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THE HOMOLOGY REGARDING TO E-EXACT SEQUENCES

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Abstract. Let R be a commutative ring with identity. Let R be an integral domain and M a torsion-free R-module. We investigate the relation between the notion of e-exactness, recently introduced by Akray and Zebari [1], and generalized the concept of homology, and establish a relation between e-exact sequences and homology of modules. We modify some applications of e-exact sequences in homology and reprove some results of homology with e-exact sequences such as horseshoe lemma, long exact sequences, connecting homomorphisms and etc. Next, we generalize two special drived functor Tor and Ext, and study some properties of them.

1. Introduction

Throughout this article, R will denote an integral domain, M a unitary torsion-free R-module. Here we use monic and epic to denote a monomorphism and an epimorphism, respectively. The homology concept has had a long and varied history. In [4], Weibel Charles described this history which is started in the nineteenth century, via the work of Riemann (1857) and Betti (1871) on homology numbers. The concept of essential exact sequences was introduced by Akray and Zebari [1] as a generalization to the notion of exact sequences of modules. They introduced the e-exact sequence of a module and proved some results in module theory and they arose two questions. Here we try to answer question one which is stated:

Question 1. One can use the above two definitions to redefine the homology, using the left e-exact functors $\mathrm{Hom}(M, -)$, $\mathrm{Hom}(-, M)$ and right e-exact functor $M \otimes -$ to define their derived functors and study properties of them.

In this paper, we generalized the concept of homology and establish a relationship between e-exact sequences and the homology. We prove some results of homology with an e-exact sequence such as horseshoe lemma, long exact sequence, connecting homomorphism, etc. Finally, we redefine the drive functor with e-exact sequences and discuss two special drive functors Tor and Ext.

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A complex or chain complex $\mathbf{A}$ is a sequence of R-modules and maps

$$\mathbf{A} \ : \quad \cdots \longrightarrow A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \longrightarrow \cdots , \quad n \in Z \ ,$$

with $d_n d_{n+1} = 0$ for all n. We will write $(\mathbf{A}, d)$ instead of $\mathbf{A}$. The n^{th} homology R-module is defined to be $H_n(\mathbf{A}) = Ker(d_n)/Im(d_{n+1})$. The elements of A_n are called n-chains, the elements of $Ker(d_n)$ are called n-cycles, and the elements of $Im(d_{n+1})$ are called n-boundaries and we symbolize by $Ker(d_n) = Z_n(\mathbf{A}) = Z_n$, $Im(d_{n+1}) = B_n(\mathbf{A}) = B_n$, and thus $H_n(\mathbf{A}) = Z_n(\mathbf{A})/B_n(\mathbf{A})$ [3, p. 169].

2. Preliminaries

In this section, we list some basic concepts and well-known results on e-exact sequences and essential submodule of/in modules which are mainly taken from [1] and [2].

Definition 1 ([2])**.** Let M be an R-module. Then a submodule N of M is called an essential submodule in M if the intersection of N with every non-zero submodule of M is not equal to zero and we denoted by $N \leq_e M$.

Equivalently, N is an essential submodule of M if $N \cap Rx \neq 0$ for all non-zero element $x \in M$ ([2, p. 75]).

Definition 2 ([1])**.** A sequence of R-modules and R-morphisms

$$\cdots \longrightarrow A_{i-1} \xrightarrow{f_{i-1}} A_i \xrightarrow{f_i} A_{i+1} \longrightarrow \cdots$$

is said to be e-exact at A_i if $Im(f_{i-1}) \leq_e Ker(f_i)$, and it is said to be e-exact sequences if it is e-exact at each A_i. Moreover, a sequence of R-modules and R-morphisms

$$0 \longrightarrow A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \longrightarrow 0$$

is called a short e-exact sequence if and only if $Ker(f_1) = 0$, $Im(f_1) \leq_e Ker(f_2)$ and $Im(f_2) \leq_e A_3$.

In the following, we have an example which is an e-exact sequence but not exact that show the class of all e-exact sequences is larger than the class of exact sequences.

Example 2.1. Consider the short e-exact sequence

$$0 \longrightarrow 4Z \xrightarrow{f_1} Z \xrightarrow{f_2} Z/4Z \longrightarrow 0$$

we define f_1 and f_2 as $f_1(4n) = 2n$ and $f_2(n) = 2n + 4Z$. But f_2 is not epic, the sequence is not exact.

A functor F is called covariant left e-exact if for every short e-exact sequence $0 \longrightarrow A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \longrightarrow 0$, the sequence

$$0 \longrightarrow F(A_1) \xrightarrow{F(f_1)} F(A_2) \xrightarrow{F(f_2)} F(A_3)$$

is e-exact and called covariant right e-exact if the sequence

$$F(A_1) \xrightarrow{F(f_1)} F(A_2) \xrightarrow{F(f_2)} F(A_3) \longrightarrow 0$$

is e-exact whenever $0 \longrightarrow A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \longrightarrow 0$ is e-exact. A functor F is called a covariant e-exact functor if it is both covariant left e-exact functor and covariant right e-exact functor.

A functor F is called contravariant left e-exact if for every short e-exact sequence $0 \longrightarrow A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \longrightarrow 0$, the sequence

$$0 \longrightarrow F(A_3) \xrightarrow{F(f_2)} F(A_2) \xrightarrow{F(f_1)} F(A_1)$$

is e-exact.

Theorem 2.2 ([1])**.** *The sequence of R-modules and R-morphisms*

$$0 \longrightarrow A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3$$

is e-exact if and only if for all R-module B, the sequence

$$0 \longrightarrow \mathrm{Hom}(B, A_1) \xrightarrow{{f_1}^*} \mathrm{Hom}(B, A_2) \xrightarrow{{f_2}^*} \mathrm{Hom}(B, A_3)$$

is e-exact.

Theorem 2.3 ([1])**.** *If a sequence of R-modules and R-morphisms*

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \longrightarrow 0$$

is e-exact, then for all torsion-free R-module B, the sequence

$$0 \longrightarrow \mathrm{Hom}(A_3, B) \xrightarrow{{f_2}^*} \mathrm{Hom}(A_2, B) \xrightarrow{{f_1}^*} \mathrm{Hom}(A_1, B)$$

is e-exact. The converse is true if $A_3/Im(f_2)$ and $A_2/Im(f_1)$ are torsion-free R-modules.

Theorem 2.4 ([1])**.** *Let $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \longrightarrow 0$ be an e-exact sequence. Then for any torsion-free R-module B, the sequence*

$$A_1 \otimes B \xrightarrow{f_1 \otimes 1} A_2 \otimes B \xrightarrow{f_2 \otimes 1} A_3 \otimes B \longrightarrow 0$$

is e-exact.

