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ON NOETHERIAN PSEUDO-PRIME SPECTRUM OF A TOPOLOGICAL LE-MODULE

Anjan Kumar Bhuniya and Manas Kumbhakar

Abstract. An le-module M over a commutative ring R is a complete lattice ordered additive monoid $(M, \leqslant, +)$ having the greatest element e together with a module like action of R. This article characterizes the le-modules ${}_RM$ such that the pseudo-prime spectrum X_M endowed with the Zariski topology is a Noetherian topological space. If the ring R is Noetherian and the pseudo-prime radical of every submodule elements of ${}_RM$ coincides with its Zariski radical, then X_M is a Noetherian topological space. Also we prove that if R is Noetherian and for every submodule element n of M there is an ideal I of R such that $V(n) = V(Ie)$, then the topological space X_M is spectral.

1. Introduction

Inspired by the abstract ideal theory [2–4, 11, 27, 28] and the theory of lattice modules [17–19, 29], we introduced le-modules over a commutative ring [8] with a desire to develop an alternative abstract submodule theory. An le-module over a commutative ring has two distinctive features, namely it abstracts the set $P(A)$ of all subsets of a module A over R and the action considered on M is of the ring R. Whereas in the existing theory of lattice modules, a lattice module stands for the set $Sub(A)$ of all submodules of A and action considered on a lattice module is of a multiplicative lattice which stands for the lattice of all ideals of R. Thus it becomes possible to characterize submodules of a module as distinguished elements in an le-module M and to study structure of rings directly. In the lattice $Sub(A)$, addition of two submodules is their lattice join, but the situation is not similar in the lattice $P(A)$. So we have to consider an 'addition' on $P(A)$ together with the complete lattice order '$\subseteq$' to catch the additive feature of A. Thus we define [8] an le-module as follows:

An *le-semigroup* $(M, +, \leqslant, e)$ is such that $(M, \leqslant)$ is a complete lattice with the greatest element e, $(M, +)$ is a commutative monoid with the zero element 0_M and for all $m, m_i \in M, i \in I$ it satisfies

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(S) $m + (\vee_{i\in I} m_i) = \vee_{i\in I}(m + m_i)$.

Let R be a commutative ring and $(M, +, \leqslant, e)$ be an le-semigroup. Then M is called an *le-module* over R if there is a mapping $R \times M \longrightarrow M$ which satisfies

(M1) $r(m_1 + m_2) = rm_1 + rm_2$,
(M2) $(r_1 + r_2)m \leqslant r_1m + r_2m$,
(M3) $(r_1r_2)m = r_1(r_2m)$,
(M4) $1_Rm = m$; $0_Rm = r0_M = 0_M$,
(M5) $r(\vee_{i\in I}(m_i)) = \vee_{i\in I}(rm_i)$ for all $r, r_1, r_2 \in R$ and $m, m_1, m_2, m_i \in M$, and $i \in I$.

We denote an le-module M over R by ${}_RM$ or simply by M. From (M5), we have,

(M5)′ $m_1 \leqslant m_2 \Rightarrow rm_1 \leqslant rm_2$ for all $r \in R$ and $m_1, m_2 \in M$.

For basic notions and results on le-modules over commutative rings we refer to [8], [22]. Here we recap only some of them from [8], [20] and [21], which we will need in this article.

An element n of an le-module ${}_RM$ is called a *submodule element* if $n+n, rn \leqslant n$ for all $r \in R$. If $n \neq e$, then n is called a proper submodule element. It follows that $0_M = 0_Rn \leqslant n$ for every submodule element n of M.

Now we fix some notations:

$$\mathbb{N} = \text{set of all natural numbers},$$
$$\mathcal{S}(M) = \text{set of all submodule elements of } M,$$
$$\sum_{i\in I} n_i = \vee\{(n_{i_1} + n_{i_2} + \cdots + n_{i_k}) : k \in \mathbb{N}, \text{ and } i_1, i_2, \ldots, i_k \in I\},\ n_i \in \mathcal{S}(M),$$
$$Ie = \vee\{\sum_{i=1}^{k} a_ie : k \in \mathbb{N}; a_1, a_2, \ldots, a_k \in I\},\ I \text{ is an ideal of } R,$$
$$(n : e) = \{r \in R : re \leqslant n\},\ n \in \mathcal{S}(M),$$
$$X_M = \{n \in \mathcal{S}(M) \mid n \neq e \text{ and } (n : e) \text{ is a prime ideal of } R\},$$
$$V(n) = \{l \in X_M : n \leqslant l\},\ n \in \mathcal{S}(M),$$
$$\mathcal{V}_R(M) = \{V(n) : n \in \mathcal{S}(M)\},$$
$$\mathbb{P}rad(n) = \wedge_{p\in V(n)}p.$$

