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SIMPLE ZEROS OF L-FUNCTIONS AND THE WEYL-TYPE SUBCONVEXITY

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Abstract. Let f be a self-dual primitive Maass or modular forms for level 4. For such a form f, we define

$$N_f^s(T) := |\{\rho \in \mathbb{C} : |\Im(\rho)| \leq T,\ \rho \text{ is a non-trivial simple zero of } L_f(s)\}|.$$

We establish an omega result for $N_f^s(T)$, which is $N_f^s(T) = \Omega\big(T^{\frac{1}{6}-\epsilon}\big)$ for any $\epsilon > 0$. For this purpose, we need to establish the Weyl-type subconvexity for L-functions attached to primitive Maass forms by following a recent work of Aggarwal, Holowinsky, Lin, and Qi.

1. Introduction

Zeros of L-functions have drawn the attention of many mathematicians. We expect that the Generalized Riemann Hypothesis for automorphic L-functions (i.e., all non-trivial zeros of automorphic L-functions lie on the critical line $\Re(s) = 1/2$) is true and the zeros are simple except for some occasional cases.

Even though we are far from verifying them, we know quite a lot about zeros of L-functions. About 40 years later, after Selberg [24] proved that a positive proportion of zeros of the Riemann zeta function $\zeta(s)$ lie on the critical line, Hafner [18,19] obtained analogous theorems for modular L-functions and Maass L-functions for the full modular group.

In the case of the Riemann zeta function, Conrey [12] showed that more than two-fifths of the zeros are simple and lie on the critical line. For the current best record on the number of zeros or simple zeros of the Riemann zeta functions on the critical line, we refer to [9, 16].

We have less knowledge of simple zeros for degree 2 L-functions. Conrey and Ghosh [13] proved that the L-function attached to the Ramanujan tau-series has infinitely many simple zeros. They showed that if an L-function attached to a modular form for the full modular group has a simple zero, it has infinitely many simple zeros. They checked that the L-function attached

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to the Ramanujan tau-series has a simple zero. Hence we know that it has infinitely many simple zeros. In [13], the infinitude of simple zeros is stated quantitatively, which depends on the subconvexity of L-functions. The first named author [11] extended Conrey and Ghosh's result to the case of Maass L-functions for the full modular group. Strömbergsson [25] showed that some specific three Maass L-functions have simple zeros. Hence, at least these three L-functions have infinitely many simple zeros. One drawback of Conrey and Ghosh's idea is that it requires the existence of a simple zero. In recent years progress has been made in studying simple zeros of $GL_2(A_{\mathbb{Q}})$ automorphic L-functions. Booker [7] showed that L-functions of holomorphic newforms have infinitely many simple zeros. The first named author, Booker, and Kim [8] generalized this result to all $GL_2(A_{\mathbb{Q}})$ automorphic L-functions.

We extend Conrey and Ghosh's result to newforms for level 4. Let f be a primitive form (i.e., a normalized Hecke newform), either a modular form or a Maass form. There are two Dirichlet characters modulo 4. One is the trivial character modulo 4, and the other, denoted by χ, is the primitive odd character of conductor 4. For Maass forms, we consider weight zero Maass forms only. For modular forms, it can be a cusp form with the trivial character or the character χ depending on the parity of weight k. In addition, we assume that our primitive forms are self-dual. This assumption is true for all primitive forms in $S_{2k}(\Gamma_0(4))$ since the Hecke operators are self-adjoint. This assumption also seems true for primitive Maass forms of weight zero for level 4. [1]If it is not self-dual, our approach does not work. See Remark 3.

Let $L_f(s)$ be the L-function attached to the primitive form f, let

$$N_f^s(T) := |\{\rho \in \mathbb{C} : |\Im(\rho)| \leq T,\ \rho \text{ is a non-trivial simple zero of } L_f(s)\}|.$$

We have the following omega result for $N_f^s(T)$.

Theorem 1. *Let f be a self-dual primitive form for level 4. Then, for any $\epsilon > 0$,*

$$N_f^s(T) = \Omega_\epsilon\left(T^{\frac{1}{6}-\epsilon}\right),$$

where the notation $f = \Omega(g)$ means that $\neq O(g)$.

Remark 1. (1) The exponent $\frac{1}{6} = \frac{1}{2} - \frac{1}{3}$ comes from the subconvexity $L_f(k/2+it) \ll_{f,\epsilon} |t|^{\frac{1}{3}+\epsilon}$ for a primitive modular form f and $L_f(1/2+it) \ll_{f,\epsilon} |t|^{\frac{1}{3}+\epsilon}$ for a primitive Maass form f of weight zero. Hence, Lindelöf Hypothesis implies that $N_f^s(T) = \Omega(T^{\frac{1}{2}-\epsilon})$, which still takes up a very little portion among all the zeros.

(2) Recently, de Faveri [14] made impressive progress for modular forms for arbitrary levels. He showed:

[1]We can find a list of self-dual primitive Maass forms of level 4 from the website LMFDB.

Theorem 1.1 (de Faveri). *Let $f \in S_k(\Gamma_0(N), \xi)$ be a primitive holomorphic modular form of arbitrary weight k, level N, and nebentypus ξ. Then*

$$N_f^s(T) = \Omega(T^{\delta})$$

for any $\delta < \frac{2}{27}$.

To obtain Theorem 1, we need the Weyl-type subconvexity of $L_f(s)$, which is

$$L_f(k/2 + it) \ll_{f,\epsilon} |t|^{\frac{1}{3}+\epsilon}$$

for any $\epsilon > 0$. For modular forms for arbitrary level, Booker, Milinovich and Ng showed that $L_f(k/2 + it) \ll |t|^{\frac{1}{3}} \log |t|$ for $|t| \geq 2$. For Maass forms of weight 0, Aggarwal [2] was the first who established the Weyl type subconvexity. Afterward, Aggarwal, Holowinsky, Lin, and Qi [3] also obtained the Weyl type subconvexity for modular forms for arbitrary levels using a simple Bessel delta method. Following [3], we reconfirmed that Maass form L-functions satisfy the Weyl type subconvexity.

Theorem 2. *Let $g \in M_\lambda(M, \xi)$ be a primitive Maass form of weight zero with eigenvalue λ, level M and nebentypus ξ. Then,*

$$L_g(1/2 + it) \ll_{M,\epsilon} t^{\frac{1}{3}+\epsilon}.$$

The following theorem is the key point of the proof of the Weyl type subconvexity.

Theorem 3. *Let $\epsilon > 0$ be an arbitrarily small constant. Let $N, T, \Delta > 1$ be parameters such that*

$$N^{\epsilon}\Delta \leq T.$$

Let $V(x) \in C_c^{\infty}(0, \infty)$ be a smooth function with support in $[1, 2]$. Assume that its total variation $Var(V) \ll 1$ and $V^{(j)} \ll_j \Delta^j$ for $j \geq 0$. For γ real, define $\phi(x) = -\log x$ and $f(x) = T\phi(x/N) + \gamma x$. Let $g \in M_\lambda(M, \xi)$ and $\lambda_g(n)$ be its Fourier coefficients. Then

$$\sum_{n=1}^{\infty} \lambda_g(n) e(f(n)) V\left(\frac{n}{N}\right) \ll T^{1/3} N^{1/2+\epsilon} + \frac{N^{1+\epsilon}}{T^{1/6}},$$

with the implied constant depending only on g, ϕ, and ϵ.

Remark 2. An anonymous referee informed us that a work of Fan and Sun [15] also shows the Wely-type subconvexity for Maass forms.

In Section 2, we prove Theorem 1 for modular forms and in Section 3 we do for Maass forms. Section 4 is devoted to the proof of the Weyl-type subconvexity for Maass forms for arbitrary level by following [3].

2. Simple zeros of modular L-functions

First, we recall the functional equation of a modular L-function and then sketch the outline of the proof.

2.1. Functional equation

Let $f(z)=\sum_{n=1}^{\infty}\lambda(n)e^{2\pi inz}\in S_k(\Gamma_0(4))$ or $S_k(4,\chi)$ be a self-dual primitive form. We associate the Dirichlet series to f:

$$L_f(s)=\sum_{n=1}^{\infty}\lambda(n)n^{-s}.$$

Since f is primitive, f is an eigenfunction of the Fricke involution W_4, which is defined by

$$W_4f=4^{-k/2}z^{-k}f\Big(-\frac{1}{4z}\Big).$$

Hence $W_4f=\eta f$ with $\eta=\pm i^k$. Especially if $f\in S_k(4,\chi)$, then we can determine η explicitly;

$$\eta=\tau(\chi)\lambda(4)4^{-k/2}.$$

Since $\lambda(4)=(\lambda(2))^2=(2^{\frac{k-1}{2}})^2$ and $\tau(\chi)=2i$, we have $\eta=(2i)2^{k-1}2^{-k}=i$. Then we have the functional equation for $L_f(s)$;

$$\Lambda_f(s)=\left(\frac{\sqrt{4}}{2\pi}\right)^s\Gamma(s)L_f(s)=-i^{k+1}\left(\frac{\sqrt{4}}{2\pi}\right)^{k-s}\Gamma(k-s)L_f(k-s). \tag{2.1}$$

For $f\in S_k(\Gamma_0(4))$, we can determine the functional equation up to a sign;

$$\left(\frac{\sqrt{4}}{2\pi}\right)^s\Gamma(s)L_f(s)=\pm\left(\frac{\sqrt{4}}{2\pi}\right)^{k-s}\Gamma(k-s)L_f(k-s).$$

2.2. Outline of the proof

For the sake of convenience we omit the subscript f from $L_f(s)$ and $\Lambda_f(s)$. Cauchy's residue theorem tells that the difference between the two integrals,

$$\begin{aligned}&\frac{1}{2\pi i}\int_{(\frac{k+1}{2}+\epsilon)}\frac{L'}{L}(k-s)\frac{L'}{L}(s)\Lambda(s)e^{i(\pi/2-\delta)s}\,ds\\&-\frac{1}{2\pi i}\int_{(\frac{k-1}{2}-\epsilon)}\frac{L'}{L}(k-s)\frac{L'}{L}(s)\Lambda(s)e^{i(\pi/2-\delta)s}\,ds\end{aligned} \tag{2.2}$$

equals

$$-\sum_{\frac{k-1}{2}<\Re(\rho)<\frac{k+1}{2}}L'(\rho)\left(\frac{\sqrt{4}}{2\pi}\right)^{\rho}\Gamma(\rho)e^{i(\pi/2-\delta)\rho}, \tag{2.3}$$

where the sum is over the non-trivial simple zeros ρ. By the functional equation (2.1), we can express (2.2) as a single contour integral:

$$\frac{1}{2\pi i}\int_{(\frac{k+1}{2}+\epsilon)} \frac{L'}{L}(k-s)\frac{L'}{L}(s)\Lambda(s)f(s,\delta)\,ds, \tag{2.4}$$

where $f(s,\delta) = e^{i(\pi/2-\delta)s} + i^{k+1}e^{i(\pi/2-\delta)(k-s)}$. We will show, in Section 2.3, that (2.4) $\gg_\epsilon (1/\delta)^{\beta-\epsilon}$ for some arbitrarily small $\delta > 0$ if $L(s)$ has a simple zero $\rho = \beta + it_0$.

