J. Korean Math. Soc. **60** (2023), No. 1, pp. 71–90
https://doi.org/10.4134/JKMS.j220095
pISSN: 0304-9914 / eISSN: 2234-3008

MINIMAL SURFACE SYSTEM IN EUCLIDEAN FOUR-SPACE

Hojoo Lee

Abstract. We construct generalized Cauchy-Riemann equations of the first order for a pair of two $\mathbb{R}$-valued functions to deform a minimal graph in $\mathbb{R}^3$ to the one parameter family of the two dimensional minimal graphs in $\mathbb{R}^4$. We construct the two parameter family of minimal graphs in $\mathbb{R}^4$, which include catenoids, helicoids, planes in $\mathbb{R}^3$, and complex logarithmic graphs in $\mathbb{C}^2$. We present higher codimensional generalizations of Scherk's periodic minimal surfaces.

1. Introduction

Extending Bernstein's Theorem that the only entire minimal graphs in $\mathbb{R}^3$ are planes, Osserman [24, Theorem 5.1] proved that any entire two dimensional minimal graph in $\mathbb{R}^4$ should be *degenerate*, in the sense that its generalized Gauss map lies on a hyperplane of the complex projective space $\mathbb{CP}^3$. Landsberg [15] investigated the systems of the first order whose solutions induce minimal varieties. The classical Cauchy–Riemann equations $(f_x, f_y) = (g_y, -g_x)$ satisfies the minimal surface system of the second order

$$\begin{cases} 0 = \left(1 + {f_y}^2 + {g_y}^2\right) f_{xx} - 2\left(f_x f_y + g_x g_y\right) f_{xy} + \left(1 + {f_x}^2 + {g_x}^2\right) f_{yy}, \\ 0 = \left(1 + {f_y}^2 + {g_y}^2\right) g_{xx} - 2\left(f_x f_y + g_x g_y\right) g_{xy} + \left(1 + {f_x}^2 + {g_x}^2\right) g_{yy}. \end{cases}$$

We construct the Osserman system of the first order, whose solution graphs become degenerate minimal surfaces in $\mathbb{R}^4$.

Theorem 1.1 (Osserman system as a generalization of Cauchy–Riemann equations)**.** *Let*

$$\Sigma = \left\{ \begin{bmatrix} x \\ y \\ f(x,y) \\ g(x,y) \end{bmatrix} \in \mathbb{R}^4 \,\middle|\, (x,y) \in \Omega \right\}$$

be the graph in $\mathbb{R}^4$ *of the pair* $(f(x,y), g(x,y))$ *of height functions defined on the domain* Ω*. Let* $g_{\Sigma} = Edx^2 + 2Fdxdy + Gdy^2$ *denote the induced metric on*

Received March 2, 2022; Revised August 28, 2022; Accepted October 27, 2022.
2020 *Mathematics Subject Classification.* Primary 53A10, 49Q05.
Key words and phrases. Minimal surface system, Gauss map, Scherk surface.

Σ. *If the pair* $(f(x,y), g(x,y))$ *obeys the Osserman system with* $\mu \in \mathbb{R} - \{0\}$:

$$(1) \quad \begin{bmatrix} f_x \\ f_y \end{bmatrix} = \mu \begin{bmatrix} \frac{E}{\omega} & \frac{F}{\omega} \\ \frac{F}{\omega} & \frac{G}{\omega} \end{bmatrix} \begin{bmatrix} g_y \\ -g_x \end{bmatrix}, \text{ or equivalently, } \begin{bmatrix} g_x \\ g_y \end{bmatrix} = -\frac{1}{\mu} \begin{bmatrix} \frac{E}{\omega} & \frac{F}{\omega} \\ \frac{F}{\omega} & \frac{G}{\omega} \end{bmatrix} \begin{bmatrix} f_y \\ -f_x \end{bmatrix},$$

where $\omega = \sqrt{EG - F^2}$, *then the two dimensional graph* Σ *is minimal in* $\mathbb{R}^4$.

The *Lagrange potentials* (Lemma 4.1 and Remark 4.2) on minimal graphs in $\mathbb{R}^3$ play a critical role in the Jenkins-Serrin construction [11, Section 3] of minimal graphs with infinite boundary values. We use the Lagrange potentials to construct explicit examples of two dimensional minimal graphs in $\mathbb{R}^4$ and three dimensional minimal graphs in $\mathbb{R}^6$.

