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A QUESTION ABOUT MAXIMAL NON ϕ-CHAINED SUBRINGS

Atul Gaur and Rahul Kumar

Abstract. Let $\mathcal{H}_0$ be the set of rings R such that $Nil(R) = Z(R)$ is a divided prime ideal of R. The concept of maximal non ϕ-chained subrings is a generalization of maximal non valuation subrings from domains to rings in $\mathcal{H}_0$. This generalization was introduced in [20] where the authors proved that if $R \in \mathcal{H}_0$ is an integrally closed ring with finite Krull dimension, then R is a maximal non ϕ-chained subring of $T(R)$ if and only if R is not local and $|[R, T(R)]| = \dim(R) + 3$. This motivates us to investigate the other natural numbers n for which R is a maximal non ϕ-chained subring of some overring S. The existence of such an overring S of R is shown for $3 \leq n \leq 6$, and no such overring exists for $n = 7$.

1. Introduction

This paper can be seen as a sequel to [20]. All rings considered below are commutative with nonzero identity and all ring extensions are unital. If R is a ring, then R is local if R has a unique maximal ideal. Also, $T(R)$ denotes the total quotient ring of R, $Nil(R)$ the set of all nilpotent elements of R, and $Z(R)$ the set of all zero-divisors of R. A ring is said to be integrally closed if it is integrally closed in its total quotient ring. Recall from [7] that a prime ideal Q of a ring R is called a divided prime ideal if Q is comparable to every ideal of R. Let $\mathcal{H}_0$ denote the set of all rings R such that $Nil(R)$ is a divided prime ideal of R with $Nil(R) = Z(R)$. This class of rings were studied by Badawi et al. in [1, 2, 8–16]. We also worked on this class in [23].

For a ring extension $R \subset T$, R is said to be a maximal non-$\mathcal{P}$ subring of T (where $\mathcal{P}$ is a ring-theoretic property) if R does not satisfy $\mathcal{P}$ but each subring of T which properly contains R satisfies $\mathcal{P}$. Recently studied properties are

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$\mathcal{P}$:= valuation domain, Noetherian domain, ACCP domain, Jaffard domain, universally catenarian domain and λ-domain, see [4, 5, 17, 22, 24, 25].

Let $\mathcal{H}$ denote the set of all rings R such that $Nil(R)$ is a divided prime ideal of R. If $R \in \mathcal{H}$, then Badawi [8] defined a ring homomorphism $\phi : T(R) \longrightarrow R_{Nil(R)}$ given by $\phi(r/s) = r/s$, where $r \in R$ and $s \in R \setminus Z(R)$, and ϕ restricted to R is also a ring homomorphism given by $\phi(r) = r/1$, where $r \in R$. A ring R is said to be a Prüfer ring if each finitely generated regular ideal of R is invertible, see [21]. A ring $R \in \mathcal{H}$ is said to be a ϕ-Prüfer ring if $\phi(R)$ is a Prüfer ring, see [1]. Recall from [10] that a ring $R \in \mathcal{H}$ is said to be a ϕ-chained ring if for each $x \in R_{Nil(R)} \setminus \phi(R)$, we have $x^{-1} \in \phi(R)$.

For a ring extension $R \subset S$, $[R, S] = \{T \mid R \subseteq T \subseteq S, T \text{ is a subring of } S\}$. For an extension $R \subset S$ of integral domains, R is a maximal non valuation subring of S [18] if R is not a valuation domain but each $T \in [R, S] \setminus \{R\}$ is a valuation domain. In [20], we generalized the concept of maximal non valuation subrings to the maximal non chained subrings and maximal non ϕ-chained subrings. A ring R is said to be a maximal non ϕ-chained subring of S if R is not a ϕ-chained ring but every $T \in [R, S] \setminus \{R\}$ is a ϕ-chained ring. This paper can also be seen as a sequel of [22] as all the results of [22] are extended to rings in $\mathcal{H}_0$. As usual, $|X|$ denotes the cardinality of a set X. If R is a ring, then Spec(R) denotes the set of all prime ideals of R, Max(R) denotes the set of all maximal ideals of R, and dim(R) refers to the Krull dimension of R.