With the Weyl-type subconvexity bound for $L(s)$

$$L\Big(\frac{k}{2}+it\Big) \ll_{f,\epsilon} (1+|t|)^{1/3+\epsilon},$$

Stirling's formula, and Phragmen-Lindelöf argument, we have

$$L'(\rho)\bigg(\frac{\sqrt{4}}{2\pi}\bigg)^{\rho}\Gamma(\rho)e^{i(\frac{\pi}{2}-\delta)\rho} \ll_{f,\epsilon} e^{-\delta|t|}|t|^{\frac{\sigma+(k+1)}{3}-\frac{1}{2}+\epsilon}$$

for $\frac{k}{2} \le \sigma < \frac{k+1}{2}$. Let $\frac{1}{\delta} = T$ and $\beta_0 = \sup\{\beta \,|\, \rho = \beta + it \text{ is a simple zero of } L(s)\}$. By the functional equation, $\beta_0 \ge \frac{k}{2}$. For sufficiently large T, the sum (2.3) is

$$\ll_{f,\epsilon} \sum_{\substack{|t|<T^{1+\epsilon}\\ \rho=\beta+it:\,\text{simple}}} T^{\frac{\beta_0+(k+1)}{3}-\frac{1}{2}+\epsilon} \ll_{f,\epsilon} N_f^s(T^{1+\epsilon})\cdot T^{\frac{\beta_0+(k+1)}{3}-\frac{1}{2}+\epsilon}.$$

Since there is an arbitrarily large $T = \frac{1}{\delta}$ such that (2.4) $\gg_\epsilon T^{\beta_0-\epsilon}$, Theorem 1 for modular forms follows.

Remark 3. If a primitive form f is not self-dual, the equation (2.2) is replaced with

$$\begin{aligned}\frac{1}{2\pi i}\int_{(\frac{k+1}{2}+\epsilon)} \frac{L'_{\overline{f}}}{L\overline{f}}(k-s)\frac{L'_f}{L_f}(s)\Lambda_f(s)e^{i(\pi/2-\delta)s}\,ds\\ -\frac{1}{2\pi i}\int_{(\frac{k-1}{2}-\epsilon)} \frac{L'_{\overline{f}}}{L_{\overline{f}}}(k-s)\frac{L'_f}{L_f}(s)\Lambda_f(s)e^{i(\pi/2-\delta)s}\,ds.\end{aligned} \tag{2.5}$$

Then, by the functional equation, the equation (2.5) is equal to

$$\begin{aligned}\frac{1}{2\pi i}\int_{(\frac{k+1}{2}+\epsilon)} \frac{L'_{\overline{f}}}{L\overline{f}}(k-s)\frac{L'_f}{L_f}(s)\Lambda_f(s)e^{i(\pi/2-\delta)s}\,ds\\ +\frac{\varepsilon_f}{2\pi i}\int_{(\frac{k+1}{2}+\epsilon)} \frac{L'_f}{L_f}(k-s)\frac{L'_{\overline{f}}}{L_{\overline{f}}}(s)\Lambda_{\overline{f}}(s)e^{i(\pi/2-\delta)(k-s)}\,ds,\end{aligned} \tag{2.6}$$

where ε_f is the root number of $L_f(s)$. If $L_f(s)$ has a simple zero $\rho = \beta + it_0$, then $L_{\overline{f}}(s)$ also has a simple zero $k-\rho$. By the arguments in Section 2.3, we can see that there is an arbitrarily small $\delta > 0$ such that the two integrals in (2.6) are $\gg_\epsilon \big(\frac{1}{\delta}\big)^{\beta-\epsilon}$ and $\gg_\epsilon \big(\frac{1}{\delta}\big)^{k-\beta-\epsilon}$, respectively. Under the Generalized

Riemann Hypothesis, we have $\beta = k - \beta = \frac{k}{2}$. We cannot exclude the possibility of the cancellation of the two integrals.

2.3. Estimates of integrals

Define $X(s)$ be $-i^{k+1}\frac{H(k-s)}{H(s)}$, where $H(s) = \left(\frac{\sqrt{4}}{2\pi}\right)^s \Gamma(s)$. Then we have an asymmetric functional equation $L(s) = X(s)L(k-s)$. Then (2.4) equals

$$\begin{aligned} (2.7) \qquad & -\frac{1}{2\pi i}\int_{(\frac{k+1}{2}+\epsilon)} \frac{L'}{L}(s)L'(s)(2\pi)^{-s}(1/2)^{-s}\Gamma(s)f(s,\delta)ds \\ & + \frac{1}{2\pi i}\int_{(\frac{k+1}{2}+\epsilon)} \frac{X'}{X}(s)L'(s)(2\pi)^{-s}(1/2)^{-s}\Gamma(s)f(s,\delta)ds. \end{aligned}$$

We introduce a key lemma which is used to show the first integral in (2.7) is divergent as $\delta \to 0^+$ when $L(s)$ has a simple zero.

Lemma 2.1 ([11, Lemma 1]). *Suppose that the Dirichlet series*

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

absolutely converges for $\sigma > \sigma_0 > 0$. Then for $l > \sigma_0$, $l + c > 0$, and $0 < \delta < \pi/4$,

$$\begin{aligned} & \frac{1}{2\pi i}\int_{(l)} F(s)(2\pi)^{-s}x^{-s}\Gamma(s+c)\, e^{\pm i(\pi/2-\delta)s}ds \\ = & \frac{1}{2\pi i}\int_{(l)} (\mp i)^c F_x(s)(2\pi)^{-s}x^{-s}\Gamma(s+c)\,(2\sin(\delta/2))^{-s-c}\left(e^{\pm i\delta/2}\right)^{-s+c} ds, \end{aligned}$$

where $F_x(s) = \sum_{n=1}^{\infty} \frac{f(n)e^{2\pi i n x}}{n^s}$.

By Lemma 2.1,

$$\frac{1}{2\pi i}\int_{(\frac{k+1}{2}+\epsilon)} \frac{L'}{L}(s)L'(s)(2\pi)^{-s}(1/2)^{-s}\Gamma(s)f(s,\delta)\,ds$$

is converted to

$$(2.8) \quad \frac{1}{2\pi i}\int_{(\frac{k+1}{2}+\epsilon)} \sum_{n=1}^{\infty} \frac{(-1)^n a(n)}{n^s}(2\pi)^{-s}(1/2)^{-s}\Gamma(s)\,(2\sin(\delta/2))^{-s}\, g(s,\delta)\,ds,$$

where

$$\begin{aligned} & \frac{L'}{L}(s)L'(s) = \sum_{n=1}^{\infty} a(n)n^{-s} \text{ and} \\ & g(s,\delta) = \left[(e^{i\delta/2})^{-s} + i^{k+1}e^{i(\pi/2-\delta)k}(e^{-i\delta/2})^{-s}\right]. \end{aligned}$$

Let us assume that $L(s)$ has a simple zero $\rho = \beta + it$ for $\frac{k-1}{2} < \beta < \frac{k+1}{2}$. From the functional equation (2.1), we can say that $\beta \geq k/2$. Then $\frac{L'}{L}(s)L'(s)$

has a pole at $s = \rho$. We claim that its twist by the additive character $(-1)^n$ still has a pole at $s = \rho$.

Lemma 2.2 ([13, Lemma 5])**.** *Suppose that, for $\Re(s) > \frac{k+1}{2}$, we have*

$$\frac{L'}{L}(s)L'(s) = \sum_{n=1}^{\infty} a(n)n^{-s}.$$

Then,

$$\sum_{n=1}^{\infty} \frac{(-1)^n a(n)}{n^s} = [1 - 2\alpha(s)]\frac{L'}{L}(s)L'(s) - 4\alpha'(s)L'(s) - 2\frac{\alpha'}{\alpha}(s)\alpha'(s)L(s),$$

where $\alpha(s) = (1 - \lambda(2)2^{-s})$.

Since $\lambda(2) = \pm 2^{\frac{k-1}{2}}$, it is easy to see that $\sum_{n=1}^{\infty} \frac{(-1)^n a(n)}{n^s}$ also has a pole at $s = \rho$. On the other hand, the zeros of $g(s,\delta)$ are $k - \frac{\frac{3\pi}{2} + \pi(2n+k)}{\delta}$ for $n \in \mathbb{Z}$, as $\delta \to 0$, the integrand of (2.8) still has a simple pole at $s = \beta + it$.

Lemma 2.3 ([6, Lemma 4])**.** *Let $\psi(s)$ be meromorphic in the complex plane and holomorphic for $\sigma > \sigma_0$ and of rapid decay in vertical strips in a right-half plane. If $\psi(s)$ has a pole at $s = \beta + it$, then for $l > \sigma_0$*

$$\frac{1}{2\pi i}\int_{(l)} \psi(s)x^{-s}ds = \Omega_\epsilon\left(x^{-(\beta-\epsilon)}\right) \quad \text{as } x \to 0$$

for all $\epsilon > 0$.

Hence by Lemma 2.3,

$$(2.8) \gg_\epsilon \left(\frac{1}{\delta}\right)^{\beta-\epsilon}$$

for some arbitrarily small δ.

The remaining integral to estimate is

$$\frac{1}{2\pi i}\int_{(\frac{k+1}{2}+\epsilon)} \frac{X'}{X}(s)L'(s)(2\pi)^{-s}(1/2)^{-s}\Gamma(s)\, f(s,\delta)\, ds. \tag{2.9}$$

Since $X(s) = -i^{k+1}\left(\frac{\sqrt{4}}{2\pi}\right)^{k-2s}\frac{\Gamma(k-s)}{\Gamma(s)}$, we have

$$\frac{X'}{X}(s) = -2\log\left(\frac{\sqrt{4}}{2\pi}\right) - \frac{\Gamma'}{\Gamma}(k-s) - \frac{\Gamma'}{\Gamma}(s). \tag{2.10}$$

By inserting (2.10) into the integral (2.9), we encounter the following three integrals:

$$\frac{1}{2\pi i}\int_{(\frac{k+1}{2}+\epsilon)} L'(s)(2\pi)^{-s}(1/2)^{-s}\Gamma(s)\, f(s,\delta)ds, \tag{2.11}$$

$$\frac{1}{2\pi i}\int_{(\frac{k+1}{2}+\epsilon)} L'(s)(2\pi)^{-s}(1/2)^{-s}\frac{\Gamma'}{\Gamma}(k-s)\Gamma(s)\, f(s,\delta)\, ds, \tag{2.12}$$

$$\frac{1}{2\pi i}\int_{(\frac{k+1}{2}+\epsilon)} L'(s)(2\pi)^{-s}(1/2)^{-s}\Gamma'(s)\, f(s,\delta)\, ds. \tag{2.13}$$

(2.11), by Lemma 2.1, equals

$$\frac{1}{2\pi i}\int_{(\frac{k+1}{2}+\epsilon)} \widehat{L}'(s)(2\pi)^{-s}(1/2)^{-s}\Gamma(s)\,(2\sin(\delta/2))^{-s}\, g(s,\delta)\, ds,$$

where $\widehat{L}'(s) = -\sum_{n=1}^{\infty}\frac{(-1)^n\lambda(n)\log n}{n^s}$. Since $\widehat{L}'(s) = \frac{d}{ds}\widehat{L}(s) = \frac{d}{ds}\left[L(s)(1-2\alpha(s))\right]$, it is entire, and we can move the contour of integral (2.11) from $\Re(s) = \frac{k+1}{2}+\epsilon$ to $\Re(s) = \epsilon$ and we have (2.11) $\ll \frac{1}{\delta^\epsilon}$. To show that (2.12) and (2.13) are negligible, we need the following lemma.