Theorem 1.2 (Two applications of Lagrange potentials of the height functions on minimal surfaces in $\mathbb{R}^3$)**.** *Let*

$$\Sigma_0 = \left\{ \begin{bmatrix} x \\ y \\ p(x,y) \end{bmatrix} \in \mathbb{R}^3 \,\middle|\, (x,y) \in \Omega \right\}$$

be the minimal graph of the function $p : \Omega \to \mathbb{R}$ *defined on a domain* $\Omega \subset \mathbb{R}^2$. *Let* $q : \Omega \to \mathbb{R}$ *denote the Lagrange potential of* $p : \Omega \to \mathbb{R}$ *such that*

$$\begin{bmatrix} q_y \\ -q_x \end{bmatrix} = \begin{bmatrix} \frac{p_x}{\sqrt{1+{p_x}^2+{p_y}^2}} \\ \frac{p_y}{\sqrt{1+{p_x}^2+{p_y}^2}} \end{bmatrix}.$$

(a) *For a constant* $\lambda \in \mathbb{R} - \{0\}$, *we consider the graph of the pair* $(f(x,y), g(x,y))$:

$$\Sigma_\lambda = \left\{ \begin{bmatrix} x \\ y \\ f(x,y) \\ g(x,y) \end{bmatrix} = \begin{bmatrix} x \\ y \\ (\cosh\lambda)\ p(x,y) \\ (\sinh\lambda)\ q(x,y) \end{bmatrix} \in \mathbb{R}^4 \,\middle|\, (x,y) \in \Omega \right\}.$$

Then, the pair $(f(x,y), g(x,y))$ *satisfies the Osserman system* (1) *in Theorem 1.1 with* $\mu = \coth\lambda$. *In particular, the graph* Σ_λ *is a minimal surface in* $\mathbb{R}^4$. *Also, we obtain the invariance of the conformally changed induced metric* $\frac{1}{\sqrt{\det\left(g_{\Sigma_\lambda}\right)}}\, g_{\Sigma_\lambda} = \frac{1}{\sqrt{\det\left(g_{\Sigma_0}\right)}}\, g_{\Sigma_0}$.

(b) *For any constant* $\lambda \in \mathbb{R} - \{0\}$, *the three dimensional graph*

$$\left\{ \begin{bmatrix} x \\ y \\ z \\ p_x + \lambda z q_x \\ p_y + \lambda z q_y \\ \lambda q \end{bmatrix} \in \mathbb{R}^6 \,\middle|\, (x,y) \in \Omega,\ z \in \mathbb{R} \right\}$$

is minimal in $\mathbb{R}^6$. *Moreover, it is a special Lagrangian graph in* $\mathbb{C}^3$.

We present examples of minimal graphs of codimension two in $\mathbb{R}^4$. In Example 2.3, we construct the two parameter family of minimal graphs in $\mathbb{R}^4$, which include catenoids, helicoids, planes in $\mathbb{R}^3$, and complex logarithmic graphs in $\mathbb{C}^2$. In Example 4.7, we give a family of codimension two minimal graphs in $\mathbb{R}^4$, which contains Scherk's doubly periodic minimal graphs in $\mathbb{R}^3$. We present higher codimensional generalizations of Scherk's periodic minimal surfaces.

2. Minimal surface system in $\mathbb{R}^4$ and Cauchy–Riemann equations

Our ambient space is the Euclidean space $\mathbb{R}^4$ equipped with the flat metric ${dx_1}^2 + {dx_2}^2 + {dx_3}^2 + {dx_4}^2$.

Proposition 2.1 (Two dimensional minimal graphs in $\mathbb{R}^4$). *Let Σ be the graph*

$$\Sigma = \left\{ \begin{bmatrix} x \\ y \\ f(x,y) \\ g(x,y) \end{bmatrix} \in \mathbb{R}^4 \;\middle|\; (x,y) \in \Omega \right\}.$$

The induced metric g_Σ on the surface Σ reads

$$g_\Sigma = E dx^2 + 2F dx dy + G dy^2,$$

where the coefficients of the first fundamental form are determined by

$$E = \Phi_x \cdot \Phi_x = 1 + {f_x}^2 + {g_x}^2, \; F = \Phi_x \cdot \Phi_y = f_x f_y + g_x g_y, \; G = \Phi_y \cdot \Phi_y = 1 + {f_y}^2 + {g_y}^2.$$

Let $\omega = \sqrt{EG - F^2}$. We introduce the minimal surface operator $\mathcal{L}_\Sigma$ and Laplace-Beltrami operator $\triangle_\Sigma$ acting on functions on Ω :

$$\mathcal{L}_\Sigma = G\frac{\partial^2}{\partial x^2} - 2F\frac{\partial^2}{\partial x \partial y} + E\frac{\partial^2}{\partial y^2},$$

$$(2) \quad \triangle_\Sigma = \triangle_{g_\Sigma} = \frac{1}{\omega}\left[\frac{\partial}{\partial x}\left(\frac{G}{\omega}\frac{\partial}{\partial x} - \frac{F}{\omega}\frac{\partial}{\partial y} \right) + \frac{\partial}{\partial y}\left(-\frac{F}{\omega}\frac{\partial}{\partial x} + \frac{E}{\omega}\frac{\partial}{\partial y} \right) \right].$$

Then, the following three conditions are equivalent.