We now recall some results on ϕ-rings which are already in literature and are frequently used in this paper. Note that the first five results are from [8] whereas as the last one is from [2]. Let $R \in \mathcal{H}$. Then

(A) $\phi(R) \in \mathcal{H}_0$.
(B) $Ker(\phi) \subseteq Nil(R)$.
(C) $Nil(T(R)) = Nil(R)$.
(D) $Nil(R_{Nil(R)}) = \phi(Nil(R)) = Nil(\phi(R)) = Z(\phi(R))$.
(E) $T(\phi(R)) = R_{Nil(R)}$ is a local ring with maximal ideal $Nil(\phi(R))$, and $R_{Nil(R)}/Nil(\phi(R)) = T(\phi(R))/Nil(\phi(R)) = T(\phi(R)/Nil(\phi(R)))$.
(F) $(R/Nil(R))' = R'/Nil(R)$ provided $R \in \mathcal{H}_0$.

2. Results

Throughout this paper we are assuming that $\mathcal{H}_1$ is the set of all rings R in $\mathcal{H}_0$ such that $|[R, T(R)]|$ is finite. Let $R \in \mathcal{H}_1$. Then dim(R) is finite as dim$(R) < |[R, T(R)]|$. Thus,

$$(*) \qquad |[R, T(R)]| = \dim(R) + n$$

for some $n \in \mathbb{N}$. In the first result we give a necessary condition and a sufficient condition for $n \geq 3$. Note that this can be seen as a generalization of [22, Proposition 2].

Proposition 2.1. *Let $R \in \mathcal{H}_1$ and $|[R, T(R)]| = \dim(R) + n$. Then the following hold*:

(i) *If R is not local, then $n \geq 3$.*
(ii) *If R is integrally closed and $n \geq 3$, then R is not local.*

Proof. Since $R \in \mathcal{H}_1$, $R/Nil(R)$ is a finite dimensional integral domain. Also, we have $T(R/Nil(R)) = T(R)/Nil(R)$ by (E). It follows that

$$|[R/Nil(R), T(R/Nil(R))]| = |[R, T(R)]| = \dim(R/Nil(R)) + n.$$

(i) If R is not local, then $R/Nil(R)$ is not local. Thus, by [22, Proposition 2], $n \geq 3$.
(ii) Let R be integrally closed and $n \geq 3$. Then $R/Nil(R)$ is integrally closed by (F). Therefore, by [22, Proposition 2], $R/Nil(R)$ is not local and thus R is not local. □

If we take $n = 3$ or 4 in $(*)$, then we have the following generalization of [22, Lemma 1].

Proposition 2.2. *Let $R \in \mathcal{H}_1$ be such that either $|[R, T(R)]| = \dim(R) + 3$ or $|[R, T(R)]| = \dim(R) + 4$. Then R is integrally closed if and only if R is not local.*

Proof. Note that by (E), we have $R/Nil(R)$ is a finite dimensional domain such that either $|[R/Nil(R), T(R/Nil(R))]| = |[R, T(R)]| = \dim(R/Nil(R)) + 3$ or $|[R/Nil(R), T(R/Nil(R))]| = |[R, T(R)]| = \dim(R/Nil(R)) + 4$. Now, if R is integrally closed, then R is not local by Proposition 2.1. Conversely, assume that R is not local. Then $R/Nil(R)$ is not local. Thus, by [22, Lemma 1], $R/Nil(R)$ is integrally closed. Hence, by (F), R is integrally closed. □

An integral domain R is said to be a treed domain if incomparable prime ideals of R are coprime, see [19]. We say that a ring $R \in \mathcal{H}$ is a ϕ-treed ring if $\phi(R)$ is a treed ring, that is, incomparable prime ideals of $\phi(R)$ are coprime.

Proposition 2.3. *Let $R \in \mathcal{H}$. Then R is a ϕ-treed ring if and only if $R/Nil(R)$ is a treed domain.*

Proof. Let R be a ϕ-treed ring. Then $\phi(R)$ is a treed ring in $\mathcal{H}_0$ by (A). We claim that $\phi(R)/Nil(\phi(R))$ is a treed domain. Let P, Q be incomparable prime ideals of $\phi(R)/Nil(\phi(R))$. Then $P = \phi(\mathfrak{p})/Nil(\phi(R))$ and $Q = \phi(\mathfrak{q})/Nil(\phi(R))$ for some incomparable prime ideals $\mathfrak{p}, \mathfrak{q}$ of R. Since $\phi(R)$ is a treed ring, $\phi(\mathfrak{p}) + \phi(\mathfrak{q}) = \phi(R)$. It follows that $P + Q = \phi(R)/Nil(\phi(R))$. Thus, our claim holds. Note that $Nil(\phi(R)) = \phi(Nil(R))$ by (D). It follows that $R/Nil(R)$ is a treed domain, by [1, Lemma 2.5].