Lemma 2.4 ([13, Lemma 2]). *Suppose that* $|\arg z| < \frac{\pi}{2}$ *and* $0 < c < k$. *Then, we have*

$$\frac{1}{2\pi i}\int_{(c)} \Gamma'(s) z^{-s}\, ds = e^{-z}\log z$$

and

$$\frac{1}{2\pi i}\int_{(c)} \Gamma(s)\frac{\Gamma'}{\Gamma}(k-s) z^{-s}\, ds = e^{-z}\left(\frac{\Gamma'}{\Gamma}(k) - \int_0^1 \frac{e^{tz}-1}{t}(1-t)^{k-1}\, dt\right).$$

Now we consider the following integral

$$\frac{1}{2\pi i}\int_{(\frac{k+1}{2}+\epsilon)} L'(s)(2\pi)^{-s}(1/2)^{-s}\Gamma'(s)\, e^{\pm i(\pi/2-\delta)s}\, ds, \tag{2.14}$$

which we rewrite in the following form:

$$\frac{1}{2\pi i}\int_{(\frac{k+1}{2}+\epsilon)} \sum_{n=1}^{\infty} b(n)(\mp 2\pi n i\frac{1}{2}e^{\pm i\delta})^{-s}\Gamma'(s)\, ds, \tag{2.15}$$

where $L'(s) = -\sum_{n=1}^{\infty}\frac{\lambda(n)\log n}{n^s} = \sum_{n=1}^{\infty}\frac{b(n)}{n^s}$. By Lemma 2.4, the integral (2.15) is equal to

$$\begin{aligned}
&= \sum_{n=1}^{\infty} b(n)\log(\mp 2\pi n i\frac{1}{2}e^{\pm i\delta})e^{\pm(2\pi n i\frac{1}{2}e^{\pm i\delta})}\\
&= \sum_{n=1}^{\infty} b(n)\log(\mp 2\pi n i\frac{1}{2}e^{\pm i\delta})e^{\pm\pi n i}e^{\pm 2\pi n i\frac{1}{2}(e^{\pm i\delta}-1)}\\
&= \sum_{n=1}^{\infty}(-1)^n b(n)\log(\mp 2\pi n i\frac{1}{2}e^{\pm i\delta})e^{-2\pi n\frac{1}{2}(2\sin(\delta/2))e^{\pm i\delta/2}}\\
&= \frac{1}{2\pi i}\int_{(\frac{k+1}{2}+\epsilon)}\sum_{n=1}^{\infty}\frac{(-1)^n b(n)\log(\mp\pi n i e^{\pm i\delta})}{n^s}(2\pi)^{-s}(1/2)^{-s}(2\sin(\delta/2))^{-s}e^{\mp i\frac{\delta}{2}s}\Gamma(s)\, ds,
\end{aligned}$$

where the inverse Mellin transform of the gamma function is applied in the last equality. Note that

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{(-1)^n b(n)\log(\mp\pi n i e^{\pm i\delta})}{n^s}\\ =\ &\sum_{n=1}^{\infty}\frac{(-1)^n b(n)\log n}{n^s}+\sum_{n=1}^{\infty}\frac{(-1)^n b(n)\log(\mp\pi i e^{\pm i\delta})}{n^s}\\ =\ &-\frac{d^2}{ds^2}\left(L(s)\left(1-2\alpha(s)\right)\right)+\log(\mp\pi i e^{\pm i\delta})\frac{d}{ds}\left(L(s)\left(1-2\alpha(s)\right)\right).\end{aligned}$$

We move the contour in (2.14) from $\Re(s)=\frac{k+1}{2}+\epsilon$ to $\Re(s)=\epsilon$, and we have (2.14) $\ll\left(\frac{1}{\delta}\right)^{\epsilon}$. This implies (2.13) $\ll\left(\frac{1}{\delta}\right)^{\epsilon}$.

2.4. An estimate of the integral (2.12)

Let

$$\omega_1(v,\delta)_\pm=\frac{1}{2\pi i}\int_{(\frac{k+1}{2}+\epsilon)}\Gamma(s)\frac{\Gamma'}{\Gamma}(k-s)(2\pi)^{-s}v^{-s}e^{\pm i(\pi/2-\delta)s}\,ds.$$

By Lemma 2.4, this equals

$$e^{-z}\left(\frac{\Gamma'}{\Gamma}(k)-\int_0^1\frac{e^{tz}-1}{t}(1-t)^{k-1}\,dt\right),$$

where $z=\mp 2\pi i v e^{\pm i\delta}=2\pi v\sin\delta\mp i2\pi v\cos\delta=x\mp iy$. Then we have

$$\omega_1(v,\delta)_\pm\ll e^{-x}\log\frac{1}{\delta}+\min\left\{1,\frac{1}{x^k}\right\}\tag{2.16}$$

and

$$\begin{aligned}&\omega_1(v+1/2,\delta)_\pm+\omega_1(v,\delta)_\pm\\ &\ll e^{-x}\delta\log\frac{1}{\delta}+\frac{1}{v}\min\left\{1,x^{1-k}\right\}+\delta\min\left\{1,\frac{1}{x^k}\right\}.\end{aligned}\tag{2.17}$$

(2.16) and (2.17) are essentially shown in [13]. Then, we find the following identity:

$$\begin{aligned}F(\delta)_\pm&=\frac{1}{2\pi i}\int_{(\frac{k+1}{2}+\epsilon)}L'(s)\Gamma(s)\frac{\Gamma'}{\Gamma}(k-s)(2\pi)^{-s}\left(\frac{1}{2}\right)^{-s}e^{\pm i(\pi/2-\delta)s}\,ds\\ &=-\sum_{n=1}^{\infty}\lambda(n)\log n\,\omega_1\left(\frac{n}{2},\delta\right)_\pm.\end{aligned}\tag{2.18}$$

Note that (2.12) $=F(\delta)_+ + i^{k+1}e^{i(\pi/2-\delta)k}F(\delta)_-$. To bound up (2.18), we introduce an upper bound on a partial sum of the coefficients of $L(s)$ and $\widehat{L}(s)$.

Lemma 2.5. *Let $S_m^\pm=\sum_{n=1}^m(\pm1)^n\lambda(n)\log n$. Then,*

$$S_m^\pm\ll m^{k/2-1/6+\epsilon}.$$

Proof. Let $\mathfrak{a}(n) = \frac{\lambda(n)}{n^{\frac{k-1}{2}}}$. If we show that $\sum_{n\le m}(\pm 1)^n \mathfrak{a}(n) \ll m^{1/3+\epsilon}$, Lemma 2.5 follows from partial summation. Friedlander and Iwaniec [17] already obtained that $\sum_{n\le m}\mathfrak{a}(n) \ll m^{1/3+\epsilon}$.

For the modular form $f(z) = \sum_{n=1}^{\infty}\lambda(n)q^n$, we consider its additive twist

$$g(z) := f(z+1/2) = f\left|\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}\right. = \sum_{n=1}^{\infty}(-1)^n\lambda(n)q^n.$$

Note that [5, Proposition 3.1] and the criterion for cusp form in [20] implies $g(z)$ is a cusp form of level 4 with Nebentypus χ. The result of [17] holds if the L-function has the standard functional equation and satisfies the Ramanujan-Petersson conjecture. Since $L_g(s)$ meets the two conditions, we have

$$\sum_{n\le m}(-1)^n\mathfrak{a}(n) \ll m^{1/3+\epsilon}.$$

□

Since the following series

$$-\sum_{n=1}^{\infty} S_n^- \left((-1)^n\omega_1\left(\frac{n+1}{2},\delta\right)_{\pm} + (-1)^n\omega_1\left(\frac{n}{2},\delta\right)_{\pm}\right)$$

absolutely converges by Lemma 2.5 and (2.17), and

$$\lim_{n\to\infty} S_n^-(-1)^n\omega_1\left(\frac{n+1}{2},\delta\right) = 0$$

by Lemma 2.5 and (2.16), we have

$$F(\delta)_{\pm} = -\sum_{n=1}^{\infty} S_n^- \left((-1)^n\omega_1\left(\frac{n+1}{2},\delta\right)_{\pm} + (-1)^n\omega_1\left(\frac{n}{2},\delta\right)_{\pm}\right).$$

Now, we have

$$\begin{aligned} F(\delta)_{\pm} &\ll \sum_{n=1}^{\infty} n^{k/2-1/6+\epsilon}\left|\omega\left(\frac{n+1}{2},\delta\right)_{\pm} + \omega\left(\frac{n}{2},\delta\right)_{\pm}\right| \\ &\ll \sum_{n=1}^{\infty} n^{k/2-1/6+\epsilon}\left[e^{-\frac{n\delta}{2}}\delta\log\frac{1}{\delta} + \frac{2}{n}\min\left\{1,\frac{1}{(n\delta)^{k-1}}\right\} + \delta\min\left\{1,\frac{1}{(n\delta)^k}\right\}\right] \\ &\ll \left(\frac{1}{\delta}\right)^{\frac{k}{2}-\frac{1}{6}+\epsilon}, \end{aligned}$$

which implies that (2.12) $\ll \left(\frac{1}{\delta}\right)^{\frac{k}{2}-\frac{1}{6}+\epsilon}$.

In conclusion, if $L(s)$ has a simple zero $\rho = \beta + it_0$, then there is an arbitrarily large $T = \frac{1}{\delta}$ such that (2.4) $\gg_\epsilon T^{\beta-\epsilon}$ and Theorem 1 for self-dual primitive forms in $S_{2k+1}(\Gamma_0(4),\chi)$ follows.

Remark 4. The step from the second inequality to the third inequality is the only place where the condition weight $k \geq 2$ is required. It would be interesting to show that $F(\delta) \ll \left(\frac{1}{\delta}\right)^{\frac{k}{2}-\frac{1}{6}+\epsilon}$ for modular forms of weight one.