Definition 3 ([1])**.** We say that an R-module P is e-projective if satisfies the following condition: for any e-epic map $f_1 : A_1 \to A_2$, and any map $f_2 : P \to A_2$, there exist $0 \neq r \in R$ and $f_3 : P \to A_1$ such that $f_1 f_3 = r f_2$:

$$\begin{tikzcd} & P \arrow[dl, dashed, "f_3"'] \arrow[d, "f_2"] & \\ A_1 \arrow[r, "f_1"] & A_2 \arrow[r] & 0 \end{tikzcd}$$

Definition 4 ([1])**.** An e-projective resolution of an R-module A is an e-exact sequence

$$\cdots \longrightarrow P_{n+1} \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

in which each P_n is e-projective.

Dually, we can define e-injective resolution as follows.

Definition 5 ([1])**.** An e-injective resolution of an R-module A is an e-exact sequence

$$0 \longrightarrow A \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots \longrightarrow E^n \longrightarrow E^{n+1} \longrightarrow \cdots$$

in which each E^n is injective.

3. E-exact sequences and homology

In this section, we have the applications of e-exact sequences in homology and prove some results of homology with e-exact sequences.

Theorem 3.1 (Connecting homomorphism with e-exact sequence)**.** *Let*

$$0 \longrightarrow \boldsymbol{A}' \xrightarrow{\ i\ } \boldsymbol{A} \xrightarrow{\ p\ } \boldsymbol{A}'' \longrightarrow 0$$

be an e-exact sequence of complexes. Then for each non-negative integer number n, there is a homomorphism

$$\sigma_n : H_n(\boldsymbol{A}'') \to H_{n-1}(\boldsymbol{A}')$$

defined by:

$$z''_n + B_n(\boldsymbol{A}'') \mapsto i^{-1}_{n-1} d_n p^{-1}_n (r z''_n) + B_{n-1}(\boldsymbol{A}'),$$

where $r \in R$.

Proof. Consider the commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & A'_n \arrow[r, "i_n"] \arrow[d, "d'_n"] & A_n \arrow[r, "p_n"] \arrow[d, "d_n"] & A''_n \arrow[r] \arrow[d, "d''_n"] & 0 \\ 0 \arrow[r] & A'_{n-1} \arrow[r, "i_{n-1}"] & A_{n-1} \arrow[r, "p_{n-1}"] & A''_{n-1} \arrow[r] & 0 \end{tikzcd}$$

Suppose that $z''_n \in A''_n$ with $d''_n(z''_n) = 0$. Since $Im(p_n) \leq_e A''_n$, there exist $a_n \in A_n$ and $0 \neq r \in R$ such that $p_n(a_n) = rz''_n$. Also, by commutativity of the diagram

$$p_{n-1}d_n(a_n) = d''_n p_n(a_n) = d''_n(rz''_n) = rd''_n(z''_n) = 0.$$

That means $d_n(a_n) \in Ker(p_{n-1})$, as $Im(i_{n-1}) \leq_e Ker(p_{n-1})$ and i_{n-1} is monic, there exist $0 \neq s \in R$ and unique $a'_{n-1} \in A'$ such that $i_{n-1}(a'_{n-1}) = sd_n(a_n)$. Let we lifted z'' to $\bar{a}_n \in A_n$. Similarly, we have a unique $\bar{a}'_{n-1} \in A'_{n-1}$ and $0 \neq \bar{s} \in R$ such that $i(\bar{a}'_{n-1}) = \bar{s}d_n(\bar{a}_n)$. It is clear that

$$p_n(a_n - \bar{a}_n) = p_n(a_n) - p_n(\bar{a}_n) = rz''_n - rz''_n = 0.$$

Then $a_n - \bar{a}_n \in Ker(p_n)$ and by e-exactness of the top row there exists $x'_n \in A'_n$ such that $a_n - \bar{a}_n = d'_n(x'_n) \in B_{n-1}(A')$. Hence the R-morphism defined by

$$Z_n(\mathbf{A}'') \longrightarrow A'_{n-1}/B_{n-1}(\mathbf{A}')$$

is well-defined. Since $Im(d_{n+1}) \subset Ker(d_n)$, then this map sends the element of $B_n(\mathbf{A}'')$ into $B_{n-1}(\mathbf{A}')$ and that $i^{-1}_{n-1}dp^{-1}_n(rz'') = a'_{n-1}$ is a cycle. Therefore the map $\sigma_n : H_n(\mathbf{A}'') \to H_{n-1}(\mathbf{A}')$ is well-defined. □

In the next, we have one of the important theorems of homology we prove with e-exact sequences.

Theorem 3.2 (Long e-exact sequence). *Let*

$$0 \longrightarrow \boldsymbol{A}' \xrightarrow{\ i\ } \boldsymbol{A} \xrightarrow{\ p\ } \boldsymbol{A}'' \longrightarrow 0$$

be an e-exact sequence of complexes. Then there is a long e-exact sequence of R-modules and R-morphisms

$$\cdots \longrightarrow H_n(\boldsymbol{A}') \xrightarrow{i_{n*}} H_n(\boldsymbol{A}) \xrightarrow{p_{n*}} H_n(\boldsymbol{A}'') \xrightarrow{\sigma_n} H_{n-1}(\boldsymbol{A}') \xrightarrow{i_{n-1}*} H_{n-1}(\boldsymbol{A}) \longrightarrow \cdots.$$

Proof. First, to show that $Im(i_{n*}) \leq_e Ker(p_{n*})$. Let x be a non-zero element of $Ker(p_{n*})$. Since $x \in H_n(\mathbf{A})$, so $x = z_n + B_n$, where $z_n \in Ker(d_n)$ and $B_n = Im(d_{n+1})$. That is $p_{n*}(x) = p_{n*}(z_n + B_n) = 0$ and $p_n(z_n) + B''_n = B''_n$. Then $p_n(z_n) \in B''_n$ and $p_n(z_n) = d''_{n+1}(a'')$, where $a'' \in A''_{n+1}$. By assumption, we have $Im(p_{n+1}) \leq_e A''_{n+1}$, so there exist $0 \neq r \in R$ and $a \in A_{n+1}$ such that $p_{n+1}(a) = ra'' \neq 0$. Also,

$$p_n(rz_n) = rp_n(z_n) = rd''_{n+1}(a'') = d''_{n+1}(ra'') = d''_{n+1}p_{n+1}(a) = p_n d_{n+1}(a).$$

It means that $p_n(rz_n - d_{n+1}(a)) = 0$ and $rz_n - d_{n+1}(a) \in Ker(p_n)$. By hypotheses and as $Im(i_n) \cap R(rz_n - d_{n+1}(a)) \neq 0$, then there exist $0 \neq s \in R$ and $a' \in A'_n$ such that $i_n(a') = s(rz_n - d_{n+1}(a)) \neq 0$. By monicness of i_{n-1} we can check that $a' \in Ker(d'_n) = Z'_n$.

$$\begin{aligned} i_{n-1}d'_n(a') = d_n i_n(a') &= d_n(srz_n - sd_{n+1}(a)) \\ &= srd_n(z_n) - sd_n d_{n+1}(a) = 0. \end{aligned}$$

Now we get:

$$\begin{aligned} i_{n*}(a' + B'_n) &= i_n(a') + B_n = srz_n - sd_{n+1}(a) + B_n \\ &= srz_n + B_n = sr(z_n + B_n) = sr(x) \neq 0. \end{aligned}$$

Since $H_n(\mathbf{A})$ has no zero divisors on R, we have $Im(i_{n*}) \cap Rx \neq 0$.