Conversely, assume that $R/Nil(R)$ is a treed domain. Then $\phi(R)/Nil(\phi(R))$ is a treed domain by (D) and [1, Lemma 2.5]. Let P, Q be incomparable prime ideals of $\phi(R)$. Then $P/Nil(\phi(R))$ and $Q/Nil(\phi(R))$ are incomparable and so $P/Nil(\phi(R)) + Q/Nil(\phi(R)) = \phi(R)/Nil(\phi(R))$. Consequently, $P + Q = \phi(R)$. Thus, $\phi(R)$ is a treed ring, that is, R is a ϕ-treed ring. □

If $n \leq 6$ in $(*)$, then we have the following generalization of [22, Lemma 2].

Proposition 2.4. *Let $R \in \mathcal{H}_1$ be such that $|[R, T(R)]| \leq \dim(R) + 6$. Then the following hold*:

(i) $|Max(R)| \leq 2$.
(ii) *If R is a non local ϕ-treed ring, then $Max(R) = \{M, N\}$ and $Spec(R) = \{Nil(R) = P_0 \subset P_1 \subset \cdots \subset P_r = M, N\}$, where $r = \dim(R)$.*

Proof. Note that by (E), we have

$$|[R/Nil(R), T(R/Nil(R))]| = |[R, T(R)]| \leq \dim(R/Nil(R)) + 6.$$

Thus, by [22, Lemma 2], $|\mathrm{Max}(R/Nil(R))| \leq 2$ and so $|\mathrm{Max}(R)| \leq 2$.

Now, suppose that R is a non local ϕ-treed ring. Then by Proposition 2.3, $R/Nil(R)$ is a non local treed domain. Again by [22, Lemma 2],

$$\mathrm{Max}(R/Nil(R)) = \{M/Nil(R), N/Nil(R)\} \text{ and}$$

$$\mathrm{Spec}(R/Nil(R)) = \{(0) \subset P_1/Nil(R) \subset \cdots \subset P_r/Nil(R) = M/Nil(R), N/Nil(R)\},$$

where $r = \dim(R/Nil(R))$. Thus, the result holds. □

If R is a ring and M is an R-module, then Nagata defined the idealization $R(+)M$ (see [26, cf. Nagata, 1962, p. 2]) as follows: its additive structure is that of the abelian group $R \oplus M$, and multiplication is defined by $(r_1, m_1)(r_2, m_2) := (r_1 r_2, r_1 m_2 + r_2 m_1)$ for all $r_1, r_2 \in R$ and $m_1, m_2 \in M$. For further study on idealization, see [3].

Remark 2.5. (i) Let A be a one dimensional Prüfer domain with exactly three maximal ideals. Then by [1, Example 2.18], $R = A(+)\mathrm{qf}(A) \in \mathcal{H}_0$ is a one dimensional ϕ-Prüfer ring. Also, R has exactly three maximal ideals by [3, Theorem 3.2(1)]. Note that by (E), we have

$$|[R, T(R)]| = |[R/Nil(R), T(R)/Nil(R)]| = |[R/Nil(R), T(R/Nil(R))]|.$$

Moreover, by [3, Theorem 4.1(3)], $T(R) = \mathrm{qf}(A)(+)\mathrm{qf}(A)$. Consequently, $|[R, T(R)]| = |[A, \mathrm{qf}(A)]|$. Now, by [6, Corollary 2.6], we conclude that

$$|[A, \mathrm{qf}(A)]| = \dim(A) + 7,$$

that is, $|[R, T(R)]| = \dim(R) + 7$. Thus, if $n > 6$ in $(*)$, then (i) of Proposition 2.4 fails, or if (i) of Proposition 2.4 does not hold, then n may be greater than 6 in $(*)$.