2.5. Newforms on $\Gamma_0(4)$ with the trivial character

Assume that f is a normalized newform for $\Gamma_0(4)$ with the trivial character. In this case, the eigenvalue of f for the $T_2(= U_2)$ is zero. It is a general phenomenon. For a given positive integer N, let f be a cusp form of weight k on $\Gamma_0(N)$. If p is a prime number with $p^2 \mid N$, then $T_p(f)$ is a cusp form on $\Gamma_0(N/p)$ (see [4, Lemma 17]). Hence, in our case, $T_2(f)$ becomes a form on $\Gamma_0(2)$, hence the eigenvalue $\lambda(2)$ should be zero.

Our computations for the form f become much simpler than the case of newforms on $\Gamma_0(4)$ with χ. For example, we need to show in Lemma 2.2 that when $\frac{L'}{L}(s) = \sum_{n=1}^{\infty} a(n)n^{-s}$ has a pole ρ, then $\sum_{n=1}^{\infty}(-1)^n a(n)n^{-s}$ still has the same pole ρ. However, if $\lambda(2) = 0$, then $\sum_{n=1}^{\infty}(-1)^n a(n)n^{-s} = -\sum_{n=1}^{\infty} a(n)n^{-s}$. Also, we have that $S_m^- = -S_m^+$. Without difficulty, we can see that Theorem 1 holds for primitive forms in $S_{2k}(\Gamma_0(4))$.

3. Simple zeros of Maass L-functions

Let f be a primitive Maass form for $\Gamma_0(4)$ of weight 0 with eigenvalue $\frac{1}{4} + r^2$ for the Laplacian operator $\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$. The Fourier-Whittaker expansion of $f(z)$ is given as follows:

$$f(x) = \sum_{n \neq 0} \lambda(n)\sqrt{y}K_{ir}(2\pi|n|y)e^{2\pi nix}.$$

Again the eigenvalue $\lambda(2) = 0$ as in Section 2.5. The L-function attached to f is defined to be

$$L_f(s) = \sum_{n=1}^{\infty} \lambda(n)n^{-s}$$

and it satisfies the functional equation

$$\Lambda_f(s) := \left(\frac{\sqrt{4}}{\pi}\right)^s \Gamma\Big(\frac{s+a+ir}{2}\Big)\Gamma\Big(\frac{s+a-ir}{2}\Big) = \pm\Lambda_f(1-s),$$

where $a = 0$ if f is even, and $a = 1$ if f is odd. We can determine the root number ± 1 in the functional equation. f is an eigenfunction of the Fricke involution W_4, where $W_4(f)(z) = f\left(\frac{-1}{4z}\right)$. Since $W_4^2(f) = f$, $W_4(f) = \pm f$. The parity of f and the eigenvalue for W_4 determine the root number as follows:

	$W_4(f) = f$	$W_4(f) = -f$
even	$+$	$-$
odd	$-$	$+$

For the sake of convenience, we omit the subscript f from $L_f(s)$ and $\Lambda_f(s)$. All the computations and estimates in this section are similar to those in [11]. For the sake of readers' convenience, we repeat them here.

From now on, we assume that f is even. As in the case of modular forms, we consider the integral

$$\frac{1}{2\pi i}\int_{(1+\epsilon)}\frac{L'}{L}(1-s)\frac{L'}{L}(s)\Lambda(s)\Big(s-\frac{1}{2}\Big)f(s,\delta)\,ds, \tag{3.1}$$

where $f(s,\delta)=e^{i(\pi/2-\delta)s}\pm e^{i(\pi/2-\delta)(1-s)}$ and $\pm$ in $f(s,\delta)$ is the opposite of the root number. The integral (3.1) is expressed as the following sum over simple zeros of $L(s)$.

$$-\sum_{0<\Re(\rho)<1}L'(\rho)H(\rho)(\rho-1/2)e^{i(\pi/2-\delta)\rho}+O_f(1), \tag{3.2}$$

where $H(s)=\left(\frac{\sqrt{4}}{\pi}\right)^s\Gamma\big(\frac{s+ir}{2}\big)\Gamma\big(\frac{s-ir}{2}\big)$ and $O_f(1)$ comes from trivial zeros of $L(1-s)$ and simple poles $\pm ir$ of $H(s)$. By the Weyl-type subconvexity, Phragmen-Lindelöf argument, and Stirling's formula, we have

$$(3.2)\ll_{f,\epsilon}\sum_{\substack{|t|<T^{1+\epsilon}\\ \rho=\beta+it:\,\text{simple}}}T^{\frac{\beta_0}{3}+\frac{2}{3}+\epsilon}\ll_{f,\epsilon}N_f^s(T^{1+\epsilon})\cdot T^{\frac{\beta_0}{3}+\frac{2}{3}+\epsilon},$$

where $\beta_0=\sup\{\beta\,|\,\rho=\beta+it\text{ is a simple zero of }L(s)\}$. As in the case of modular forms, we show that $(3.1)\gg\big(\frac{1}{\delta}\big)^{\beta+1/2-\epsilon}$ for some arbitrarily small δ if there is a non-trivial simple zero $\rho=\beta+it_0$. This implies Theorem 1 for Maass L-functions.

Via Stirling's formula, we can see

$$\begin{aligned}H(s)(s-1/2)&=\left(\frac{\sqrt{4}}{\pi}\right)^s\Gamma\Big(\frac{s+ir}{2}\Big)\Gamma\Big(\frac{s-ir}{2}\Big)(s-1/2)\\&=\sqrt{8\pi}(1/2)^{-s}(2\pi)^{-s}\Gamma(s+1/2)+b(1/2)^{-s}(2\pi)^{-s}\Gamma(s-1/2)\\&\quad+(1/2)^{-s}(2\pi)^{-s}\Gamma(s-1/2)E_{(1,r)}(s),\end{aligned}$$

where b is some constant and $E_{(1,r)}(s)$ is holomorphic and $O\big(\frac{1}{|s|}\big)$ for $\Re(s)>1$.

Let $X(s)=\pm\frac{H(1-s)}{H(s)}$. Then $\frac{L'}{L}(1-s)=\frac{X'}{X}(s)-\frac{L'}{L}(s)$ and the integral (3.1) equals

$$\begin{aligned}&-\frac{1}{2\pi i}\int_{(1+\epsilon)}\frac{L'}{L}(s)L'(s)(1/2)^{-s}(2\pi)^{-s}G_1(s)f(s,\delta)\,ds\\&+\frac{1}{2\pi i}\int_{(1+\epsilon)}\frac{X'}{X}(s)L'(s)(2\pi)^{-s}G_1(s)f(s,\delta)\,ds,\end{aligned}$$

where $G_1(s)=\sqrt{8\pi}\Gamma(s+1/2)+b\Gamma(s-1/2)+\Gamma(s-1/2)E_{(1,r)}(s)$.

Lemma 2.1 transforms

$$\frac{1}{2\pi i}\int_{(1+\epsilon)}\frac{L'}{L}(s)L'(s)(1/2)^{-s}(2\pi)^{-s}\Gamma\Big(s+\frac{1}{2}\Big)f(s,\delta)\,ds$$

into

$$\frac{1}{2\pi i}\int_{(1+\epsilon)}\sum_{n=1}^{\infty}\frac{(-1)^n a(n)}{n^s}(\pi)^{-s}\Gamma\Big(s+\frac{1}{2}\Big)\Big(2\sin\frac{\delta}{2}\Big)^{-s-1/2}g(s,\delta)\,ds, \tag{3.3}$$

where

$$\frac{L'}{L}(s)L'(s)=\sum_{n=1}^{\infty}a(n)n^{-s},$$
$$g(s,\delta)=(-i)^{1/2}(e^{i\delta/2})^{-s+1/2}\pm e^{i(\pi/2-\delta)}i^{1/2}(e^{-i\delta/2})^{-s+1/2}.$$

However, as we pointed out in Section 2.5, $\sum_{n=1}^{\infty}\frac{(-1)^n a(n)}{n^s}=-\sum_{n=1}^{\infty}\frac{a(n)}{n^s}$ because $\lambda(2)=0$. If $L(s)$ has a simple zero $\rho=\beta+it_0$, the integrand in the integral (3.3) has a pole at $s=\rho$. Hence by Lemma 2.3, (3.3) $\gg\big(\frac{1}{\delta}\big)^{\beta+1/2-\epsilon}$ for some arbitrarily small $\delta>0$. The integral takes up the main term for (3.1).

Next, we consider

$$\frac{1}{2\pi i}\int_{(1+\epsilon)}\frac{L'}{L}(s)L'(s)(1/2)^{-s}\Gamma\Big(s-\frac{1}{2}\Big)f(s,\delta)\,ds.$$

Again by Lemma 2.1, this integral is $\ll(1/\delta)^{1/2+\epsilon}$.

For the integral

$$\frac{1}{2\pi i}\int_{(1+\epsilon)}\frac{L'}{L}(s)L'(s)(1/2)^{-s}(2\pi)^{-s}\Gamma\Big(s-\frac{1}{2}\Big)E_{(1,r)}(s)f(s,\delta)\,ds,$$

by taking the absolute value on the integrand, we can see that it is $O((1/\delta)^{\epsilon})$.

Next, let us consider the integral

$$\frac{1}{2\pi i}\int_{(1+\epsilon)}\frac{X'}{X}(s)L'(s)(1/2)^{-s}(2\pi)^{-s}(b\Gamma(s-1/2)+\Gamma(s-1/2)E_{(1,r)}(s))f(s,\delta)\,ds$$

which is dominated by

$$\frac{1}{2\pi i}\int_{(1+\epsilon)}\frac{X'}{X}(s)L'(s)(2\pi)^{-s}\Gamma(s-1/2)f(s,\delta)\,ds.$$

If we shift the contour of the integral to $1/2+\epsilon$, then

$$\frac{1}{2\pi i}\int_{(1+\epsilon)}\frac{X'}{X}(s)L'(s)(1/2)^{-s}(2\pi)^{-s}\Gamma(s-1/2)e^{i(\pi/2-\delta)(-s)}\,ds$$
$$=\frac{1}{2\pi i}\int_{(1/2+\epsilon)}\frac{X'}{X}(s)L'(s)(1/2)^{-s}(2\pi)^{-s}\Gamma(s-1/2)e^{i(\pi/2-\delta)(-s)}\,ds.$$

By the Weyl-type subconvexity bound and Stirling's formula, the integrand is $O(|t|^{\frac{1}{3}+\epsilon-1}e^{-\delta|t|})$ and this integral is $O((1/\delta)^{\frac{1}{3}+\epsilon})$.