Second, to prove that $Im(p_{n*}) \leq_e Ker(\sigma_n)$. Suppose that x is a non-zero element of $Ker(\sigma_n)$. Then $x = z''_n + B''_n$, where $z''_n \in Ker(d''_n)$ and $B''_n = Im(d''_{n+1})$. That is $\sigma_n(x) = 0$ and $\sigma_n(z''_n + B''_n) = B'_{n-1}$. By Theorem 3.1 we have $i^{-1}_{n-1}d_np^{-1}_n(rz''_n) + B'_{n-1} = B'_{n-1}$, where $r \in R$ and $B'_{n-1} = Im(d'_n)$. Which implies that $i^{-1}_{n-1}d_np^{-1}_n(rz''_n) \in B'_{n-1}$. There exists $a' \in A'_n$ such that $i^{-1}_{n-1}d_np^{-1}_n(rz''_n) = d'_n(a')$, so that $d_np^{-1}_n(rz''_n) = i_{n-1}d'_n(a') = d_ni_n(a')$. Thus $d_n(p^{-1}_n(rz''_n - i_n(a'))) = 0$, and we get $p^{-1}_n(rz''_n - i_n(a')) \in Ker(d_n)$. Therefore

$$p_{n*}(p^{-1}_n(rz''_n) - i_n(a') + B_n) = p_np^{-1}_n(rz''_n) - p_ni_n(a') + B''_n = z''_n + B''_n = x \neq 0.$$

Hence $Im(p_*) \cap Rx \neq 0$.

Finally, to show that $Im(\sigma_n) \leq_e Ker(i_{n-1*})$. Let x be a non-zero element of $Ker(i_{n-1*})$ and $x = z'_{n-1} + B'_{n-1}$, where $z'_{n-1} \in Ker(d_{n-1})$ and $B'_{n-1} = Im(d'_n)$. That is $i_{n-1*}(x) = 0$ and $i_{n-1*}(z'_{n-1} + B'_{n-1}) = B_{n-1}$. Which implies that $i_{n-1}(z'_{n-1}) + B_{n-1} = B_{n-1}$. It means $i_{n-1}(z'_{n-1}) \in B_{n-1}$, where $B_{n-1} = Im(d_n)$. Then there exists $a \in A_n$ such that $i_{n-1}(z'_{n-1}) = d_n(a)$. But $d''_np_n(a) = p_{n-1}d_n(a) = p_{n-1}i_{n-1}(z'_{n-1}) = 0$. From thus, we get $p_n(a) \in Ker(d''_n) = Z''_n$. Therefore $\sigma_n(p_n(a) + B'_n) = i^{-1}_{n-1}d_np^{-1}_np_n(ra) + B' = i^{-1}_{n-1}d_n(ra) + B' = ri^{-1}_{n-1}i_{n-1}(z'_{n-1}) + B' = r(z'_{n-1} + B'_{n-1}) = rx$, where $r \in R$. Hence $Im(\sigma) \cap Rx \neq 0$ and we have $Im(\sigma) \leq_e Ker(i_*)$. □

Remark 3.3 (Naturality of σ with e-exact sequence). Consider the commutative diagram of complexes with e-exact rows:

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{A}' \arrow[r, "i"] \arrow[d, "f"] & \mathbf{A} \arrow[r, "p"] \arrow[d, "g"] & \mathbf{A}'' \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & \mathbf{B}' \arrow[r, "j"] & \mathbf{B} \arrow[r, "q"] & \mathbf{B}'' \arrow[r] & 0 \end{tikzcd}$$

Then there is a commutative diagram of R-modules and R-morphisms with e-exact rows:

$$\begin{tikzcd} \cdots \arrow[r] & H_n(\mathbf{A}') \arrow[r, "i_*"] \arrow[d, "f_*"] & H_n(\mathbf{A}) \arrow[r, "p_*"] \arrow[d, "g_*"] & H_n(\mathbf{A}'') \arrow[r, "\sigma"] \arrow[d, "h_*"] & H_{n-1}(\mathbf{A}') \arrow[r] \arrow[d, "f_*"] & \cdots \\ \cdots \arrow[r] & H_n(\mathbf{B}') \arrow[r, "j_*"] & H_n(\mathbf{B}) \arrow[r, "q_*"] & H_n(\mathbf{B}'') \arrow[r, "\sigma'"] & H_{n-1}(\mathbf{B}') \arrow[r] & \cdots \end{tikzcd}$$

The proof of this is easy. Since by Theorem 3.2 the rows are e-exact. Also, H_n is a functor and using Theorem 3.1 we get each square is commute.

Definition 6. For any diagram of R-modules and R-morphisms, the triangle

$$\begin{tikzcd} A_1 \arrow[r, "f_1"] \arrow[d, "f_2"'] & A_2 \\ B_1 \arrow[ur, "f_3"'] & \end{tikzcd}$$

is e-commute if and only if there exists $0 \neq r \in R$ such that $f_3 \circ f_2 = rf_1$. Also, the diagram

$$\begin{tikzcd} A_1 \arrow[r, "f_1"] \arrow[d, "t_1"] & A_2 \arrow[d, "t_2"] \\ B_1 \arrow[r, "g_1"] & B_2 \end{tikzcd}$$

is e-commute if and only if there exists $0 \neq r \in R$ such that $g_1 \circ t_1 = r(t_2 \circ f_1)$. That is, the diagram is e-commutative if each of its triangles and squares is e-commute.

Definition 7. Let $f, g : \mathbf{A} \longrightarrow \mathbf{B}$ be chain maps. Then f is e-homotopic to g if there are maps $s_n : A_n \longrightarrow B_{n+1}$ and non-zero elements r, p and q in R such that

$$r(f_n - g_n) = p(d'_{n+1}s_n) + q(s_{n-1}d_n) \quad \text{for all } n.$$

Theorem 3.4. *If $f, g : \boldsymbol{A} \longrightarrow \boldsymbol{B}$ are e-homotopic chain maps, then*

$$f_* = g_* : H_n(\boldsymbol{A}) \longrightarrow H_n(\boldsymbol{B})$$

for all integer number n.

