrak{q}$ are the only distinct prime ideals of R that are minimal over $(N :_R M)$. In the rest of this paper, it will be supposed that $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ have been numbered (renumbered if necessary) such that $\mathfrak{p} = \mathfrak{p}_1, \mathfrak{q} = \mathfrak{p}_2$ and either $\mathfrak{p}_1 \subset \mathfrak{p}_3 \subset \cdots \subset \mathfrak{p}_n$ or $\mathfrak{p}_2 \subset \mathfrak{p}_3 \subset \cdots \subset \mathfrak{p}_n$.

Corollary 2.11. *Let N be a 2-absorbing submodule of M such that $r(N :_R M) = \mathfrak{p} \cap \mathfrak{q}$, where $\mathfrak{p}, \mathfrak{q}$ are the only distinct prime ideals of R that are minimal over $(N :_R M)$. Then $\{(N :_M a) : a \in R \backslash \mathfrak{p}_1 \cup \mathfrak{p}_2\} = \{N = \cap_{i=1}^{n} Q_i, \cap_{i=1}^{n-1} Q_i, \ldots, Q_1 \cap Q_2\}$ and $\{(N :_M a) : a \in \mathfrak{p}_2 \setminus \mathfrak{p}_1\} = \{\cap_{i=1, i\neq 2}^{n} Q_i, \cap_{i=1, i\neq 2}^{n-1} Q_i, \ldots, Q_1 \cap Q_3, Q_1\}$ whenever $\mathfrak{p}_1 \subset \mathfrak{p}_3 \subset \cdots \subset \mathfrak{p}_n$.*

Proof. The proof is similar to that of Theorem 2.6 □

3. Zero-divisor graph of equivalences classes of zero-divisors

Let R be a commutative ring and M be a Noetherian R-module. The zero-divisor graph of M, denoted by $\Gamma(M)$, is a simple undirected graph whose vertex set is $Z_R(M) \setminus \mathrm{Ann}_R(M)$ and two distinct vertices a and b are adjacent if and only if $abM = 0$, see [8]. In the following we define the zero-divisor graph for equivalences classes of zero divisors of M. For $a, b \in R$, we say that $a \sim b$ if and only if $\mathrm{Ann}_M(a) = \mathrm{Ann}_M(b)$. It is clear that $\sim$ is an equivalence relation. If $[a]$ denotes the class of a, then $[a] = \mathrm{Ann}_R(M)$ for all $a \in \mathrm{Ann}_R(M)$ and $[a] = R \setminus Z(M)$ for all $a \in R \setminus Z(M)$; the other equivalence classes form a partition of $Z(M) \setminus \mathrm{Ann}_R(M)$.

Definition. The zero-divisor graph of equivalence classes of zero divisors of M, denoted $\Gamma_E(M)$, is a simple graph associated to M whose vertices are the equivalence classes of the elements of $Z(M) \setminus \mathrm{Ann}_R(M)$, and each pair of distinct classes such as $[a]$ and $[b]$ are adjacent if and only if $abM = 0$.

Lemma 3.1. *Let $x, y \in Z(M) \setminus \mathrm{Ann}_R(M)$. If $\mathrm{Ann}_M(x) \subset \mathrm{Ann}_M(y)$, then* $\deg[x] \leq \deg[y]$.

Proof. If $[z] \in \Gamma_E(M)$ is such that $zxM = 0$, then clearly $zyM = 0$. So if $[z]$ is adjacent to $[x]$, then $[z]$ is adjacent to $[y]$. Thus $\deg[x] \leq \deg[y]$. □

Let $G = (V, E)$ be a graph. A subset S of V is called an independent set of G if no two vertices in S are adjacent.

Theorem 3.2. *Let the zero submodule of M be a 2-absorbing submodule such that $r(0 :_R M) = \mathfrak{p}$ is a prime ideal of R. Then $\Gamma_E(M)$ has an independent set of vertices such as $\{[a_1], \ldots, [a_{n-1}]\}$, where* $\deg[a_{n-1}] \leq \cdots \leq \deg[a_1]$.

Proof. Suppose that $0 = Q_1 \cap \cdots \cap Q_n$ $(n \geq 2)$ with $r(Q_i :_R M) = \mathfrak{p}_i$ for $i = 1, \ldots, n$, is a minimal primary decomposition of the zero submodule of M. By Theorem 2.6, there is a subset of elements of $Z(M) \setminus \mathrm{Ann}(M)$ such as $\{a_1, \ldots, a_{n-1}\}$, where $\{\mathrm{Ann}_M(a_{n-1}) = \cap_{i=1}^{n-1} Q_i, \ldots, \mathrm{Ann}_M(a_1) = Q_1\}$. Thus the set $\{[a_1], \ldots, [a_{n-1}]\}$ is an independent set of vertices of $\Gamma_E(M)$. Since if $a_k a_j M = 0$ for some k and j with $1 \leq k < j \leq n-1$, then $a_j M \subseteq \mathrm{Ann}_M(a_k) = \cap_{i=1}^{k} Q_i$ which implies that $a_j M \subseteq Q_1$ so $a_j \in \mathfrak{p}_1$, contrary to choose of a_j in Theorem 2.6. The second assertion follows by Lemma 3.1. □

The graph $H = (V_0, E_0)$ is a subgraph of $G = (V, E)$ if $V_0 \subseteq V$ and $E_0 \subseteq E$. Moreover, H is called an induced subgraph by V_0, denoted by $G[V_0]$, if $V_0 \subseteq V$ and $E_0 = \{\{u, v\} \in E \mid u, v \in V_0\}$.