The remaining integral to estimate is

$$\frac{1}{2\pi i}\int_{(1+\epsilon)}\frac{X'}{X}(s)L'(s)(1/2)^{-s}(2\pi)^{-s}\Gamma(s+1/2)f(s,\delta)\,ds. \tag{3.4}$$

Recall that $H(s)=\left(\frac{\sqrt{4}}{\pi}\right)^s\Gamma\left(\frac{s+ir}{2}\right)\left(\frac{s-ir}{2}\right)$. By Stirling's formula, we have $H(s)$ $=(\pi)^{-s}\Gamma\left(s-\frac{1}{2}\right)E_{(2.r)}(s)$, where $E_{(2,r)}(s)$ is holomorphic in the complex plane except when $s=\pm ir-2n$ for $n=0,1,2,\dots$ and $E_{(2,r)}(s)=\sqrt{8\pi}+O(1/s)$. Then, we have

$$\begin{aligned}X(s)=\frac{H(1-s)}{H(s)}&=(\pi)^{2s-1}\frac{\Gamma(1/2-s)E_{(2,r)}(1-s)}{\Gamma(s-1/2)E_{(2,r)}(s)}\\&=(\pi)^{2s-1}\frac{\Gamma(3/2-s)(s-1/2)E_{(2,r)}(1-s)}{\Gamma(s+1/2)(1/2-s)E_{(2,r)}(s)}\\&=-(\pi)^{2s-1}\frac{\Gamma(3/2-s)E_{(2,r)}(1-s)}{\Gamma(s+1/2)E_{(2,r)}(s)}.\end{aligned}$$

Hence we have

$$\frac{X'}{X}(s)=2\log\pi-\frac{\Gamma'}{\Gamma}(3/2-s)-\frac{\Gamma'}{\Gamma}(s+1/2)-\frac{E'_{(2,r)}}{E_{(2,r)}}(1-s)-\frac{E'_{(2,r)}}{E_{(2,r)}}(s).$$

Since $\frac{\Gamma'}{\Gamma}(z)=\log z-\frac{1}{2z}+O(1/|z|^2)$, it is easy to see that $\frac{E'_{(2,r)}}{E_{(2,r)}}(s)=O\left(\frac{1}{|s|}\right)$. Then, the contribution of $\frac{E'_{(2,r)}}{E_{(2,r)}}(1-s)+\frac{E'_{(2,r)}}{E_{(2,r)}}(s)$ to the integral (3.4) is $O((1/\delta)^{\frac{1}{3}})$ when we move the contour of integration from $c=1+\epsilon$ to $c=1/2$.

We put the remaining part of $\frac{X'}{X}(s)$ into the integral (3.4), and we have

$$\frac{1}{2\pi i}\int_{(1+\epsilon}L'(s)(1/2)^{-s}(2\pi)^{-s}G_2(s)f(s,\delta)\,ds,$$

where $G_2(s)=2\log\pi\Gamma(s+1/2)-\frac{\Gamma'}{\Gamma}(3/2-s)\Gamma(s+1/2)-\Gamma'(s+1/2)$. As in the case of modular forms, we can show that

$$\frac{1}{2\pi i}\int_{(1+\epsilon)}L'(s)(1/2)^{-s}(2\pi)^{-s}\Gamma(s+1/2)f(s,\delta)\,ds\ll\left(\frac{1}{\delta}\right)^{1/2+\epsilon}$$

and

$$\frac{1}{2\pi i}\int_{(1+\epsilon)}L'(s)(1/2)^{-s}(2\pi)^{-s}\Gamma'(s+1/2)f(s,\delta)\,ds\ll\left(\frac{1}{\delta}\right)^{1/2+\epsilon}.$$

We define

$$\omega_2(v,\delta)_\pm=\frac{1}{2\pi i}\int_{(1+\epsilon)}\frac{\Gamma'}{\Gamma}(3/2-s)\Gamma(s+1/2)(2\pi)^{-s}v^{-s}e^{\pm i(\pi/2-\delta)s}\,ds.$$

The only left integral to estimate is

$$\frac{1}{2\pi i}\int_{(1+\epsilon)} L'(s)(1/2)^{-s}(2\pi)^{-s}\frac{\Gamma'}{\Gamma}(3/2-s)\Gamma(s+1/2)f(s,\delta)\,ds,$$

which equals

$$-\sum_{n=1}^{\infty}\lambda(n)\log n\left(\omega_2\left(\frac{n}{2},\delta\right)_+ \pm e^{i(\pi/2-\delta)}\omega_2\left(\frac{n}{2},\delta\right)_-\right). \tag{3.5}$$

In [11], we extended (2.16) and (2.17) to the following form.

Lemma 3.1. *Suppose $v \geq 1/2$.*

(1)

$$\omega_2(v,\delta)_\pm \ll v^{1/2}\left(e^{-x}\log\frac{1}{\delta}+\min\{1,\frac{1}{x^2}\}\right),\ x=2\pi v\sin\delta.$$

(2)

$$\omega_2(v+1/2,\delta)_\pm+\omega_2(v,\delta)_\pm \ll e^{-x}\left(\delta\log\frac{1}{\delta}+\frac{1}{v^{1/2}}\log\frac{1}{\delta}\right) + \delta\min\{1,\frac{1}{x^2}\}+\frac{1}{v^{1/2}}\min\{1,\frac{1}{x}\}.$$

Since the Petersson-Ramanujan conjecture for Maass forms has not been established, we can not use the result of Friedlander and Iwaniec [17]. A result of Chandrasekharan and Narasimhan [10] combined with the bound of the Fourier coefficients by Kim and Sarnak [22] gives the following lemma.

Lemma 3.2. *Let $L(s,f)=\sum_{n=1}^{\infty}\frac{\lambda(n)}{n^s}$ be an L-function attached to a cuspidal automorphic representation of $GL_2(A_{\mathbb{Q}})$ with $\lambda(n)\ll_{f,\epsilon} n^{\theta+\epsilon}$. Then,*

$$\sum_{n\leq x}\lambda(n)\ll_{f,\epsilon} x^{\frac{1}{2}+\theta-\frac{4\theta+1}{8\theta+6}+\epsilon}.$$

In particular, when f is a Maass form, by taking $\theta=\frac{7}{64}$, we have

$$\sum_{n\leq x}\lambda(n)\ll_{f,\epsilon} x^{0.4002841+\epsilon}.$$

Proof. We apply [10, Theorem 4.1] to the L-function $L(s,f)$. In our case, the constants δ, A and β correspond to 1, 1 and $1+\theta+\epsilon$, respectively. Since $L(s,f)$ is entire, we do not have the second big O-term. The first big O-term is $O\left(x^{\frac{1}{4}+2\eta(\frac{1}{4}+\theta+\epsilon)}\right)$. For the third O-term, we have

$$\sum_{x<\frac{n}{\sqrt{q_f}}\ll x+x^{\frac{1}{2}-\eta}}|\lambda(n)|\ll_f (x+x^{\frac{1}{2}-\eta})^{1+\theta+\epsilon}-x^{1+\theta+\epsilon}\ll x^{\frac{1}{2}+\theta-\eta+\epsilon}.$$

By taking $\eta=\frac{4\theta+1}{8\theta+6}$, the O-terms are $\ll x^{\frac{1}{2}+\theta-\frac{4\theta+1}{8\theta+6}+\epsilon}$. □

Remark 5. In [11, Lemma 6], the first named author used the bound $\sum_{n\le x}\lambda(n) \ll_\epsilon x^{1/3+\epsilon}$. However, this bound is true under the Petersson-Ramanujan conjecture. Lemma 3.2 can replace [11, Lemma 6].

By Lemma 3.1 and Lemma 3.2,

$$\begin{aligned}
&\sum_{n=1}^{\infty}\lambda(n)\log n\omega_2\Big(\frac{n}{2},\delta\Big)_{\pm}\\
&=\sum_{n=1}^{\infty}S_n^-(-1)^n\bigg(\omega_2\Big(\frac{n+1}{2},\delta\Big)_{\pm}+\omega_2\Big(\frac{n}{2},\delta\Big)_{\pm}\bigg)\\
&\ll\sum_{n=1}^{\infty}n^{0.4002841+\epsilon}\bigg[e^{-n\delta}\Big(\delta\log\frac{1}{\delta}+\frac{1}{n^{1/2}}\log\frac{1}{\delta}\Big)+\delta\min\Big\{1,\frac{1}{(n\delta)^2}\Big\}+\frac{1}{n^{1/2}}\min\Big\{1,\frac{1}{n\delta}\Big\}\bigg]\\
&\ll\delta\Big(\log\frac{1}{\delta}\Big)\Big(\frac{1}{\delta}\Big)^{1.4002841+\epsilon}+\Big(\log\frac{1}{\delta}\Big)\Big(\frac{1}{\delta}\Big)^{0.9002841+\epsilon}+\Big(\frac{1}{\delta}\Big)^{0.4002841+\epsilon}+\Big(\frac{1}{\delta}\Big)^{0.9002841+\epsilon}\\
&\ll\Big(\frac{1}{\delta}\Big)^{0.9002841+\epsilon}
\end{aligned}$$

and this implies (3.5) $\ll \left(\frac{1}{\delta}\right)^{0.9002841+\epsilon}$. Here we use that $S_n^- = -S_n^+$. Our computations are summarized that (3.1) $\gg \left(\frac{1}{\delta}\right)^{1/2+\beta-\epsilon}$ for some arbitrarily small δ of $L(s)$ has a simple zero $\rho=\beta+it_0$, and Theorem 1 for even Maass forms follows.

3.1. Odd Maass forms

When f is odd, all the computations can be carried out similarly as in the case of even Maass forms. There is a slight change in the following main integral;

$$\frac{1}{2\pi i}\int_{(1+\epsilon)}\frac{L'}{L}(1-s)L'(s)H(s)(e^{i(\pi/2-\delta)s}\mp e^{i(\pi/2-\delta)(1-s)})ds, \tag{3.6}$$

where $H(s)=\left(\frac{\sqrt{4}}{\pi}\right)^s\Gamma\left(\frac{s+1+ir}{2}\right)\Gamma\left(\frac{s+1-ir}{2}\right)$ and $\mp$ is the opposite with the sign in the functional equation.

Via Stirling's formula, we also see

$$\begin{aligned}
&\pi^{-s}\Gamma\left(\frac{s+1+ir}{2}\right)\Gamma\left(\frac{s+1-ir}{2}\right)\\
&=\sqrt{2\pi}(1/2)^{-s}(2\pi)^{-s}\Gamma(s+1/2)+b(1/2)^{-s}(2\pi)^{-s}\Gamma(s-1/2)\\
&\quad+(1/2)^{-s}(2\pi)^{-s}\Gamma(s-1/2)E_{(1,r)}(s),
\end{aligned}$$

where b is some constant and $E_{(1,r)}(s)$ is holomorphic and $O\big(\frac{1}{|s|}\big)$ in $\mathrm{Re}(s)>1$.