Proof. We have to show that $f_*(z_n + B_n(\mathbf{A})) = g_*(z_n + B_n(\mathbf{A}))$ for all $z_n + B_n(\mathbf{A}) \in H_n(\mathbf{A})$. Since $rf(z_n) - rg(z_n) = r(f - g)(z_n)$ and by definition of e-homotopic we have $rf(z_n) - rg(z_n) = p(d'_{n+1}s_n)(z_n) + q(s_{n-1}d_n)(z_n) = d'_{n+1}s_n(pz_n) \in B_n(\mathbf{B})$. Therefore $r(f(z_n) - g(z_n)) + B_n(\mathbf{A}) = B_n(\mathbf{B})$. Hence $f_*(z_n + B_n(\mathbf{A})) = g_*(z_n + B_n(\mathbf{A}))$. $\square$

Theorem 3.5 (Comparison Theorem with e-exact sequence)**.** *Consider the diagram*

$$\begin{tikzcd} \cdots \arrow[r] & X_2 \arrow[r, "d_2"] \arrow[d, dashed] & X_1 \arrow[r, "d_1"] \arrow[d, dashed] & X_0 \arrow[r, "\sigma"] \arrow[d, dashed] & A \arrow[r] \arrow[d, "f"] & 0 \\ \cdots \arrow[r] & X'_2 \arrow[r, "d'_2"] & X'_1 \arrow[r, "d'_1"] & X'_0 \arrow[r, "\sigma'"] & A' \arrow[r] & 0 \end{tikzcd}$$

where the top row is e-projective resolution and the bottom row is e-exact sequences. Then there is a chain map $\bar{f} : X_A \longrightarrow X'{}_{A'}$ (the dashed arrows) making the completed diagram e-commute. Moreover, any two such maps are e-homotopic.

Proof. By induction on n, if $n = 0$, then we have the diagram:

$$\begin{tikzcd} & X_0 \arrow[d, "f\sigma"] & \\ X'_0 \arrow[r, "\sigma'"] & A' \arrow[r] & 0 \end{tikzcd}$$

Since X_0 is e-projective and σ' is e-epic, then there exist $\bar{f}_0 : X_0 \longrightarrow X'_0$ and a non-zero element $r \in R$ such that $\sigma' \bar{f}_0 = rf\sigma$. For the indicative step, consider the diagram:

$$\begin{tikzcd} & X_{n+1} \arrow[d, "\bar{f}_n d_{n+1}"] & \\ X'_{n+1} \arrow[r, "d'_{n+1}"] & Ker(d'_n) \arrow[r] & 0 \end{tikzcd}$$

which is defined when $Im(\bar{f}_n d_{n+1}) \subset Ker(d'_n)$. To prove this, suppose that $x'_n \in Im(\bar{f}_n d_{n+1})$. Then, there exists $x_{n+1} \in X_{n+1}$ such that $\bar{f}_n d_{n+1}(x_{n+1}) = x'_n$. Also, we have

$$d'_n(x'_n) = d'_n \bar{f}_n d_{n+1}(x_{n+1}) = r\bar{f}_{n-1} d_n d_{n+1}(x_{n+1}) = 0.$$

By e-exactness of the bottom row, we have $Im(d'_{n+1}) \leq_e Ker(d'_n)$ and as X_{n+1} is e-projective, there exist $r_n \in R$ and $\bar{f}_{n+1} : X_{n+1} \longrightarrow X'_{n+1}$ such that

$$d'_{n+1} \bar{f}_{n+1} = r_n \bar{f}_n d_{n+1}.$$

Now, to show the uniqueness of $\bar{f}$ up to e-homotopy, suppose that $h : X_A \longrightarrow {X'}_{A'}$ is a second chain map satisfying $\sigma' h_0 = rf\sigma$. We construct an e-homotopy s by induction. We define $s_{-1} : X_{-1} \longrightarrow X'_0$ as the zero map (there is no choice here, because X_{-1} is zero). For the inductive step (and also for s_0). Now, we want to show that $Im(r(h_{n+1} - \bar{f}_{n+1}) - r' p s_n d_{n+1}) \subset Ker(d'_{n+1})$, where r, r', q and p are non-zero elements of R. Then

$$\begin{aligned} & d'_{n+1}(r(h_{n+1} - \bar{f}_{n+1}) - r' p s_n d_{n+1}) \\ &= d'_{n+1} r(h_{n+1} - \bar{f}_{n+1}) - d'_{n+1}(r' p s_n d_{n+1}) \\ &= d'_{n+1} r(h_{n+1} - \bar{f}_{n+1}) - r'(r(h_n - \bar{f}_n) - q s_{n-1} d_n) d_{n+1} \\ &= d'_{n+1} r(h_{n+1} - \bar{f}_{n+1}) - r' r(h_n - \bar{f}_n) d_{n+1} - r' p(s_{n-1} d_n) d_{n+1} \\ &= d'_{n+1} r(h_{n+1} - \bar{f}_{n+1}) - d'_{n+1} r(h_{n+1} - \bar{f}_{n+1}) \\ &= 0, \end{aligned}$$

where $d'_{n+1}(h_{n+1} - \bar{f}_{n+1}) = r'(h_n - \bar{f}_n) d_{n+1}$. Therefore, we have the diagram:

$$\begin{tikzcd} & X_{n+1} \arrow[d, "r(h_{n+1} - \bar{f}_{n+1}) - r' p s_n d_{n+1}"] & \\ X'_{n+2} \arrow[r, "d'_{n+2}"] & Ker(d'_{n+1}) \arrow[r] & 0 \end{tikzcd}$$

Since X_{n+1} is e-projective, there exist a map s_{n+1} and $r'' \in R$ such that

$$d'_{n+2}s_{n+1} = r''(r(h_{n+1} - \bar{f}_{n+1}) - r'ps_nd_{n+1}).$$

Therefore

$$r''r(h_{n+1} - \bar{f}_{n+1}) = d'_{n+2}s_{n+1} + r''r'ps_nd_{n+1}.$$

Hence f and h are e-homotopic. □

The dual of this theorem is true for e-injective resolution.

The Akray and Zebari generalized 3×3 lemma with e-exact rows and columns [1]. In the following, we have a generalize of 3×3 lemma where the diagram is e-commute, the rows and columns are e-exact sequences. We use it to prove the next results.