(a) *The height functions $f(x,y)$ and $g(x,y)$ are harmonic on the graph Σ:*

$$\triangle_\Sigma f = 0 \quad \text{and} \quad \triangle_\Sigma g = 0.$$

(b) *The graph Σ is minimal in $\mathbb{R}^4$.*

(c) *The height functions $f(x,y)$ and $g(x,y)$ solve the minimal surface system:*

$$\mathcal{L}_\Sigma f = 0 \quad \text{and} \quad \mathcal{L}_\Sigma g = 0.$$

Proof. Though the equivalences of (a), (b), (c) are well-known, we sketch the proof for the convenience of the readers. The equivalence of (a) and (b) follows from [24, Equation (3.14) in Section 2], which indicates that the Euler-Lagrange system for the area functional of the graph is

$$\frac{\partial}{\partial x}\left(\frac{G}{\omega}\begin{bmatrix} f_x \\ g_x \end{bmatrix} - \frac{F}{\omega}\begin{bmatrix} f_y \\ g_y \end{bmatrix} \right) + \frac{\partial}{\partial y}\left(-\frac{F}{\omega}\begin{bmatrix} f_x \\ g_x \end{bmatrix} + \frac{E}{\omega}\begin{bmatrix} f_y \\ g_y \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which is equivalent to

$$\triangle_{\Sigma} \begin{bmatrix} f_x \\ g_x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

There are several ways to establish the equivalence of (b) and (c): [24, Section 2, p. 16–17], [17, Section 2], [1, Section 1.2] (for arbitrary codimension), [20, Appendix: The minimal surface system], and [7, Example 1] (for more general ambient spaces). Here, we adopt the argument in the proof of [22, Theorem 2.2]. We use the formula (2) and introduce

$$(\mathcal{P}, \mathcal{Q}) := \left(\frac{\partial}{\partial x}\left(\frac{G}{\omega}\right) - \frac{\partial}{\partial y}\left(\frac{F}{\omega}\right), \frac{\partial}{\partial y}\left(\frac{E}{\omega}\right) - \frac{\partial}{\partial x}\left(\frac{F}{\omega}\right) \right)$$

to obtain the identity for the mean curvature vector $H(x, y)$:

$$H = \triangle_{\Sigma} \begin{bmatrix} x \\ y \\ f(x,y) \\ g(x,y) \end{bmatrix} = \frac{1}{\omega} \begin{bmatrix} \mathcal{P} \\ \mathcal{Q} \\ \mathcal{P} f_x + \mathcal{Q} f_y + \frac{1}{\omega}\mathcal{L}_{\Sigma} f \\ \mathcal{P} g_x + \mathcal{Q} g_y + \frac{1}{\omega}\mathcal{L}_{\Sigma} g \end{bmatrix} = \frac{\mathcal{P}}{\omega}\Phi_x + \frac{\mathcal{Q}}{\omega}\Phi_y + \frac{1}{\omega^2} \begin{bmatrix} 0 \\ 0 \\ \mathcal{L}_{\Sigma} f \\ \mathcal{L}_{\Sigma} g \end{bmatrix}.$$

First, we assume (b). Since the mean curvature vector $H(x, y)$ vanishes on the minimal surface, the four quantities $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{L}_{\Sigma} f$, $\mathcal{L}_{\Sigma} g$ vanish. So, (c) holds. Second, we assume (c). Since $\mathcal{L}_{\Sigma} f = 0$ and $\mathcal{L}_{\Sigma} g = 0$, the mean curvature vector $H(x, y)$ is equal to the tangent vector $\frac{\mathcal{P}}{\omega}\Phi_x + \frac{\mathcal{Q}}{\omega}\Phi_y$. As the mean curvature vector $H(x, y)$ is normal to the graph Σ, $H(x, y)$ vanishes. So, (b) holds. □