(ii) Let A be a Prüfer domain with exactly two maximal ideals M and N such that $\mathrm{Spec}(A) = \{(0) \subset P_1 \subset M, (0) \subset P_2 \subset N\}$. Then $R = A(+)\mathrm{qf}(A) \in \mathcal{H}_0$ is a ϕ-Prüfer ring with exactly two maximal ideals $M(+)\mathrm{qf}(A)$ and $N(+)\mathrm{qf}(A)$ such that

$$Spec(R) = \{(0)(+)\mathrm{qf}(A) \subset P_1(+)\mathrm{qf}(A) \subset M(+)\mathrm{qf}(A), (0)(+)\mathrm{qf}(A) \subset P_2(+)\mathrm{qf}(A) \subset N(+)\mathrm{qf}(A)\}.$$

Now, by [6, Corollary 2.6], $|[A, \mathrm{qf}(A)]| = \dim(A) + 7$. It follows that $|[R, T(R)]| = \dim(R) + 7$. Thus, if (ii) of Proposition 2.4 does not hold, then n may be greater than 6 in $(*)$.

Let $R \in \mathcal{H}$ be a ϕ-Prüfer ring with exactly two maximal ideals, say M and N. Then by [1, Theorem 2.6], $R/Nil(R)$ is a Prüfer domain with exactly two maximal ideals, namely $M/Nil(R)$ and $N/Nil(R)$. Thus, the set of prime ideals of $R/Nil(R)$ contained in $(M/Nil(R)) \cap (N/Nil(R))$ has a unique maximal element. Consequently, the same holds in R. We denote this a unique prime ideal of R by $M * N$.

Note that the ring R in (i) of above remark is not a maximal non ϕ-chained subring of any overing of R, by [20, Theorem 2.6]. However, when $3 \leq n \leq 6$, then there exists an overring S of R (depending on n) such that R is a maximal non ϕ-chained subring of S. This we show in the remaining paper. We start with $n = 3$.

Theorem 2.6. *For a ring $R \in \mathcal{H}_1$, the following are equivalent*:

1. *R is integrally closed and $|[R, T(R)]| = \dim(R) + 3$;*
2. *R is not local and $|[R, T(R)]| = \dim(R) + 3$;*
3. *R is a ϕ-Prüfer ring with exactly two maximal ideals M and N and*
$$Spec(R) = \{Nil(R) = P_0 \subset P_1 \subset \cdots \subset P_{r-1} \subset P_r = M, P_{r-1} \subset N\};$$
4. *R is a maximal non ϕ-chained subring of R_{M*N} and $ht(N) = ht(M) = \dim(R)$.*

Proof. (1) $\Leftrightarrow$ (2): It follows from Proposition 2.2.

(2) $\Rightarrow$ (3): We have $R/Nil(R)$ is not local and $|[R/Nil(R), T(R)/Nil(R)]| = \dim(R/Nil(R)) + 3$. Now, by (E), it follows that
$$|[R/Nil(R), T(R/Nil(R))]| = \dim(R/Nil(R)) + 3.$$
Thus, by [22, Theorem 1], $R/Nil(R)$ is a Prüfer domain with exactly two maximal ideals $M/Nil(R)$ and $N/Nil(R)$ and
$$\mathrm{Spec}(R/Nil(R)) = \{(0) \subset P_1/Nil(R) \subset \cdots \subset P_{r-1}/Nil(R) \subset P_r/Nil(R) = M/Nil(R), P_{r-1}/Nil(R) \subset N/Nil(R)\}.$$
Finally, R is ϕ-Prüfer, by [1, Theorem 2.6] and hence (3) holds.

(3) $\Rightarrow$ (4): Since $R \in \mathcal{H}_1$, R is a Prüfer ring and $R \subseteq R_{M*N} \subseteq T(R)$. It follows that $R_{M*N} \in \mathcal{H}$. Also, by (C), $Nil(R_{M*N}) = Nil(R)$. Thus, (4) follows from [20, Theorem 2.6].

(4) $\Rightarrow$ (1): Note that $Nil(R_{M*N}) = Nil(R)$ and $R_{M*N} \in \mathcal{H}$. Thus, $R/Nil(R)$ is a maximal non valuation subring of $R_{M*N}/Nil(R)$, by [20, Theorem 2.4]. Consequently, by [22, Theorem 1], we have $R/Nil(R)$ is integrally closed and $|[R/Nil(R), T(R/Nil(R))]| = \dim(R/Nil(R)) + 3$. Now, (1) follows by (E) and (F). $\square$

For $n = 4$ in $(*)$, we have the following generalization of [22, Theorem 2].