Theorem 3.6 (3×3 lemma with e-commute)**.** *Consider the e-commutative diagram of R-modules and R-morphisms*:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_1 \arrow[r, "f_1"] \arrow[d, "i_1"] & A_2 \arrow[r, "f_2"] \arrow[d, "j_1"] & A_3 \arrow[r] \arrow[d, "p_1"] & 0 \\
0 \arrow[r] & B_1 \arrow[r, "g_1"] \arrow[d, "i_2"] & B_2 \arrow[r, "g_2"] \arrow[d, "j_2"] & B_3 \arrow[r] \arrow[d, "p_2"] & 0 \\
0 \arrow[r] & C_1 \arrow[r, "h_1"] \arrow[d] & C_2 \arrow[r, "h_2"] \arrow[d] & C_3 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

If the columns and the two bottom rows are e-exact, then the top row is also e-exact.

Proof. To prove that the top row is e-exact we have to check the following three conditions:

(1) Let $a_1 \in Ker(f_1)$. Then $g_1i_1(a_1) = rj_1f_1(a_1) = 0$, where $r \in R$ and so $i_1(a_1) \in Ker(g_1)$. Since g_1 and i_1 are monic, then the result follows that $a_1 = 0$. Therefore $Ker(f_1) = 0$.

(2) To prove that $Im(f_1) \leq_e Ker(f_2)$. We must to show that $Im(f_1) \subseteq Ker(f_2)$. Let $a_2 \in Im(f_1)$. Then, there exists $a_1 \in A_1$ such that $f_1(a_1) = a_2$ and by e-commutative of the diagram there exists $0 \neq r \in R$ such that $g_1i_1(a_1) = rj_1f_1(a_1) = rj_1(a_2)$, which implies that $rj_1(a_2) \in Im(g_1) \subseteq Ker(g_2)$ and $0 = rg_2j_1(a_2) = rr'p_1f_2(a_2)$, where $r' \in R$. Hence $rr'f_2(a_2) \in Ker(p_1) = 0$ and $Ker(p_1)$ is a torsion-free module. Therefore $f_2(a_2) = 0$. Now to prove that $Im(f_1)$ is an essential submodule of $Ker(f_2)$. Let a_2 be a non-zero element of $Ker(f_2)$. Then $g_2j_1(a_2) = rp_1f_2(a_2) = 0$ with $r \in R$. So $j_1(a_2) \in$

$Ker(g_2)$ and as $Im(g_1) \leq_e Ker(g_2)$, there exist $0 \neq s \in R$ and $b_1 \in B_1$ such that $g_1(b_1) = sj_1(a_2)$. Also $h_1 i_2(b_1) = r' j_2 g_1(b_1) = r' s j_2 j_1(a_2) = 0$, where $r' \in R$. Thus $i_2(b_1) \in Ker(h_1) = 0$. By e-exactness of first column we have $Im(i_1) \cap Rb_1 \neq 0$. Then, there exist $a_1 \in A_1$ and $k \in R$ such that $i_1(a_1) = kb_1$ and by e-commutative of the diagram $ksj_1(a_2) = kg_1(b_1) = g_1(kb_1) = g_1 i_1(a_1) = r'' j_1 f_1(a_1)$ with $r'' \in R$. Thus $j_1(ksa_2 - r'' f_1(a_1)) = 0$ and as $Ker(j_1) = 0$, Then $ksa_2 = f_1(r'' a_1)$. Therefore we have $Im(f_1) \leq_e Ker(f_2)$.

(3) Let a_3 be a non-zero element of A_3. So there exist $b_2 \in B_2$ and $s \in R$ such that $g_2(b_2) = sp_1(a_3)$. By e-commutative of the diagram there exists $r \in R$ with $h_2 j_2(b_2) = rp_2 g_2(b_2) = rsp_2 p_1(a_3) = 0$. Which implies that $j_2(b_2) \in Ker(h_2)$ and by e-exactness of the bottom row, there exist $c_1 \in C_1$ and $n \in R$ such that $h_1(c_1) = nj_2(b_2)$. Again by e-exactness of the first column there exist $m \in R$ and $b_1 \in B_1$ such that $i_2(b_1) = mc_1$. Using the e-commutativity, we have $mnj_2(b_2) = mh_1(c_1) = h_1(mc_1) = h_1 i_2(b_1) = r' j_2 g_1(b_1)$, where $r' \in R$. Therefore $j_2(r' g_1(b_1) - mnb_2) = 0$, so $r' g_1(b_1) - mnb_2 \in Ker(j_2)$. Since $Im(j_1) \leq_e Ker(j_2)$, there exist $a_2 \in A_2$ and $0 \neq k \in R$ such that $j_1(a_2) = k(r' g_1(b_1) - mnb_2)$. By hypotheses and e-commute of the diagram, we have

$$\begin{aligned} -kmnsp_1(a_3) = -kmng_2(b_2) &= kr' g_2 g_1(b_1) - kmng_2(b_2) \\ &= g_2(k(r' g_1(b_1) - mnb_2)) \\ &= g_2 j_1(a_2) = r'' p_1 f_2(a_2). \end{aligned}$$

This is equivalent to $p_1(r'' f(a_2) + kmns(a_3)) = 0$. But p_1 is monic, so $f(r'' a_2) = -kmns(a_3)$. Hence $Im(f_2) \leq_e A_3$. $\square$

Theorem 3.7 (Horseshoe lemma with e-exact sequence)**.** *Consider the diagram of R-modules and R-morphisms*:

$$\begin{tikzcd}
 & {} \arrow[d] & & {} \arrow[d] & \\
 & P_1' \arrow[d, "d_1'"'] & & P_1'' \arrow[d, "d_1''"] & \\
 & P_0' \arrow[d, "\sigma'"'] & & P_0'' \arrow[d, "\sigma''"] & \\
0 \arrow[r] & A' \arrow[r, "i"'] \arrow[d] & A \arrow[r, "p"'] & A'' \arrow[r] \arrow[d] & 0 \\
 & 0 & & 0 &
\end{tikzcd}$$

where the columns are e-projective resolutions and the row is e-exact. Then there exist an e-projective resolution of A and a chain R-maps so that the columns form an e-exact sequence of complexes.