Remark 2.2 (Minimal surface operator $\mathcal{L}_{\Sigma}$ and Laplace-Beltrami operator $\triangle_{\Sigma}$). We assume that the two dimensional minimal graph Σ is minimal in $\mathbb{R}^4$. Then,

$$\triangle_{\Sigma} = \frac{1}{\omega^2}\mathcal{L}_{\Sigma}.$$

Indeed, the minimality of the graph Σ implies the two interesting identities

$$\text{(3)} \qquad \frac{\partial}{\partial y}\left(\frac{F}{\omega}\right) = \frac{\partial}{\partial x}\left(\frac{G}{\omega}\right) \quad \text{and} \quad \frac{\partial}{\partial y}\left(\frac{E}{\omega}\right) = \frac{\partial}{\partial x}\left(\frac{F}{\omega}\right),$$

which imply

$$\begin{aligned} \triangle_{\Sigma} &= \frac{1}{\omega^2}\mathcal{L}_{\Sigma} + \left[\frac{\partial}{\partial x}\left(\frac{G}{\omega}\right) - \frac{\partial}{\partial y}\left(\frac{F}{\omega}\right)\right]\frac{\partial}{\partial x} + \left[\frac{\partial}{\partial y}\left(\frac{E}{\omega}\right) - \frac{\partial}{\partial x}\left(\frac{F}{\omega}\right)\right]\frac{\partial}{\partial y} \\ &= \frac{1}{\omega^2}\mathcal{L}_{\Sigma}. \end{aligned}$$

A geometric meaning of (3) is given in Rado's book [26, p. 108]. A variational proof of (3) can be found in Osserman's book [24, Chapter 3]. An interpretation of (3) (via the conjugate minimal surface) is illustrated in Remark 4.2.

Example 2.3 (Two parameter family of minimal graphs in $\mathbb{R}^4$ connecting complex logarithmic graphs in $\mathbb{C}^2$, catenoids, and helicoids in $\mathbb{R}^3$)**.** Given a pair $(\alpha,\beta)\in\mathbb{R}^+\times\mathbb{R}^+$, we define the two dimensional graph $\Sigma_{(\alpha,\beta)}$ in $\mathbb{R}^4$:

$$\Sigma_{(\alpha,\beta)}=\left\{\left[\begin{array}{c} x\\ y\\ \alpha\ln\left(\frac{\sqrt{x^2+y^2}+\sqrt{x^2+y^2+\beta^2-\alpha^2}}{2}\right)\\ \beta\arctan\left(\frac{y}{x}\right)\end{array}\right]\in\mathbb{R}^4\;\middle|\;(x,y)\in\Omega\right\}.$$

The domain Ω depends on the choice of (α,β). We distinguish the three cases.

(a) We consider the case when $\alpha>\beta>0$. So, $\sqrt{\alpha^2-\beta^2}>0$. Observing that $\left(\frac{\alpha}{\sqrt{\alpha^2-\beta^2}}\right)^2-\left(\frac{\beta}{\sqrt{\alpha^2-\beta^2}}\right)^2=1$, we can take the constant $\lambda>0$ with $(\cosh\lambda,\sinh\lambda)=\left(\frac{\alpha}{\sqrt{\alpha^2-\beta^2}},\frac{\beta}{\sqrt{\alpha^2-\beta^2}}\right)$. We introduce the new coordinates $(\widetilde{x},\widetilde{y})=\left(\frac{x}{\sqrt{\alpha^2-\beta^2}},\frac{y}{\sqrt{\alpha^2-\beta^2}}\right)$. Recalling the identity $\operatorname{arcosh} r=\ln\left(r+\sqrt{r^2-1}\right)$, $r\geq 1$, we find that, up to translations, the rescaled graph $\frac{1}{\sqrt{\alpha^2-\beta^2}}\Sigma_{(\alpha,\beta)}$ is congruent to the surface

$$\left\{\left[\begin{array}{c}\widetilde{x}\\ \widetilde{y}\\ (\cosh\lambda)\ \operatorname{arcosh}\left(\sqrt{\widetilde{x}^{\,2}+\widetilde{y}^{\,2}}\right)\\ (\sinh\lambda)\ \arctan\left(\frac{\widetilde{y}}{\widetilde{x}}\right)\end{array}\right]\in\mathbb{R}^4\;\middle|\;\widetilde{x}^{\,2}+\widetilde{y}^{\,2}\geq 1,\ \widetilde{x}\neq 0\right\}.$$

The limit case $\beta=0$ (or $\lambda=0$) recovers a catenoid in $\mathbb{R}^3$.