Then $\sum_{i\in I} n_i$ is a submodule element of M, which we call the sum of $\{n_i\}_{i\in I}$. Also for every ideal I of R and a submodule element n of M, Ie is a submodule element of M and $(n : e)$ is an ideal of R. Moreover, $Ie \leqslant n$ if and only if $I \subseteq (n : e)$. For any two ideals I and J of R, $I \subseteq J$ implies that $Ie \leqslant Je$. If $n, l \in \mathcal{S}(M)$ are such that $n \leqslant l$, then $(n : e) \subseteq (l : e)$. Also if $\{n_i\}_{i\in I}$ is an arbitrary family of submodule elements in ${}_RM$, then $(\wedge_{i\in I} n_i : e) = \cap_{i\in I}(n_i : e)$. This results, proved in [8], are useful here.

Every element of X_M is called a *pseudo-prime submodule element* and X_M is called the *pseudo-prime spectrum* of ${}_RM$. A submodule element n of M is said to be *pseudo-semiprime* if n is a meet of some pseudo-prime submodule elements of M. A pseudo-prime submodule element p of M is called *extraordinary* if for any two pseudo-semiprime submodule elements n and l of M, $n \wedge l \leqslant p$ implies that either $n \leqslant p$ or $l \leqslant p$. If $X_M = \emptyset$ or every pseudo-prime submodule element of M is extraordinary, then ${}_RM$ is called a *topological le-module.*

There are many functorial constructions associating topological spaces with a ring or a module. It helps to interpret arithmetical properties of a ring R or a module ${}_RM$ in the geometric language on the associated topological spaces. Inspired by several enlightening interplay between the Zariski topology on the pseudo-prime spectrum of a module M and algebraic properties of M [1, 6, 7, 12–15, 23–26], we introduced the Zariski topology on the pseudo-prime spectrum X_M of an le-module ${}_RM$ over a ring R in [21]. There we studied the le-modules ${}_RM$ such that X_M is an irreducible topological space. In this article, we characterize the le-modules ${}_RM$ such that X_M is a Noetherian topological space.

The set $\mathcal{V}_R(M)$ satisfies all axioms of a topological space for the closed subsets if and only if ${}_RM$ is a topological le-module [21]. If ${}_RM$ is a topological le-module, then the topology induced by $\mathcal{V}_R(M)$ is called the *Zariski topology* on X_M.

Henceforth, in this article, we assume that every le-module ${}_RM$ is a topological le-module.

For every $n \in M$, $\mathbb{P}rad(n)$ is a submodule element of M and is called the *pseudo-prime radical* of n. If $V(n) = \emptyset$, then we set $\mathbb{P}rad(n) = e$. Note that $n \leqslant \mathbb{P}rad(n)$ and that $\mathbb{P}rad(n) = e$ or $\mathbb{P}rad(n)$ is a pseudo-semiprime submodule element of M. Also $V(n) = V(\mathbb{P}rad(n))$. A submodule element n of M is said to be a *pseudo-prime radical submodule element* if $n = \mathbb{P}rad(n)$. It is easy to check that $\mathbb{P}rad(\mathbb{P}rad(n)) = \mathbb{P}rad(n)$, i.e., $\mathbb{P}rad(n)$ is a pseudo-prime radical submodule element of M.

For each subset Y of X_M, we denote the closure of Y in X_M by $\overline{Y}$, and meet of the elements of Y by $\Im(Y)$, i.e., $\Im(Y) = \wedge_{p \in Y} p$. If $Y = \emptyset$, then we take $\Im(Y) = e$.

Now we recall the following results from [20] and [21], which have some use in this article.