Proof. By induction on n it suffices to complete 3×3-diagram. Consider the diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & \\
 & K'_0 \arrow[d] & & K''_0 \arrow[d] & \\
 & P'_0 \arrow[d, "\sigma'"'] & & P''_0 \arrow[d, "\sigma''"] & \\
0 \arrow[r] & A' \arrow[r, "i"'] \arrow[d] & A \arrow[r, "p"'] & A'' \arrow[r] \arrow[d] & 0 \\
 & 0 & & 0 &
\end{tikzcd}$$

where $K'_0 = Ker(\sigma')$ and $K''_0 = Ker(\sigma'')$. Then the rows and columns are e-exacts. Since P'_0 and P''_0 are e-projective, we define $P_0 = P'_0 \oplus P''_0$, $i_0 : P'_0 \longrightarrow P_0$ by $x' \mapsto (x', 0)$, and $p_0 : P_0 \longrightarrow P''_0$ by $(x', x'') \mapsto x''$. Since P_0 is sum of two e-projective, then it is also e-projective. It is clear

$$0 \longrightarrow P'_0 \xrightarrow{i_0} P_0 \xrightarrow{p_0} P''_0 \longrightarrow 0$$

is an e-exact sequence. Since P''_0 is e-projective, there exist a map $h : P''_0 \longrightarrow A$ and $r \in R$ such that $ph = r\sigma''$. Now we define $\sigma : P_0 \longrightarrow A$ by

$$\sigma : (x', x'') \mapsto i\sigma'(x') + h(x'').$$

We take $K_0 = Ker(\sigma)$, then the diagram is e-commute and columns are e-exact. Also the two bottom rows are e-exact. Then by Theorem 3.6 the top row is also e-exact. $\square$

The dual of Theorem 3.7 is true for e-injective resolution.

4. E-derived functors

Let T be a functor between categories of R-modules. In this section, we want to describe its left and right e-derived functors on the e-projective and e-injective resolutions.

Definition 8. For each R-module A, its left e-derived functors are defined by

$$(L_nT)A = H_n(T\mathbf{P}_A) = Ker(Td_n)/Im(Td_{n+1}),$$

where

$$\mathbf{P} \ : \ \cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \longrightarrow A \longrightarrow 0$$

is the e-projective resolution of A chosen once for all.

Definition 9. If T is a covariant functor, its right e-derived functors R^nT are defined on an R-module A by

$$(R^nT)A = H^n(T\mathbf{E}_A) = Ker(Td^n)/Im(Td^{n-1}),$$

where

$$\mathbf{E} \ : \ 0 \longrightarrow A \longrightarrow E^0 \xrightarrow{d_0} E^1 \xrightarrow{d_1} E^2 \longrightarrow \cdots$$

is the e-injective resolution of A chosen once for all. Also, we used a convenient notation for the e-injective resolution.

$$(R^nT)A = H_{-n}(T\mathbf{E}_A) = Ker(Td_{-n})/Im(Td_{-n+1}).$$

Definition 10. If T is a contravarint functor, then

$$(R^nT)A = Ker(Td_{n+1})/Im(Td_n),$$

where

$$\mathbf{P} \ : \ \cdots \ \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \longrightarrow A \longrightarrow 0$$

is the e-projective resolution of A chosen once for all.

We need the next results to make the definitions of left and right e-derived functors well-defined. We prove these definitions are independent of the choice of e-projective resolutions and e-injective resolutions.

Theorem 4.1. *For a functor T, left e-derived functors are additive functors for every n.*

Proof. Let $f : A \longrightarrow B$ be a map. Then by Theorem 3.5 there is a chain map $\bar{f} : \mathbf{P}_A \longrightarrow \mathbf{P}_B$, where $\mathbf{P}_A$ and $\mathbf{P}_B$ are deleted complexes of e-projective resolutions for A and B, respectively. To prove that $(L_nT)f$ is well-defined for all integer numbers n. Suppose that $h : \mathbf{P}_A \longrightarrow \mathbf{P}_B$ is a second chain map over f. Then by Theorem 3.5 again we have $\bar{f}$ and h are e-homotopic, and $T\bar{f}$ and Th are also e-homotopic. By Theorem 3.4 we can say that $(T\bar{f})_* = (Th)_*$. Now, to prove that L_nT is additive functor for all n. We have $(L_nT)(f+g) = H_n(T(f+g)) = H_n(Tf+Tg) = H_n(Tf) + H_n(Tg) = (L_nT)f + (L_nT)g$. Therefore left e-derived functors are additive functors for every n. □

Theorem 4.2. *For any functor T, the left e-derived functors L_nT and $\hat{L}_nT$ are naturally equivalent. In particular for each A,*

$$(L_nT)A \cong (\hat{L}_nT)A.$$

Proof. Let

$$\mathbf{P} \ : \ \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \ \longrightarrow 0$$

be an e-projective resolution for A that used to define L_nT. Let

$$\hat{\mathbf{P}} \ : \ \cdots \longrightarrow \hat{P}_2 \longrightarrow \hat{P}_1 \longrightarrow \hat{P}_0 \longrightarrow A \ \longrightarrow 0$$

be another e-projective resolution for A. It used to define $\hat{L}_nT$. Consider the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r, "\sigma"] & A \arrow[r] \arrow[d, "1_A"] & 0 \\
\cdots \arrow[r] & \hat{P}_2 \arrow[r] & \hat{P}_1 \arrow[r] & \hat{P}_0 \arrow[r, "\hat{\sigma}"] & A \arrow[r] & 0
\end{tikzcd}$$

where $1_A : A \longrightarrow A$ is the identity map. By Theorem 3.5 there is a chain map $i : \mathbf{P}_A \longrightarrow \hat{\mathbf{P}}_A$ over 1_A which is unique up to e-homotopy. When we applying a functor T gives a chain map $Ti : T\mathbf{P}_A \longrightarrow T\hat{\mathbf{P}}_A$ over 1_{TA}. Now, we symbolize this chain map to define

$$\tau_A = (Ti)_* : (L_nT)A \longrightarrow (\hat{L}_nT)A.$$

To prove that τ_A is an isomorphism, consider the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \hat{P}_2 \arrow[r] & \hat{P}_1 \arrow[r] & \hat{P}_0 \arrow[r, "\hat{\sigma}"] & A \arrow[r] \arrow[d, "1_A"] & 0 \\
\cdots \arrow[r] & P_2 \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r, "\sigma"] & A \arrow[r] & 0
\end{tikzcd}$$

and Theorem 3.5 gives a chain map $j : \hat{\mathbf{P}}_A \longrightarrow \mathbf{P}_A$ over 1_A. The composite map $ji : \mathbf{P}_A \longrightarrow \mathbf{P}_A$ and the identity chain map $1_{\mathbf{P}} : \mathbf{P}_A \longrightarrow \mathbf{P}_A$ are also chain maps over 1_A. By Theorem 3.5 ji and $1_{\mathbf{P}}$ are e-homotopic. Therefore $j_*i_* = 1$ and similarly, we have $i_*j_* = 1$. That is i_* is an isomorphism, and so is $\tau_A = (Ti)_*$.