Again, by Stirling's formula, we have that $H(s)=(\pi)^{-s}\Gamma\left(s+\frac{1}{2}\right)E_{(2,r)}(s)$ where $E_{(2,r)}(s)=\sqrt{2\pi}+O\big(\frac{1}{|s|}\big)$ is holomorphic in $\mathrm{Re}(s)>0$. Then, $X(s)=$

$\frac{H(1-s)}{H(s)}$ equals

$$(\pi)^{2s-1}\frac{\Gamma(3/2-s)E_{(2,r)}(1-s)}{\Gamma(s+1/2)E_{(2,r)}(s)},$$

and

$$\frac{X'}{X}(s) = 2\log\pi - \frac{\Gamma'}{\Gamma}(3/2-s) - \frac{\Gamma'}{\Gamma}(1/2+s) - \frac{E'_{(2,r)}}{E_{(2,r)}}(1-s) - \frac{E'_{(2,r)}}{E_{(2,r)}}(s).$$

We leave it to the readers to show that (3.6) $\gg \big(\frac{1}{\delta}\big)^{\beta+1/2-\epsilon}$ for some arbitrarily small δ if $L(s)$ has a simple zero $\rho = \beta + it_0$.

4. The Weyl-type subconvexity

To prove Theorem 3, we follow [3] closely, where it says that their approach also works for Maass forms with some effort. In [3], they introduced a simplified Bessel δ-method which arises naturally from the Voronoi summation formula. Let $M_\lambda(M,\xi)$ denote the set of normalized Maass forms of weight zero with level M, nebentypus ξ and eigenvalue $\lambda = \frac{1}{4}+r^2$ for $r \in (0,\infty)$. The following Voronoi summation formula is a special case of [23, Theorem A.4]. Note that $g_M = \bar{g} \in M_\lambda(M,\bar{\xi})$ in their notation.

Lemma 4.1 (The Voronoi summation Formula)**.** *Let $g \in M_\lambda(M,\xi)$. Let a, $\bar{a}$, c be integers such that $c \geq 1$, $(a,c)=1$, $a\bar{a} = 1 \,(\mathrm{mod}\, c)$ and $(c,M)=1$. Let $F(x) \in C_c^\infty(0,\infty)$. Then there exists a complex number η_f of modulus* 1 (*the Atkin-Lehner pseudo-eigenvalue of g*) *such that*

$$\sum_{n=1}^{\infty}\lambda_g(n)e\Big(\frac{an}{c}\Big)F(n) = \frac{\eta_g\xi(-c)}{c\sqrt{M}}\sum_{n=1}^{\infty}\overline{\lambda_g(n)}e\Big(-\frac{\overline{aM}n}{c}\Big)\breve{F}_1\Big(\frac{n}{c^2M}\Big) + \frac{\eta_g\xi(c)}{c\sqrt{M}}\sum_{n=1}^{\infty}\overline{\lambda_g(n)}e\Big(\frac{\overline{aM}n}{c}\Big)\breve{F}_2\Big(\frac{n}{c^2M}\Big),$$

where $\breve{F}_1(y)$ and $\breve{F}_2(y)$ is defined by

$$\breve{F}_1(y) = -\frac{\pi}{\sin\pi ir}\int_0^\infty F(x)\{J_{2ir}(4\pi\sqrt{xy}) - J_{-2ir}(4\pi\sqrt{xy})\}\,\mathrm{d}x$$

and

$$\breve{F}_2(y) = 4\epsilon_f\cosh\pi r\int_0^\infty F(x)K_{2ir}(4\pi\sqrt{xy})\,\mathrm{d}x.$$

Here ϵ_g be the eigenvalue of g under the reflection operator.

The Voronoi summation formula in [23, Theorem A.4] is more general, where it is only required that $\big((c,M),\frac{M}{(c,M)}\big)=1$. However, in our setting, $c=p$ will be a large prime while M is fixed, so our condition $(c,M)=1$ in Lemma 4.1 is justified.

Now, we establish a δ-identity for Maass forms. For this purpose, we define the following Bessel integrals:

$$\begin{aligned} &J_r(a,b;X)\\ &= -\frac{2\pi i}{(e^{-\pi r}-e^{\pi r})}\int_0^\infty U(x/X)e(2a\sqrt{x})\{J_{2ir}(4\pi b\sqrt{x})-J_{-2ir}(4\pi b\sqrt{x})\}\,dx \end{aligned}$$

and

$$K_r(a,b;X) = 4\epsilon_g \cosh \pi r \int_0^\infty U(x/X)e(2a\sqrt{x})K_{2ir}(4\pi b\sqrt{x})\,dx$$

for a fixed non-negative valued bump function $U \in C_c^\infty(0,\infty)$ with support in $[1,2]$, $a,b>0$ and $X>1$. Now, we can state a δ-identity for Maass forms.

Lemma 4.2 (δ-identity for Maass forms)**.** *Let p be prime and $N, X > 1$ be such that $X > p^2/N$ and $X^{1-\epsilon} > N$. Let m and n be integers in the dyadic interval $[N,2N]$. For any $A \geq 0$, we have*

$$\begin{aligned} &\frac{C_U m^{1/4}}{p^{1/2}X^{3/4}}\cdot\frac{1}{p}\sum_{a\,(\mathrm{mod}\,p)} e\left(\frac{a(m-n)}{p}\right)\cdot J_r\left(\frac{\sqrt{m}}{p},\frac{\sqrt{n}}{p};X\right)\\ &\quad+\frac{C_U m^{1/4}\xi(-1)}{p^{1/2}X^{3/4}}\cdot\frac{1}{p}\sum_{a\,(\mathrm{mod}\,p)} e\left(\frac{a(n+m)}{p}\right)\cdot K_r\left(\frac{\sqrt{m}}{p},\frac{\sqrt{n}}{p};X\right)\\ &= \delta(m=n)\left(1+O_{r,U}\left(\frac{p}{\sqrt{NX}}\right)\right)+O_{r,U,A}(X^{-A}), \end{aligned} \tag{4.1}$$

where $C_U = (1+i)/\tilde{U}(3/4)$, the $\delta(m=n)$ is the Kronecker δ that detects the condition $m=n$, and the implied constants depend only on r, U and A.

For the proof of the δ-identity, we need several lemmas and notations. First, we have

$$\begin{aligned} &\int_0^\infty e^{iax}J_\nu(ax)x^{\mu-1}\,dx\\ &= \frac{e^{\pi i(\nu+\mu)/2}}{\sqrt{\pi}(2a)^\mu}\frac{\Gamma(\nu+\mu)\Gamma(1/2-\mu)}{\Gamma(\nu-\mu+1)}, \quad -\mathrm{Re}\,\nu < \mathrm{Re}\,\mu < \frac{1}{2} \end{aligned} \tag{4.2}$$

which is [3, (3.5)]. Next, we define $J_r^{\pm}(a,a;X)$ to be

$$J_r^{\pm}(a,a;X) = -\frac{2\pi i}{(e^{-\pi r}-e^{\pi r})}\int_0^\infty U(x/X)e(2a\sqrt{x})J_{\pm 2ir}(4\pi a\sqrt{x})\,dx,$$

so $J_r(a,a;X) = J_r^+(a,a;X) - J_r^-(a,a;X)$. By applying the inverse Mellin inversion of the function U and (4.2) to $J_r^+(a,a;X)$, we have

$$J_r^+(a,a;X) = -\frac{X}{(e^{-\pi r}-e^{\pi r})}\int_{(\sigma)}\frac{2e^{-\pi r}\tilde{U}(s)}{\sqrt{\pi}(-8\pi i a\sqrt{X})^{2-2s}}\frac{\Gamma(2ir+2-2s)\Gamma(2s-3/2)}{\Gamma(2ir+2s-1)}\,ds,$$

where $\tilde{U}(s)$ is the Mellin transform of the function U, and (σ) is the contour $\Re(s) = \sigma$ with $3/4 < \sigma < 1$. By shifting the contour to the line $\Re(s) = 0$, due to the poles at $s = 1/4$ and $3/4$, we have the following estimate for $J_r^+(a, a; X)$,

$$J_r^+(a, a; X) = -\frac{2\pi i e^{-\pi r}}{(e^{-\pi r} - e^{\pi r})} \frac{(1+i)\tilde{U}(3/4)X}{4\pi(a^2X)^{1/4}} + O\left(\frac{X}{(a^2X)^{3/4}}\right).$$

Similarly, we have

$$J_r^-(a, a; X) = -\frac{2\pi i e^{\pi r}}{(e^{-\pi r} - e^{\pi r})} \frac{(1+i)\tilde{U}(3/4)X}{4\pi(a^2X)^{1/4}} + O\left(\frac{X}{(a^2X)^{3/4}}\right)$$

and the following lemma follows.

Lemma 4.3. *We have*

$$J_r(a, a; X) = \frac{(1-i)\tilde{U}(3/4)X}{2(a^2X)^{1/4}} + O\left(\frac{X}{(a^2X)^{3/4}}\right)$$

with the implied constant depending only on r and U.

For the case $a \neq b$, $J_r(a, b; X)$ is negligible under a certain condition for a, b, and X. The following lemma is an analogue of [3, Lemma 3.2], and we skip its proof.

Lemma 4.4. *Suppose that $b^2X > 1$. Then $J_r(a, b; X) = O(X^{-A})$ for any $A \geq 0$ if $|a - b|\sqrt{X} > X^\epsilon$.*

The analysis of $K_r(a, b, X)$ is quite simple. By the asymptotic $K_\nu(x) \sim \sqrt{\frac{\pi}{2x}}\, e^{-x}$ for sufficiently large $x \gg 1$ (see [1, 9.7.2]), we have:

Lemma 4.5. *We have $K_r(a, b, X) = O(X^{-A})$ for any $a, b > 0$ and $A \geq 0$. The implied constant depends only on A.*

Now, we are ready to prove the δ-identity. We see that the second sum is absorbed into the error term $O_{r,U,A}(X^{-A})$ by Lemma 4.5. For the first sum, Lemma 4.3 gives the δ term when $m = n$. Note that Lemma 4.4 implies that $J_r(\sqrt{m}/p, \sqrt{n}/p; X) = O_{r,U,A}(X^{-A})$ unless $|m - n| \leq X^\epsilon p\sqrt{N/X}$. However, the condition on m and n does not hold except for the trivial case $m = n$ because m is congruent to n modulo p and $X^\epsilon p\sqrt{N/X} < p$. Hence, we proved Lemma 4.2.