Lemma 1.1 ([20,21]). *Let ${}_RM$ be an le-module. Then the following statements hold.*

(1) *For every family of submodule elements $\{n_i\}_{i \in I}$ of M, $\cap_{i \in I} V(n_i) = V(\sum_{i \in I} n_i)$.*
(2) *If for every submodule element n of M there exists an ideal I of R such that $V(n) = V(Ie)$, then M is topological.*

A topological space X is *irreducible* if for every pair of closed subsets Y_1, Y_2 of X, $X = Y_1 \cup Y_2$ implies $X = Y_1$ or $X = Y_2$. A nonempty subset Y of a topological space X is called an *irreducible subset* if the subspace Y of X is irreducible. If a subset Y of X is irreducible, so is its closure $\overline{Y}$. An element $y \in Y$ is called a *generic point* of Y if $Y = \overline{\{y\}}$. Now we state another useful result from [21].

Lemma 1.2 ([21]). *Let $_RM$ be an le-module. Then the following statements hold.*

(1) *X_M is T_0.*
(2) *For every $Y \subseteq X_M$, $\overline{Y} = V(\Im(Y))$ and hence Y is closed if and only if $Y = V(\Im(Y))$. In particular, $\overline{\{l\}} = V(l)$ for every $l \in X_M$.*
(3) *For $Y \subseteq X_M$, Y is an irreducible closed subset of X_M if and only if $Y = V(p)$ for some $p \in X_M$. Thus every irreducible closed subset of X_M has a generic point.*

Also we refer to [5], [10] for background on commutative ring theory, [9] for fundamentals on topology.

2. Noetherian pseudo-prime spectrum of an le-module

A topological space X is called *quasi-compact* if every open cover of X has a finite subcover. A subset Y of X is said to be quasi-compact if the subspace Y is quasi-compact. To avoid ambiguity, we would like to mention that a compact topological space is a quasi-compact Hausdorff space. To keep uniformity in terminology used in the commutative ring theory we continue with the term quasi-compact. A topological space X is said to be *Noetherian* if the open subsets of X satisfy the ascending chain condition. Thus X is Noetherian if and only if the closed subsets of X satisfy the descending chain condition. This is equivalent to each of the conditions that every open subspace of X is quasi-compact and every subspace of X is quasi-compact.

In the following result we establish a relationship between the Noetherianness of the pseudo-prime spectrum of an le-module $_RM$ with a chain condition on the le-module M.

Theorem 2.1. *An le-module $_RM$ has a Noetherian pseudo-prime spectrum if and only if the ACC holds for pseudo-prime radical submodule elements of M.*

Proof. Assume that the *ACC* holds for pseudo-prime radical submodule elements of M. Let

$$V(n_1) \supseteq V(n_2) \supseteq \cdots$$

be a descending chain of closed subsets of X_M, where each n_i is a submodule element of M. Then

$$\Im(V(n_1)) \leqslant \Im(V(n_2)) \leqslant \cdots$$

is an ascending chain of pseudo-prime radical submodule elements $\Im(V(n_i)) = \mathbb{P}rad(n_i)$ of M. Thus, by assumption there exists $k \in \mathbb{N}$ such that for all $i \in \mathbb{N}$,

$$\Im(V(n_k)) = \Im(V(n_{k+i})).$$

Now, by Lemma 1.2(2), we have

$$V(n_k) = V(\Im(V(n_k))) = V(\Im(V(n_{k+i}))) = V(n_{k+i}).$$

Hence X_M is a Noetherian topological space.

Conversely, let M has a Noetherian pseudo-prime spectrum. Also let

$$n_1 \leqslant n_2 \leqslant \cdots$$

be an ascending chain of pseudo-prime radical submodule elements of M. Then $n_i = \mathbb{P}rad(n_i) = \Im(V(n_i))$. Also

$$V(n_1) \supseteq V(n_2) \supseteq \cdots$$

is a descending chain of closed subsets of X_M. Since M has a Noetherian pseudo-prime spectrum there is $k \in \mathbb{N}$ such that for all $i \in \mathbb{N}$, $V(n_k) = V(n_{k+i})$. Thus

$$n_k = \mathbb{P}rad(n_k) = \Im(V(n_k)) = \Im(V(n_{k+i})) = \mathbb{P}rad(n_{k+i}) = n_{k+i}.$$

Hence ACC holds for pseudo-prime radical submodule elements of M. □

Let R be a ring and $Spec(R)$ be the set of all prime ideals of R. Then there is a topology on $Spec(R)$, called the *Zariski topology* on $Spec(R)$, such that the closed sets are of the form

$$V^R(I) = \{P \in Spec(R) : I \subseteq P\},$$

where I is an ideal of R.