Let $f : A \longrightarrow B$ be a map. For natural transformation of τ_A we have to prove that the diagram

$$\begin{tikzcd}
(L_nT)A \arrow[r, "\tau_A"] \arrow[d, "Tf"'] & (\hat{L}_nT)A \arrow[d, "Tf"] \\
(L_nT)B \arrow[r, "\tau_B"'] & (\hat{L}_nT)B
\end{tikzcd}$$

is commutative. From $(L_nT)A$ with clockwise direction, consider the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r, "\sigma"] & A \arrow[r] \arrow[d, "1_A"] & 0 \\
\cdots \arrow[r] & \hat{P}_2 \arrow[r] & \hat{P}_1 \arrow[r] & \hat{P}_0 \arrow[r, "\hat{\sigma}"] & A \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r] & \hat{Q}_2 \arrow[r] & \hat{Q}_1 \arrow[r] & \hat{Q}_0 \arrow[r, "\sigma"] & B \arrow[r] & 0
\end{tikzcd}$$

by Theorem 3.5 there exists a chain map $\mathbf{P}_A \longrightarrow \hat{\mathbf{Q}}_B$ over $f \circ 1_A = f$. Similarly for counterclockwise direction, we have a chain map $\mathbf{P}_A \longrightarrow \hat{\mathbf{Q}}_B$ over $1_B \circ f =$

f and by Theorem 3.5 these chain maps are e-homotopic. When we apply a functor T, we get e-homotopic chain maps $T\mathbf{P}_A \longrightarrow T\hat{\mathbf{Q}}_B$ over Tf. By Theorem 3.4 these two induced maps are equal. □

Theorem 4.3. *If T is covariant, each right e-derived functor R^nT is an additive functor whose definition is independent of the choice of e-injective resolutions. Similarly, if T is contravariant, then each right e-derived R^nT is an additive contravariant functor whose definition is independent of the choice of e-projective resolutions.*

Proof. The proof is dual to the proof of Theorem 4.2. □

Theorem 4.4. *Let $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ be an e-exact sequence of R-modules. If T is a covariant functor, there is an e-exact sequence of R-modules*

$$\cdots \longrightarrow (L_nT)A' \longrightarrow (L_nT)A \longrightarrow (L_nT)A'' \xrightarrow{\sigma} (L_{n-1}T)A'' \longrightarrow$$

$$\cdots \longrightarrow (L_0T)A' \longrightarrow (L_0T)A \longrightarrow (L_0T)A'' \longrightarrow 0.$$

Proof. Assume that

$$\cdots \longrightarrow P_2' \longrightarrow P_1' \longrightarrow P_0' \longrightarrow A' \longrightarrow 0$$

and

$$\cdots \longrightarrow P_2'' \longrightarrow P_1'' \longrightarrow P_0'' \longrightarrow A'' \longrightarrow 0$$

be e-projective resolutions for A' and A'', respectively. Then by Theorem 3.7 we can construct an e-projective resolution

$$\cdots \longrightarrow \hat{P}_1 \longrightarrow \hat{P}_0 \longrightarrow A \longrightarrow 0$$

for A. The e-exact sequence of deleting complexes is

$$0 \longrightarrow \mathbf{P}'_{A'} \longrightarrow \hat{\mathbf{P}}_A \longrightarrow \mathbf{P}''_{A''} \longrightarrow 0$$

and when, we applying covariant functor T we get

$$0 \longrightarrow T\mathbf{P}'_{A'} \longrightarrow T\hat{\mathbf{P}}_A \longrightarrow T\mathbf{P}''_{A''} \longrightarrow 0.$$

By Theorem 3.3 we have an e-exact sequence

$$\cdots \longrightarrow H_n(T\mathbf{P}'_{A'}) \longrightarrow H_n(T\hat{\mathbf{P}}_A) \longrightarrow H_n(T\mathbf{P}''_{A''}) \xrightarrow{\sigma} H_{n-1}(T\mathbf{P}'_{A'}) \longrightarrow \cdots$$

and by the definition of left e-derived functor, we have

$$\cdots \longrightarrow (L_nT)A' \longrightarrow (\hat{L}_nT)A \longrightarrow (L_nT)A'' \xrightarrow{\sigma} L_{n-1}TA' \longrightarrow \cdots .$$

We write $(\hat{L}_nT)A$ instated of $(L_nT)A$, because the e-projective resolution of A constructed with Theorem 3.7 and which is not the originally e-projective reticulation. But Theorem 4.2 is saying both of them are equal and we get

$$\cdots \longrightarrow (L_nT)A' \longrightarrow (L_nT)A \longrightarrow (L_nT)A'' \xrightarrow{\sigma} (L_{n-1}T)A' \longrightarrow \cdots .$$

Also we know that $(L_nT)A' = 0$ for all negative integer number n. Therefore

$$\cdots \longrightarrow (L_nT)A' \longrightarrow (L_nT)A \longrightarrow (L_nT)A'' \xrightarrow{\sigma} (L_{n-1}T)A'' \longrightarrow$$

$$\cdots \longrightarrow (L_0T)A' \longrightarrow (L_0T)A \longrightarrow (L_0T)A'' \longrightarrow 0. \qquad \square$$

Theorem 4.5. *Let* $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ *be an e-exact sequence of R-modules. If T is a covariant functor, there is an e-exact sequence of R-modules*

$$0 \longrightarrow (R^0T)A' \longrightarrow (R^0T)A \longrightarrow (R^0T)A'' \xrightarrow{\sigma} (R^1T)A'' \longrightarrow$$

$$\cdots \longrightarrow (R^nT)A' \longrightarrow (R^nT)A \longrightarrow (R^nT)A'' \longrightarrow \cdots .$$

Proof. Suppose that

$$0 \longrightarrow A' \longrightarrow E'_0 \longrightarrow E'_1 \longrightarrow \cdots \longrightarrow E'_n \longrightarrow E'_{n+1} \longrightarrow \cdots$$

and

$$0 \longrightarrow A'' \longrightarrow E''_0 \longrightarrow E''_1 \longrightarrow \cdots \longrightarrow E''_n \longrightarrow E''_{n+1} \longrightarrow \cdots$$

are e-injective resolutions for A' and A'', respectively. Then by dual of Theorem 3.7 we have

$$0 \longrightarrow A \longrightarrow \hat{E}_0 \longrightarrow \hat{E}_1 \longrightarrow \cdots \longrightarrow \hat{E}_n \longrightarrow \hat{E}_{n+1} \longrightarrow \cdots$$

an e-injective resolution for A such that

$$0 \longrightarrow \mathbf{E}'_{A'} \longrightarrow \hat{\mathbf{E}}_A \longrightarrow \mathbf{E}''_{A''} \longrightarrow 0$$

is the e-exact sequence of deleting complexes. When we applying a covariant functor T we get

$$0 \longrightarrow T\mathbf{E}'_{A'} \longrightarrow T\hat{\mathbf{E}}_A \longrightarrow T\mathbf{E}''_{A''} \longrightarrow 0$$

and by Theorem 3.3 we have the e-exact sequence

$$\cdots \longrightarrow H_n(T\mathbf{E}'_{A'}) \longrightarrow H_n(T\hat{\mathbf{E}}_A) \longrightarrow H_n(T\mathbf{E}''_{A''}) \xrightarrow{\sigma} H_{n-1}(T\mathbf{E}'_{A'}) \longrightarrow \cdots .$$