Theorem 2.7. *For $R \in \mathcal{H}_1$, the following are equivalent*:

(1) *R is integrally closed and* $|[R, T(R)]| = \dim(R) + 4$;
(2) *R is not local and* $|[R, T(R)]| = \dim(R) + 4$;
(3) *R is a ϕ-Prüfer ring with exactly two maximal ideals M and N,* $\dim(R) \geq 2$*, and*

$$Spec(R) = \{Nil(R) \subset P_1 \subset \cdots \subset P_{r-1} \subset P_r = M, P_{r-1} \not\subseteq N, P_{r-2} \subset N\};$$

(4) *R is a maximal non ϕ-chained subring of R_M,* $\dim(R) \geq 2$ *and $ht(N) = ht(M) - 1 =$* $\dim(R) - 1$.

Proof. (1) $\Leftrightarrow$ (2): It follows from Proposition 2.2.

(2) $\Rightarrow$ (3): Note that $R/Nil(R)$ is not local and $|[R/Nil(R), T(R)/Nil(R)]| = \dim(R/Nil(R)) + 4$. Therefore, by (E), we have

$$|[R/Nil(R), T(R/Nil(R))]| = \dim(R/Nil(R)) + 4.$$

Now, by [22, Theorem 2], it follows that $R/Nil(R)$ is a Prüfer domain with exactly two maximal ideals $M/Nil(R)$ and $N/Nil(R)$, $\dim(R/Nil(R)) \geq 2$ and

$$\begin{aligned}\mathrm{Spec}(R/Nil(R)) = \{(0) \subset P_1/Nil(R) \subset \cdots \\ \subset P_{r-1}/Nil(R) \subset P_r/Nil(R) = M/Nil(R), \\ P_{r-1}/Nil(R) \not\subseteq N/Nil(R), P_{r-2}/Nil(R) \subset N/Nil(R)\}.\end{aligned}$$

Finally, R is ϕ-Prüfer, by [1, Theorem 2.6] and hence (3) holds.

(3) $\Rightarrow$ (4): Since $R \in \mathcal{H}_1$, R is a Prüfer ring and $R \subseteq R_M \subseteq T(R)$. It follows that $R_M \in \mathcal{H}$. Also, by (C), $Nil(R_M) = Nil(R)$. Thus, (4) follows from [20, Theorem 2.6].

(4) $\Rightarrow$ (1): Note that $Nil(R_M) = Nil(R)$ and $R_M \in \mathcal{H}$. Thus, $R/Nil(R)$ is a maximal non valuation subring of $R_M/Nil(R)$, by [20, Theorem 2.4]. Consequently, by [22, Theorem 2], we conclude that $R/Nil(R)$ is integrally closed and $|[R/Nil(R), T(R/Nil(R))]| = \dim(R/Nil(R)) + 4$. Now, (1) follows by (E) and (F). □

For $n = 5$ in $(*)$, we have the following generalization of [22, Theorem 3].

Theorem 2.8. *For $R \in \mathcal{H}_1$, the following are equivalent*:

(1) *R is integrally closed and* $|[R, T(R)]| = \dim(R) + 5$;
(2) *R is a ϕ-Prüfer ring with exactly two maximal ideals M and N*, $\dim(R) \geq 3$*, and*

$$Spec(R) = \{Nil(R) \subset P_1 \subset \cdots \subset P_{r-1} \subset P_r = M, P_{r-2} \not\subseteq N, P_{r-3} \subset N\};$$

(3) *R is a maximal non ϕ-chained subring of R_M,* $\dim(R) \geq 3$*, and $ht(N) = ht(M) - 2 =$* $\dim(R) - 2$.

Proof. (1) $\Rightarrow$ (2): Note that $R/Nil(R)$ is an integrally closed domain and $|[R/Nil(R), T(R)/Nil(R)]| = \dim(R/Nil(R)) + 5$. Therefore, by (E), we have

$$|[R/Nil(R), T(R/Nil(R))]| = \dim(R/Nil(R)) + 5.$$

Now, by [22, Theorem 3], it follows that $R/Nil(R)$ is a Prüfer domain with exactly two maximal ideals $M/Nil(R)$ and $N/Nil(R)$, $\dim(R/Nil(R)) \geq 3$, and

$$\begin{aligned}\text{Spec}(R/Nil(R)) = \{(0) \subset P_1/Nil(R) \subset \cdots \\ \subset P_{r-1}/Nil(R) \subset P_r/Nil(R) = M/Nil(R), \\ P_{r-2}/Nil(R) \not\subseteq N/Nil(R), P_{r-3}/Nil(R) \subset N/Nil(R)\}.\end{aligned}$$

Finally, R is ϕ-Prüfer, by [1, Theorem 2.6] and hence (2) holds.