Recall that if I is an ideal of a ring R, then the *radical* of I is defined by

$$\mathrm{Rad}(I) = \{a \in R : a^n \in I \text{ for some positive integer } n\}.$$

Then $\mathrm{Rad}(I)$ is also an ideal of R and $I \subseteq Rad(I)$. An ideal I of R is called a *radical ideal* if $I = Rad(I)$.

A topological space X is called *spectral* if it is homeomorphic to $\mathrm{Spec}(R)$ for some commutative ring R. It is well known that a topological space is spectral if and only if it is T_0, quasi-compact, the quasi-compact open subsets of X are closed under finite intersection and form an open basis, and each irreducible closed subset of X has a generic point. A Noetherian topological space is spectral if and only if it is T_0 and every non-empty irreducible closed subset has a generic point [16]. From Lemma 1.2, it follows that X_M is always T_0 and every non-empty irreducible closed subset of X_M has a generic point.

Now we present some algebraic conditions under which the pseudo-prime spectrum X_M of an le-module M is spectral. Recall that a ring R is called *Noetherian* if ascending chain condition holds for ideals in R. Also for every family $\{I_\lambda\}_{\lambda\in\Lambda}$ of ideals in R, we have $\sum_{\lambda\in\Lambda} I_\lambda e = (\sum_{\lambda\in\Lambda} I_\lambda)e$.

Theorem 2.2. *Let $_RM$ be an le-module. If R is a Noetherian ring and for every submodule element n of M there exists an ideal I of R such that $V(n) = V(Ie)$, then X_M is a spectral space.*

Proof. We will show that every open subset of X_M is quasi-compact. Let H be an open subset of X_M and let $\{E_\lambda\}_{\lambda\in\Lambda}$ be an open cover of H. Then there are submodule elements n and n_λ, where $H = X_M \setminus V(n)$ and $E_\lambda = X_M \setminus V(n_\lambda)$ for each $\lambda \in \Lambda$, such that

$$H \subseteq \cup_{\lambda\in\Lambda} E_\lambda = X_M \setminus \cap_{\lambda\in\Lambda} V(n_\lambda).$$

By hypothesis, for each $\lambda \in \Lambda$ there exists an ideal I_λ in R such that $V(n_\lambda) = V(I_\lambda e)$. Then

$$H \subseteq X_M \setminus V(\textstyle\sum_{\lambda\in\Lambda} I_\lambda e) = X_M \setminus V((\textstyle\sum_{\lambda\in\Lambda} I_\lambda)e).$$

Since R is a Noetherian ring, there exists a finite subset Λ' of Λ such that

$$H \subseteq \cup_{\lambda\in\Lambda'} E_\lambda.$$

Thus X_M is a Noetherian topological space and hence a spectral space. □

Let n be a submodule element of $_RM$. We denote,

$$c(n) = \cap\{I : I \text{ is an ideal of } R \text{ and } n \leqslant Ie\}.$$

Then $_RM$ is called a *content le-module* if $n \leqslant c(n)e$ for every submodule element n of M [20]. We call an le-module $_RM$ a *multiplication le-module* if every submodule element n of M can be expressed as $n = Ie$ for some ideal I of R [20]. An le-module $_RM$ is said to be a *pseudo-prime multiplication le-module* if for every pseudo-prime submodule element n of M, there exists an ideal I of R such that $n = Ie$. Clearly every multiplication le-module is a weak multiplication le-module [20].

Suppose that $_RM$ is a pseudo-prime multiplication le-module and n is a pseudo-prime submodule element of M. Then there is an ideal I of R such that $n = Ie$, and so $I \subseteq (n : e)$. Hence $n = Ie \leqslant (n : e)e$. Also $(n : e)e \leqslant n$ and it follows that $n = (n : e)e$. Thus we prove that an le-module $_RM$ is pseudo-prime multiplication if and only if $n = (n : e)e$ for every pseudo-prime submodule element n of M.