Then by the definition of right e-derived e-exact functor, we have

$$\cdots \longrightarrow (R^nT)A' \longrightarrow (\hat{R}^nT)A \longrightarrow (R^nT)A'' \xrightarrow{\sigma} (R^{n-1}T)A' \longrightarrow \cdots .$$

By Theorem 4.3 for all right e-derived functor we have $(\hat{R}T)A = (RT)A$. So, we have

$$\cdots \longrightarrow (R^nT)A' \longrightarrow (R^nT)A \longrightarrow (R^nT)A'' \xrightarrow{\sigma} (R^{n-1}T)A' \longrightarrow \cdots$$

and since $(R^nT)A = 0$ for all negative integer n, we get the e-exact sequence

$$0 \longrightarrow (R^0T)A' \longrightarrow (R^0T)A \longrightarrow (R^0T)A'' \xrightarrow{\sigma} (R^1T)A'' \longrightarrow$$

$$\cdots \longrightarrow (R^nT)A' \longrightarrow (R^nT)A \longrightarrow (R^nT)A'' \longrightarrow \cdots .$$ □

5. Functors e-Tor and e-Ext

In this section, we generalize two special derived functors Tor and Ext and discuss some properties of them.

Recall that, for each R-module A and B we have

$$e - Ext^n(A, B) = H_{-n}(\mathrm{Hom}(A, \mathbf{E}_B)),$$

where $\mathbf{E}_B$ is a deleted e-injective resolution of B and

$$e - Ext^n(A, B) = H_{-n}(\mathrm{Hom}(\mathbf{P}_A, B)),$$

where $\mathbf{P}_A$ is a deleted e-projective resolution of A. Also,

$$e - Tor_n^R(A, B) = H_n(\mathbf{P}_A \otimes B) = H_n(A \otimes \mathbf{Q}_B),$$

where $\mathbf{P}_A$ and $\mathbf{Q}_B$ are deleted e-projective resolutions of A and B, respectively.

Theorem 5.1. *If n is a negative integer, then $e - Ext(A, B) = 0$ for all R-module A and B.*

Proof. Suppose that

$$\mathbf{P} \;:\quad \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

is an e-projective resolution for A. Then the deleted complex of A is

$$\mathbf{P}_A \;:\quad \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0 .$$

When we apply $\mathrm{Hom}(\;, B)$ on the deleted complex we get

$$0 \longrightarrow \mathrm{Hom}(P_0, B) \longrightarrow \mathrm{Hom}(P_1, B) \longrightarrow \mathrm{Hom}(P_2, B) \longrightarrow \cdots$$

which implies that $\mathrm{Hom}(P_n, B) = 0$ for all negative integer number n. Hence $e - Ext(A, B) = 0$ for all negative integer number n. □

Theorem 5.2. *If n is a negative integer, then $e - Tor(A, B) = 0$ for all R-modules A and B.*

Proof. Let

$$\mathbf{P}\ :\quad \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow B \longrightarrow 0$$

be an e-projective resolution for B. The deleted complex of B is

$$\mathbf{P}_B\ :\quad \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0.$$

When we apply $A\otimes$ on the above deleted complex we get

$$\cdots \longrightarrow A\otimes P_2 \longrightarrow A\otimes P_1 \longrightarrow A\otimes P_0 \longrightarrow 0.$$

Since $H_n(A\otimes P_B) = 0$ for all negative integer number n. Therefore $e-Tor(A,B) = 0$ for all negative integer number n. ☐

Theorem 5.3. *If $n = 0$, then $e - Ext^n(A,\ ) \cong \mathrm{Hom}(A,\ )$.*

Proof. Let

$$\mathbf{E}\ :\quad 0 \longrightarrow B \xrightarrow{\sigma} E_0 \xrightarrow{d_0} E_{-1} \xrightarrow{d_{-1}} E_{-2} \longrightarrow \cdots$$

be an e-injective resolution for B and

$$\mathbf{E}_B\ :\quad 0 \xrightarrow{d_1} E_0 \xrightarrow{d_0} E_{-1} \xrightarrow{d_{-1}} E_{-2} \longrightarrow \cdots$$

is an deleted e-injective resolution for B and also when we apply $\mathrm{Hom}(A,\ )$ on the e-injective resolution we get

$$0 \longrightarrow \mathrm{Hom}(A,B) \xrightarrow{\sigma^*} \mathrm{Hom}(A,E_0) \xrightarrow{d_0^*} \mathrm{Hom}(A,E_{-1}) \longrightarrow \cdots\ .$$

Then $e - Ext^0(A,B) = H_0(\mathrm{Hom}(A,\mathbf{E}_B)) = Ker(d_0^*)/Im(d_1^*) = Ker(d_0^*)$. We define $\sigma^* : \mathrm{Hom}(A,E_0) \longrightarrow Ker(d_0^*)$. Since $Im(\sigma^*) \leq_e Ker(d_0^*)$, σ^* is well-defined and since $\mathrm{Hom}(A,\ )$ is a left e-exact functor, then σ^* is monic. Now, we want to prove that σ^* is epic. Let $f \in Ker(d_0^*)$. Then $0 = d_0^*(f) = d_0(f) = d_0(f(a))$ for all $a \in A$ therefore $f(a) \in Ker(d_0)$. By e-exactness of e-injective resolution we have $Im(\sigma) \leq_e Ker(d_0)$, so there exist $b \in B$ and $0 \neq r \in R$ such that $\sigma(b) = rf(a)$. We define $g : A \longrightarrow B$ by $rg(a) = b$ for fixed $r \in R$. Let $a_1, a_2 \in A$ and $a_1 = a_2$. Then $rf(a_1) = rf(a_2)$ which means that $\sigma(b) = \sigma(b')$ and by monicness of σ we have $b = b'$. Hence $rg(a_1) = rg(a_2)$ and g is well-defined. Now, we have $rf(a) = \sigma(b) = \sigma(rg(a)) = r\sigma(g(a))$ which is equivalent to $\sigma^*(g) = f$. Hence σ^* is an isomorphism and since $e-Ext^0(A,B) = Ker(d_0^*)$. Therefore $e - Ext^0(A,\ )$ is isomorphic to $\mathrm{Hom}(A,\ )$. ☐

Question 2. One can use the definition of e-exact sequences and their application in homology to redefine the cohomology, using the e-derived functors to discuss all five generalizations of cohomology study properties.

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