Theorem 3.3. *Let the zero submodule of M be a 2-absorbing submodule such that $r(0 :_R M) = \mathfrak{p}$ is a prime ideal of R. Then $\Gamma_E(M)[r(0 :_R M)]$, the induced subgraph of $\Gamma_E(M)$ by $r(0 :_R M)$, is complete.*

Proof. In view of [10, Theorem 2.4], $(0 :_R M)$ is a 2-absorbing ideal of R and by [3, Theorem 2.4], $\mathfrak{p}^2 \subseteq (0 :_R M)$. Now, suppose that $x, y \in \mathfrak{p} \setminus \mathrm{Ann}_R(M)$ and $\mathrm{Ann}_M(x) \neq \mathrm{Ann}_M(y)$. Thus $xyM = 0$ so $[x]$ and $[y]$ are adjacent in $\Gamma_E(M)$ and the results follows. □

Remark 3.4. If $m \in \cap_{i=1}^{k-1} Q_i \setminus Q_k, 2 \leq k \leq n-1$ and $a \in \mathfrak{p}_k \setminus \mathfrak{p}_1$, then $a^t m \in \cap_{i=1}^{k} Q_i$ for some positive integer t, but $m \notin \cap_{i=1}^{k} Q_i$ and $a \notin \mathfrak{p}_1 = r(\cap_{i=1}^{k} Q_i :_R M) \supseteq (\cap_{i=1}^{k} Q_i :_R M)$. Thus $\mathrm{Ann}_M(a_i)$, $i = 2, \ldots, n-1$, is not a prime submodule. Hence, non of element of $\{\mathrm{Ann}_M(a_2) = Q_1 \cap Q_2, \ldots, \mathrm{Ann}_M(a_{n-1}) = \cap_{i=1}^{n-1} Q_i\}$ is a prime submodule, see the proof of Theorem 3.2.

Let $\mathrm{Spec}_R(M)$ denote the set of all prime submodules of M and $\mathrm{m}-\mathrm{Ass}_R(M) = \{P \in \mathrm{Spec}_R(M) : P = \mathrm{Ann}_M(a) \text{ for some } a \in Z(M) \setminus \mathrm{Ann}_R(M)\}$. The properties of prime submodules and $\mathrm{m}-\mathrm{Ass}_R(M)$ are studied in [1,5,6]. By [5, Proposition 3.2], any maximal element of $\Delta = \{\mathrm{Ann}_M(a) : a \in Z(M) \setminus \mathrm{Ann}_R(M)\}$ is a prime submodule of M. Thus for a Noetherian R-module M, $\mathrm{m}-\mathrm{Ass}_R(M)$ is a nonempty set.

Corollary 3.5. *Let the zero submodule of M be a 2-absorbing submodule such that $r(0 :_R M) = \mathfrak{p}$ is a prime ideal of R. Then the following statements are true.*

(i) *If $\mathrm{m}-\mathrm{Ass}_R(M) = \{Q_1\}$, then $V(\Gamma_E(M)) = \{[a_1], \ldots, [a_{n-1}]\}$ and $\Gamma_E(M)$ is a disconnected graph.*

(ii) *If $Q_1 \in \mathrm{m}-\mathrm{Ass}_R(M)$, then $\Gamma_E(M)[r(0 :_R M) \cup \{a_1\}]$, the induced subgraph of $\Gamma_E(M)$ by $r(0 :_R M) \cup \{a_1\}$, is complete.*

(iii) *If $Q_1 = \mathrm{Ann}_M(a_1) \notin \mathrm{m}-\mathrm{Ass}_R(M)$, then* $\deg[a_1] \leq 2$.

Proof. (i) By the hypotheses $Q_1 = \mathrm{Ann}_M(a_1)$ is the only maximal element of Δ. Thus for every $x \in \mathfrak{p}$ we have $\mathrm{Ann}_M(x) \subseteq \mathrm{Ann}_M(a_1)$. If for some $x \in \mathfrak{p}$, $\mathrm{Ann}_M(x) \subset \mathrm{Ann}_M(a_1)$, then $xM \subseteq \mathrm{Ann}_M(x) \subseteq \mathrm{Ann}_M(a_1)$ implies that $a_1 \in \mathfrak{p}$ contrary to choose of a_1 in Theorem 2.6. Hence, $\mathrm{Ann}_M(x) = \mathrm{Ann}_M(a_1)$ for all $x \in \mathfrak{p}$ so $V(\Gamma_E(M)) = \{[a_1], \ldots, [a_{n-1}]\}$ and $\Gamma_E(M)$ is a disconnected graph by Theorem 3.2.

(ii) By (i) it follows that for all $x \in \mathfrak{p}$ either $\mathrm{Ann}_M(a_1) = \mathrm{Ann}_M(x)$ or $\mathrm{Ann}_M(x) \nsubseteq \mathrm{Ann}_M(a_1)$. In the first case there is nothing to prove. If $\mathrm{Ann}_M(x) \nsubseteq \mathrm{Ann}_M(a_1)$ for some $x \in \mathfrak{p}$, then there is $m \in \mathrm{Ann}_M(x) \setminus \mathrm{Ann}_M(a_1) = Q_1$ so $xm = 0 \in Q_1$ implies that $a_1 x M = 0$. Thus $[a_1]$ is adjacent to $[x]$. Hence, in view of Theorem 3.3 the result follows.

(iii) The result follows by [9, Theorems 4.1, 4.4 and Corollary 4.3]. □

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