It is proved in [20] that if $_RM$ is a content and pseudo-prime multiplication le-module, then $\mathbb{P}rad(n) = (\mathbb{P}rad(n) : e)e$ for every submodule element n of M and hence $_RM$ is topological.

Theorem 2.3. *Let $_RM$ be a content and pseudo-prime multiplication le-module. If Spec(R) is a Noetherian topological space, then X_M is a spectral space.*

Proof. We show that X_M is a Noetherian topological space. Let

$$V(n_1) \supseteq V(n_2) \supseteq \cdots$$

be a descending chain of closed subsets of X_M. Then

$$\mathbb{P}rad(n_1) \leqslant \mathbb{P}rad(n_2) \leqslant \cdots.$$

Thus we have the ascending chain

$$(\mathbb{P}rad(n_1) : e) \subseteq (\mathbb{P}rad(n_2) : e) \subseteq \cdots$$

of radical ideals. Since $\mathrm{Spec}(R)$ is Noetherian there is $k \in \mathbb{N}$ such that for each $i = 1, 2, \ldots,$

$$(\mathbb{P}rad(n_k) : e) = (\mathbb{P}rad(n_{k+i}) : e) = \cdots.$$

Since ${}_RM$ is a content and pseudo-prime multiplication le-module, for each $\lambda \in \mathbb{N}$,

$$\mathbb{P}rad(n_\lambda) = (\mathbb{P}rad(n_\lambda) : e)e.$$

Thus for each $i = 1, 2, \ldots$ we have $\mathbb{P}rad(n_k) = \mathbb{P}rad(n_{k+i}) = \cdots$. This implies that

$$V(n_k) = V(\mathbb{P}rad(n_k)) = V(\mathbb{P}rad(n_{k+i})) = V(n_{k+i}) = \cdots.$$

Therefore X_M is a Noetherian topological space and whence a spectral space. □

For a submodule element n of M, the *Zariski radical* of n, denoted by $\mathbb{Z}rad(n)$, is defined by

$$\mathbb{Z}rad(n) = \wedge\{p \in X_M : (n : e) \subseteq (p : e)\}.$$

Now we associate Noetherianness of pseudo-prime spectrum X_M of ${}_RM$ and of the ring R.

Theorem 2.4. *If $\mathbb{P}rad(n) = \mathbb{Z}rad(n)$ for each submodule element n of an le-module ${}_RM$, then M is topological. Moreover, if R is Noetherian, then X_M is a Noetherian topological space and so spectral space.*

Proof. Let n be a submodule element of M. Then we have

$$\begin{aligned} V(n) = V(\mathbb{P}rad(n)) &= V(\mathbb{Z}rad(n)) \\ &= V(\mathbb{Z}rad((n : e)e)) \\ &= V(\mathbb{P}rad((n : e)e)) \\ &= V((n : e)e). \end{aligned}$$

Hence by Lemma 1.1(2), M is a topological le-module.

Let R be Noetherian. We show that every open subset of X_M is quasi-compact. Let H be an open subset of X_M and let $\{E_\lambda\}_{\lambda\in\Lambda}$ be an open cover of H. Then there are submodule elements n and n_λ, where $H = X_M \setminus V(n)$ and $E_\lambda = X_M \setminus V(n_\lambda)$ for each $\lambda \in \Lambda$, such that

$$H \subseteq \cup_{\lambda\in\Lambda} E_\lambda = X_M \setminus \cap_{\lambda\in\Lambda} V(n_\lambda).$$

By the first part of proof we have $V(n_\lambda) = V((n_\lambda : e)e)$ for each $\lambda \in \Lambda$. Since R is a Noetherian ring, similarly as in the proof of Theorem 2.2, there exists a finite subset Λ' of Λ such that

$$H \subseteq \cup_{\lambda \in \Lambda'} E_\lambda.$$

Thus X_M is a Noetherian topological space and hence a spectral space. □

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Anjan Kumar Bhuniya
Department of Mathematics
Visva-Bharati
Santiniketan-731235, India
Email address: anjankbhuniya@gmail.com

Manas Kumbhakar
Department of Mathematics
Nistarini College
Purulia-723101, India
Email address: manaskumbhakar@gmail.com