(2) $\Rightarrow$ (3): Since $R \in \mathcal{H}_1$, R is a Prüfer ring and $R \subseteq R_M \subseteq T(R)$. It follows that $R_M \in \mathcal{H}$. Also, by (C), $Nil(R_M) = Nil(R)$. Thus, (3) follows from [20, Theorem 2.6].

(3) $\Rightarrow$ (1): Note that $Nil(R_M) = Nil(R)$ and $R_M \in \mathcal{H}$. Thus, $R/Nil(R)$ is a maximal non valuation subring of $R_M/Nil(R)$, by [20, Theorem 2.4]. Consequently, by [22, Theorem 3], we conclude that $R/Nil(R)$ is integrally closed and $|[R/Nil(R), T(R/Nil(R))]| = \dim(R/Nil(R)) + 5$. Now, (1) follows by (E) and (F). $\square$

For $n = 6$ in $(*)$, we have the following generalization of [22, Theorem 4].

Theorem 2.9. *For $R \in \mathcal{H}_1$, the following are equivalent:*

1. *R is integrally closed and $|[R, T(R)]| = \dim(R) + 6$;*
2. *R is a ϕ-Prüfer ring with exactly two maximal ideals M and N, $\dim(R) \geq 4$, and*

$$\text{Spec}(R) = \{Nil(R) \subset P_1 \subset \cdots \subset P_{r-1} \subset P_r = M, P_{r-3} \not\subseteq N, P_{r-4} \subset N\};$$

3. *R is a maximal non ϕ-chained subring of R_M, $\dim(R) \geq 4$, and $ht(N) = ht(M) - 3 = \dim(R) - 3$.*

Proof. (1) $\Rightarrow$ (2): Clearly, $R/Nil(R)$ is an integrally closed domain and $|[R/Nil(R), T(R)/Nil(R)]| = \dim(R/Nil(R)) + 6$. Therefore, by (E), we have

$$|[R/Nil(R), T(R/Nil(R))]| = \dim(R/Nil(R)) + 6.$$

Now, by [22, Theorem 4], it follows that $R/Nil(R)$ is a Prüfer domain with exactly two maximal ideals $M/Nil(R)$ and $N/Nil(R)$, $\dim(R/Nil(R)) \geq 4$, and

$$\begin{aligned}\text{Spec}(R/Nil(R)) = \{(0) \subset P_1/Nil(R) \subset \cdots \\ \subset P_{r-1}/Nil(R) \subset P_r/Nil(R) = M/Nil(R), \\ P_{r-3}/Nil(R) \not\subseteq N/Nil(R), P_{r-4}/Nil(R) \subset N/Nil(R)\}.\end{aligned}$$

Finally, R is ϕ-Prüfer, by [1, Theorem 2.6] and hence (2) holds.

(2) $\Rightarrow$ (3): Since $R \in \mathcal{H}_1$, R is a Prüfer ring and $R \subseteq R_M \subseteq T(R)$. It follows that $R_M \in \mathcal{H}$. Also, by (C), $Nil(R_M) = Nil(R)$. Thus, (3) follows from [20, Theorem 2.6].

(3) $\Rightarrow$ (1): Note that $Nil(R_M) = Nil(R)$ and $R_M \in \mathcal{H}$. Thus, $R/Nil(R)$ is a maximal non valuation subring of $R_M/Nil(R)$, by [20, Theorem 2.4]. Consequently, by [22, Theorem 4], we conclude that $R/Nil(R)$ is integrally closed and $|[R/Nil(R), T(R/Nil(R))]| = \dim(R/Nil(R)) + 6$. Now, (1) follows by (E) and (F). $\square$

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Atul Gaur
Department of Mathematics
University of Delhi
Delhi 110007, India
Email address: gaursatul@gmail.com

Rahul Kumar
Department of Mathematics
Birla Institute of Technology and Science Pilani
Pilani 333031, India
Email address: rahulkmr977@gmail.com