Let us recall that

$$\begin{aligned} S(N) &= \sum_{n=1}^{\infty} \lambda_g(n) e(f(n)) V\left(\frac{n}{N}\right) \\ &= \sum_{m=1}^{\infty} e(f(m)) V\left(\frac{m}{N}\right) \sum_{n=1}^{\infty} \lambda_g(n) \delta(m = n). \end{aligned}$$

Here, $V(x)$ and $\lambda_g(n)$ were defined in Theorem 3. Applying Lemma 4.2 and dividing the a-sum in (4.1) according whether $(a, p) = 1$ or not, we have

$$S(N) = S_p^*(N, X) + S_p^0(N, X) + R_p(N, X) + O(X^{-A}),$$

with

$$S_p^*(N,X) = \frac{M^{1/2}N^{1/4}}{\eta_g p^{3/2}X^{3/4}} \sum_{m=1}^{\infty} e(f(m))V_\natural\Big(\frac{m}{N}\Big) \sum_{a\,(\mathrm{mod}\,p)}{}^{*} e\Big(\frac{am}{p}\Big)$$
$$\cdot \Big\{ \sum_{n=1}^{\infty} \lambda_g(n) e\Big(-\frac{an}{p}\Big) J_r\Big(\frac{\sqrt{m}}{p}, \frac{\sqrt{n}}{p}; X\Big)$$
$$+ \xi(-1) \sum_{n=1}^{\infty} \lambda_g(n) e\Big(\frac{an}{p}\Big) K_r\Big(\frac{\sqrt{m}}{p}, \frac{\sqrt{n}}{p}; X\Big) \Big\}$$

and

$$S_p^0(N,X) = \frac{M^{1/2}N^{1/4}}{\eta_g p^{3/2}X^{3/4}} \sum_{m=1}^{\infty} e(f(m))V_\natural\Big(\frac{m}{N}\Big)$$
$$\cdot \Big\{ \sum_{n=1}^{\infty} \lambda_g(n) J_r\Big(\frac{\sqrt{m}}{p}, \frac{\sqrt{n}}{p}; X\Big)$$
$$+ \xi(-1) \sum_{n=1}^{\infty} \lambda_g(n) K_r\Big(\frac{\sqrt{m}}{p}, \frac{\sqrt{n}}{p}; X\Big) \Big\},$$

where $V_\natural(x) = C_U \eta_g M^{-1/2} \cdot x^{1/4} V(x)$ and $\sum^*$ means that the a-sum is subject to $(a,p)=1$, and

$$R_p(N,X) = O\Big(\frac{p}{\sqrt{NX}} \sum_{n\sim N} \big|\lambda_g(n)\big|\Big) = O\Big(p\sqrt{\frac{N}{X}}\Big).$$

Setting $p > M$, we can apply the Voronoi summation in Lemma 4.1 to the n-variable. By applying the Voronoi summation with $c = p$ and $F(x) = M^{-1}U\big(\frac{x}{MX}\big)e\big(\frac{2\sqrt{mx}}{p\sqrt{M}}\big)$, we have that

$$S_p^*(N,X)$$
$$= \frac{\xi(-p)N^{1/4}}{p^{1/2}X^{3/4}} \sum_{m=1}^{\infty} e(f(m))V_\natural\Big(\frac{m}{N}\Big) \sum_{n=1}^{\infty} \overline{\lambda_g(n)} S(n,m;p) e\Big(\frac{2\sqrt{nm}}{\sqrt{M}p}\Big) U\Big(\frac{n}{MX}\Big),$$

where

$$S(n,m;p) = \sum_{a\,(\mathrm{mod}\,p)}{}^{*} e\Big(\frac{am}{p} + \frac{\overline{aM}n}{p}\Big).$$

Similarly, we have

$$S_p^0(N,X) = \frac{\xi(-1)p^{1/2}N^{1/4}}{X^{3/4}} \sum_{m=1}^{\infty} e(f(m))V_\natural\Big(\frac{m}{N}\Big) \sum_{n=1}^{\infty} \overline{\lambda_g(n)} e\Big(\frac{2\sqrt{nm}}{\sqrt{M}}\Big) U\Big(\frac{p^2 n}{MX}\Big)$$

after the application of the Voronoi summation with modulus $c = 1$ and $F(x) = p^2M^{-1}U\big(\frac{p^2x}{MX}\big)e\big(\frac{2\sqrt{mx}}{\sqrt{M}}\big)$. By a trivial estimate, we obtain

$$S_p^0(N,X) \ll \frac{N^{5/4}X^{1/4}}{p^{3/2}}.$$

Finally, after averaging over primes p in $[P, 2P]$ for a large parameter P, there are $\asymp P/\log P$ many such p's. Hence $S(N)$ can be written as follows.

Proposition 4.6. *Let $V(x) \in C_c^\infty(0,\infty)$ be supported in $[1,2]$ with $Var(V) \ll 1$ and $V^{(j)}(x) \ll_j \Delta^j$ for $j \geq 0$. Let parameters $N, X, P > N^\epsilon$ be such that*

$$P^2/N < X, \quad N < X^{1-\epsilon}. \tag{4.3}$$

Let P^* be the number of primes in $[P, 2P]$. We have

$$S(N) = \sum_{n=1}^\infty \lambda_g(n)e(f(n))V\Big(\frac{n}{N}\Big) = S(N,X,P) + O\Big(\frac{P\sqrt{N}}{\sqrt{X}} + \frac{N^{5/4}X^{1/4}}{P^{3/2}}\Big),$$

with

$$\begin{aligned} S(N,X,P) =& \frac{N^{1/4}}{P^*X^{3/4}} \sum_{p\sim P} \frac{\xi(-p)}{\sqrt{p}} \sum_{m=1}^\infty e(f(m))V_\natural\Big(\frac{m}{N}\Big) \\ &\cdot \sum_{n=1}^\infty \overline{\lambda_g(n)}S(n,m;p)e\Big(\frac{2\sqrt{nm}}{\sqrt{M}p}\Big)U\Big(\frac{n}{MX}\Big), \end{aligned} \tag{4.4}$$

where $V_\natural(x) = C_U\eta_gM^{-1/2}\cdot x^{1/4}V(x)$ is supported in $[1,2]$, satisfying $\mathrm{Var}(V_\natural) \ll 1$ and $V_\natural^{(j)}(x) \ll_j \Delta^j$.

From now on, we follow almost the same flow in [3, Section 5] and skip some details. Thanks to Proposition 4.6, we only need to estimate the sum $S(N,X,P)$ in (4.4) to study $S(N)$. For convenience, let

$$X = P^2K^2/N, \quad N^\epsilon < K < T^{1-\epsilon} \tag{4.5}$$

with the parameter $K > 1$ to be optimized later. Then the first assumption in (4.3) is justified, while the second assumption amounts to

$$P > N^{1+\epsilon}/K. \tag{4.6}$$

Let $f(x) = T\phi(x/N) + \gamma x$ with $\phi(x) = -\log x$. By applying the Poisson summation to the m-sum in (4.4), we have

$$\sum_{m=1}^\infty e(f(m))S(n,m;p)e\Big(\frac{2\sqrt{nm}}{\sqrt{M}p}\Big)V_\natural\Big(\frac{m}{N}\Big) = N \sum_{(m,p)=1} e\Big(-\frac{\overline{mM}n}{p}\Big)\mathscr{I}(n,m,p),$$

where

$$\mathscr{I}(y,m,p) = \int_0^\infty V_\natural(x)e\Big(T\phi(x) + \gamma Nx + \frac{2\sqrt{Nxy}}{\sqrt{M}p} - \frac{mNx}{p}\Big)\,dx. \tag{4.7}$$

By following [3], we only need to consider the m-sum for $|m-\gamma p| \ll R$, where $R = PT/N$, with a negligible error. Furthermore, we have the following estimate for $\mathscr{I}(y,r,p)$ by the same way in [3, Lemma 5.1].

Lemma 4.7. *For $1 \le y/MX \le 2$, we have*

$$\mathscr{I}(y,r,p) \ll \frac{1}{\sqrt{T}}.$$

Hence, (4.4) can be written as

$$S(N,X,P) = \frac{N^2}{P^*(PK)^{3/2}} \sum_{n=1}^{\infty} \overline{\lambda_g(n)} U\Big(\frac{n}{MX}\Big) \sum_{p\sim P} \frac{\xi(-p)}{\sqrt{p}} \cdot \sum_{\substack{(m,p)=1 \\ |m-\gamma p|\ll R}} e\Big(-\frac{\overline{mM}n}{p}\Big) \mathscr{I}(n,m,p) + O(N^{-A}). \tag{4.8}$$

Next, we apply Cauchy inequality and the Ramanujan bound on average for $\lambda_g(n)$ to (4.8). Then,

$$S(N,X,P) \ll_g \frac{N^{3/2}}{P^*\sqrt{PK}} \left(\sum_{n=1}^{\infty} \left| \sum_{p\sim P} \frac{\xi(-p)}{\sqrt{p}} \sum_{\substack{(m,p)=1 \\ |m-\gamma p|\ll R}} e\Big(-\frac{\overline{mM}n}{p}\Big) \mathscr{I}(n,m,p) \right|^2 U\Big(\frac{n}{MX}\Big) \right)^{1/2}.$$

Note that the square of the right-hand side is

$$\frac{N^3}{P^{*2}PK} \sum\sum_{p_1,p_2\sim P} \frac{\xi(p_1\overline{p_2})}{\sqrt{p_1p_2}} \sum\sum_{\substack{(m_i,p_i)=1 \\ |m_i-\gamma p_i|\ll R}} \cdot \sum_{n=1}^{\infty} e\Big(\frac{\overline{m_2M}n}{p_2} - \frac{\overline{m_1M}n}{p_1}\Big) \mathscr{I}(n,m_1,p_1)\overline{\mathscr{I}(n,m_2,p_2)} U\Big(\frac{n}{MX}\Big). \tag{4.9}$$

After applying the Poisson summation with modulus p_1p_2 to the n-sum in (4.9), we obtain

$$\sum_{n=1}^{\infty} e\Big(\frac{\overline{m_2M}n}{p_2} - \frac{\overline{m_1M}n}{p_1}\Big) \mathscr{I}(n,m_1,p_1)\overline{\mathscr{I}(n,m_2,p_2)} U\Big(\frac{n}{MX}\Big) = MX \sum_{n\equiv \overline{m_1M}p_2 - \overline{m_2M}p_1 \,(\mathrm{mod}\, p_1p_2)} \mathscr{L}\Big(\frac{MXn}{p_1p_2}; m_1,m_2,p_1,p_2\Big), \tag{4.10}$$

where

$$\mathscr{L}(x) = \mathscr{L}(x;m_1,m_2,p_1,p_2) = \int_0^{\infty} U(y)\mathscr{I}(MXy,m_1,p_1)\overline{\mathscr{I}(MXy,m_2,p_2)} e(-xy)\,dy. \tag{4.11}$$

By using (4.5) and (4.7), we can transform (4.11) into the triple integral;

$$\begin{aligned}\mathscr{L}(x) = &\int_0^\infty \int_0^\infty V_\natural(\nu_1)\overline{V_\natural(\nu_2)} \\ &e\left(T(\phi(\nu_1)-\phi(\nu_2)) + \gamma N(\nu_1-\nu_2) - \frac{Nm_1\nu_1}{p_1} + \frac{Nm_2\nu_2}{p_2}\right) \\ &\cdot \int_0^\infty U(y) e\left(2PK\left(\frac{\sqrt{\nu_1}}{p_1} - \frac{\sqrt{\nu_2}}{p_2}\right)\sqrt{y} - xy\right) dy\, d\nu_2\, d\nu_1. \end{aligned} \tag{4.12}$$

By following [3], we have an estimate of $\mathscr{L}(x)$ which is an analogue of [3, Lemma 5.5]. Unfortunately, there might be an error in the calculation for the estimate of $\partial f_0(w,\nu_2)/\partial\nu_2$ in [3]. But this doesn't affect our result when we set $\phi(x) := -\log x$, which satisfies the general conditions of $\phi(x)$ in [3, Theorem 1.1]. This is why we explicitly set the function $\phi(x)$ and repeat the overall process in [3] in this paper.

Lemma 4.8. *Let N, T, K, $P > 1$ be parameters with $N^\epsilon < K \ll T$ and $N^{1+\epsilon} < PK$. Let $p_i \sim P$ and $|m_i - \gamma p_i| \ll PT/N\,(i = 1,2)$. Let the integral $\mathscr{L}(x)$ be as in (4.12).*

(1) *We have $\mathscr{L}(x) = O(N^{-A})$ if $|x| \geq K$.*

(2) *Assume that $K^2/T > N^\epsilon$. For $K^2/T \ll |x| < K$, we have*

$$\mathscr{L}(x) \ll \frac{1}{T\sqrt{x}}.$$

For $|x| \ll K^2/T$, we have

$$\mathscr{L}(x) \ll \frac{1}{T}.$$

(3) *Let $p_1 = p_2 = p$. Then*

$$\mathscr{L}(0) \ll \min\left\{\frac{1}{T}, \frac{PN^\epsilon}{KN|m_1-m_2|}\right\}.$$

Combining (4.9) and (4.10), together with Lemma 4.8, we obtain

$$S(N,X,P) \ll_M \sqrt{S^2_{\mathrm{diag}}(N,X,P)} + \sqrt{S^2_{\mathrm{off}}(N,X,P)} + O(N^{-A}) \tag{4.13}$$

with

$$S^2_{\mathrm{diag}}(N,X,P) = \frac{N^3X}{P^{*2}P^2K}\sum_{p\sim P}\mathop{\sum\sum}_{\substack{(m_1m_2,p)=1\\ |m_1-\gamma p|,|m_2-\gamma p|\ll R\\ m_1\equiv m_2\,(\mathrm{mod}\,p)}} \min\left\{\frac{1}{T}, \frac{PN^\epsilon}{KN|m_1-m_2|}\right\},$$

and

$$S^2_{\text{off}}(N,X,P) = \frac{N^3X}{P^{*2}P^2K}\sum\sum_{p_1,p_2\sim P}\ \sum\sum_{\substack{(m_i,p_i)=1\\|m_i-\gamma p_i|\ll R}}\left(\sum_{\substack{N/T\ll|n|\ll N/K\\ n\equiv c_1\,(\mathrm{mod}\,p_1p_2)}}\frac{\sqrt{p_1p_2}}{T\sqrt{X|n|}}+\sum_{\substack{0\ll|n|\ll N/T\\ n\equiv c_1\,(\mathrm{mod}\,p_1p_2)}}\frac{1}{T}\right),$$

where $c_1=\overline{m_1M}p_2-\overline{m_2M}p_1$.

For $S^2_{\text{diag}}(N,X,P)$, we split the sum over m_1 and m_2 according to $m_1=m_2$ or not, and we have

$$S_{\text{diag}^2}(N,X,P) \ll \frac{N^3X}{P^{*2}P^2K}\left(\sum_{p\sim P}\ \sum_{\substack{(m,p)=1\\|m-\gamma p|\ll R}}\frac{1}{T}+\sum_{p\sim P}\ \sum\sum_{\substack{(m_1m_2,p)=1\\|m_1-\gamma p|,\\|m_2-\gamma p|\ll R\\ m_1\equiv m_2\,(\mathrm{mod}\,p)\\ m_1\neq m_2}}\frac{PN^{\epsilon}}{KN|m_1-m_2|}\right).$$

Hence,

$$\begin{aligned}S^2_{\text{diag}}(N,X,P) &\ll \frac{N^3X}{P^{*2}P^2K}\left(\frac{P^*R}{T}+\frac{P^*R^2N^{\epsilon}}{PKN}\right)\\ &\ll \left(KN+\frac{T^2}{N^{1-\epsilon}}\right)\log P.\end{aligned}\tag{4.14}$$

Here, we use $|m_1-m_2|\geq p$ and the number of the pairs (m_1,m_2) is bounded by R^2/P to estimate the second term in the parenthesis.

Next, for $S^2_{\text{off}}(N,X,P)$, p_1 must be different from p_2 due to assumption (4.6) and the length N/K of the n-sum. Note that for fixed n, the congruence $n\equiv\overline{m_1M}p_2-\overline{m_2M}p_1\,(\mathrm{mod}\,p_1p_2)$ is equivalent to $m_1\equiv\overline{nM}p_2\,(\mathrm{mod}\,p_1)$ and $m_2\equiv-\overline{nM}p_1\,(\mathrm{mod}\,p_2)$. After interchanging the sum over n and the sums over m_1, m_2, we have

$$S^2_{\text{off}}(N,X,P) = \frac{N^3X}{P^{*2}P^2K}\sum\sum_{\substack{p_1,p_2\sim P\\ p_1\neq p_2}}\left(\sum_{N/T\ll|n|\ll N/K}\ \sum\sum_{\substack{|m_1-\gamma p|,\\|m_2-\gamma p|\ll R\\ m_1\equiv c_2\,(\mathrm{mod}\,p_1)\\ m_2\equiv c_3\,(\mathrm{mod}\,p_2)}}\frac{\sqrt{p_1p_2}}{T\sqrt{X|n|}}+\sum_{0<|n|\ll N/T}\ \sum\sum_{\substack{|m_1-\gamma p|,\\|m_2-\gamma p|\ll R\\ m_1c_2\,(\mathrm{mod}\,p_1)\\ m_2\equiv c_3\,(\mathrm{mod}\,p_2)}}\frac{1}{T}\right),$$

where $c_2 = \overline{nM}p_2$ and $c_3 = -\overline{nM}p_1$. When $T \geq N$, we have

$$S^2_{\text{off}}(N, X, P) \ll \frac{N^3 X}{P^{*2} P^2 K} P^{*2} \left(\frac{P}{T\sqrt{X}} \sqrt{\frac{N}{K}} + \frac{N}{T^2} \right) \left(\frac{R}{P} \right)^2 = \frac{NT}{\sqrt{K}} + KN. \tag{4.15}$$

When $T < N$, the $(R/P)^2$ in (4.15) may be replaced to 1. However, we can rearrange the sum S^2_{off} as follows:

$$\frac{N^3 X}{P^{*2} P^2 K} \sum_{p_1 \sim P} \mathop{\sum\sum}_{\substack{p_1, p_2 \sim P \\ p_1 \neq p_2}} \left(\sum_{N/T \ll |n| \ll N/K} \mathop{\sum\sum}_{\substack{|m_1 - \gamma p|, \\ |m_2 - \gamma p| \ll R \\ m_1 \equiv c_2 \,(\mathrm{mod}\, p_1) \\ m_2 \equiv c_3 \,(\mathrm{mod}\, p_2)}} \frac{\sqrt{p_1 p_2}}{T\sqrt{X|n|}} + \sum_{0 < |n| \ll N/T} \mathop{\sum\sum}_{\substack{|m_1 - \gamma p|, \\ |m_2 - \gamma p| \ll R \\ m_1 \equiv c_2 \,(\mathrm{mod}\, p_1) \\ m_2 \equiv c_3 \,(\mathrm{mod}\, p_2)}} \frac{1}{T} \right).$$

Thus for $T < N$, we have

$$S^2_{\text{off}}(N, X, P) \ll \frac{N^3 X}{P^{*2} P^2 K} P^* R \left(\frac{P}{T\sqrt{X}} \sqrt{\frac{N}{K}} + \frac{N}{T^2} \right) \ll \left(\frac{NT}{\sqrt{K}} + KN \right) \frac{N}{T} \log P. \tag{4.16}$$

Combining (4.15) and (4.16), we have

$$S^2_{\text{off}}(N, X, P) \ll \frac{N^3 X}{P^{*2} P^2 K} P^* R \left(\frac{NT}{\sqrt{K}} + KN \right) \left(1 + \frac{N}{T} \right) \log P. \tag{4.17}$$

We conclude from (4.13), (4.14) and (4.17) that

$$S(N, X, P) \ll \left(\frac{T}{\sqrt{N}} + \left(\sqrt{KN} + \frac{\sqrt{NT}}{K^{1/4}} \right) \left(1 + \sqrt{\frac{N}{T}} \right) \right) N^{\epsilon}. \tag{4.18}$$

Here we assumed $P < N^A$, where A is a large fixed constant so that $\log P < N^{\epsilon}$.

Finally, by Proposition 4.6 and (4.18), we have

$$S(N) \ll \frac{T}{\sqrt{N}} N^{\epsilon} + \left(\sqrt{KN} + \frac{\sqrt{NT}}{K^{1/4}} \right) \left(1 + \sqrt{\frac{N}{T}} \right) N^{\epsilon} + \frac{N}{K} + \frac{N\sqrt{K}}{P}.$$

Under assumption of $N^{\epsilon} < T < N^{3/2-\epsilon}$, by taking $K = T^{2/3}$ and $P = N/T^{1/3}$,

$$\ll \frac{T}{\sqrt{N}} N^{\epsilon} + T^{1/3} N^{1/2+\epsilon} \left(1 + \sqrt{\frac{N}{T}} \right) + \frac{N}{T^{2/3}} + T^{2/3}.$$

Noting that

$$\frac{TN^{\epsilon}}{\sqrt{N}},\, T^{2/3} \ll T^{1/3} N^{1/2+\epsilon} \quad \text{and} \quad \frac{N}{T^{2/3}} \ll \frac{N^{1+\epsilon}}{T^{1/6}},$$

we have the conclusion that

$$S(N) \ll T^{1/3}N^{1/2+\epsilon} + \frac{N^{1+\epsilon}}{T^{1/6}}.$$

4.1. Proof of Theorem 2

First, we derive the Weyl-type subconvexity for Maass form L-functions from Theorem 3. The process is very standard [3, Sec. 6], but we include this for completeness. Let $g \in M_\lambda(M,\xi)$ be a Maass form of weight zero with level M and nebentypus ξ. Then, by the approximate functional equation [21, Theorem 5.3] and dividing the interval of summation dyadically, we have

$$L_g(1/2+it) \ll t^\epsilon \left| \frac{S(N)}{\sqrt{N}} \right|$$

for some $N < t^{1+\epsilon}$ and $S(N) = \sum_{n=1}^\infty \lambda_g(n)n^{-it}V\left(\frac{n}{N}\right)$. It is left to show that $S(N) \ll \sqrt{N}t^{1/3+\epsilon}$ in the range $t^{2/3+\epsilon} < N < t^{1+\epsilon}$. This can be verified by taking $\gamma = 0$, $T = \frac{t}{2\pi}$ in Theorem 3, and Theorem 2 follows.

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