(b) When $\alpha=\beta>0$, we take $\lambda=\alpha=\beta$, the minimal surface $\Sigma_{(\alpha,\beta)}$ in $\mathbb{R}^4$ can be identified as the complex logarithmic graph in $\mathbb{C}^2$:

$$\left\{\left[\begin{array}{c}\zeta\\ \lambda\log\zeta\end{array}\right]\in\mathbb{C}^2\;\middle|\;\zeta=x+iy\in\mathbb{C}-\{0\}\right\}.$$

The limit case $\alpha=\beta=0$ (or $\lambda=0$) recovers a plane in $\mathbb{R}^3$.

(c) We assume that $\beta>\alpha>0$. So, $\sqrt{\beta^2-\alpha^2}>0$. Observing that

$$\left(\frac{\beta}{\sqrt{\alpha^2-\beta^2}}\right)^2-\left(\frac{\alpha}{\sqrt{\alpha^2-\beta^2}}\right)^2=1,$$

we can take the constant $\lambda>0$ with $(\cosh\lambda,\sinh\lambda)=\left(\frac{\beta}{\sqrt{\beta^2-\alpha^2}},\frac{\alpha}{\sqrt{\beta^2-\alpha^2}}\right)$. We introduce the new coordinates $(\widetilde{x},\widetilde{y})=\left(\frac{x}{\sqrt{\beta^2-\alpha^2}},\frac{y}{\sqrt{\beta^2-\alpha^2}}\right)$. Recalling the identity $\operatorname{arsinh} r = \ln\left(r + \sqrt{r^2+1}\right)$, $r \in \mathbb{R}$, we find that, up to translations, the rescaled graph $\frac{1}{\sqrt{\beta^2-\alpha^2}}\Sigma_{(\alpha,\beta)}$ is congruent to

$$\left\{ \begin{bmatrix} \widetilde{x} \\ \widetilde{y} \\ (\sinh\lambda)\ \operatorname{arsinh}\left(\sqrt{\widetilde{x}^{\,2}+\widetilde{y}^{\,2}}\right) \\ (\cosh\lambda)\ \arctan\left(\frac{\widetilde{y}}{\widetilde{x}}\right) \end{bmatrix} \in \mathbb{R}^4 \;\middle|\; \widetilde{x} \in \mathbb{R}-\{0\},\, \widetilde{y} \in \mathbb{R} \right\}.$$

The limit case $\alpha = 0$ (or $\lambda = 0$) recovers a helicoid in $\mathbb{R}^3$.

Proposition 2.4 (Cauchy–Riemann equations on the minimal graph)**.** *Let*

$$\Sigma = \left\{ \begin{bmatrix} x \\ y \\ f(x,y) \\ g(x,y) \end{bmatrix} \in \mathbb{R}^4 \mid (x,y) \in \Omega \right\}$$

be the two dimensional minimal graph in $\mathbb{R}^4$*. If the system*

$$(4)\qquad \begin{bmatrix} \mathcal{A}_x \\ \mathcal{A}_y \end{bmatrix} = \begin{bmatrix} \frac{E}{\omega} & \frac{F}{\omega} \\ \frac{F}{\omega} & \frac{G}{\omega} \end{bmatrix} \begin{bmatrix} \mathcal{B}_y \\ -\mathcal{B}_x \end{bmatrix}, \quad \textit{or equivalently,} \quad \begin{bmatrix} \mathcal{B}_x \\ \mathcal{B}_y \end{bmatrix} = -\begin{bmatrix} \frac{E}{\omega} & \frac{F}{\omega} \\ \frac{F}{\omega} & \frac{G}{\omega} \end{bmatrix} \begin{bmatrix} \mathcal{A}_y \\ -\mathcal{A}_x \end{bmatrix}$$

holds on Ω*, then the function* $\mathcal{A}(x,y) + i\mathcal{B}(x,y)$ *is holomorphic on* Σ*.*

Proof. We observe the two identities (3) in Remark 2.2:

$$\frac{\partial}{\partial y}\left(\frac{F}{\omega}\right) = \frac{\partial}{\partial x}\left(\frac{G}{\omega}\right) \quad \text{and} \quad \frac{\partial}{\partial y}\left(\frac{E}{\omega}\right) = \frac{\partial}{\partial x}\left(\frac{F}{\omega}\right).$$

Hence, we can find the potential functions $M(x,y)$ and $N(x,y)$ so that

$$(M_x, M_y) = \left(\frac{E}{\omega}, \frac{F}{\omega}\right) \quad \text{and} \quad (N_x, N_y) = \left(\frac{F}{\omega}, \frac{G}{\omega}\right),$$

in a simply connected neighborhood of any point in the domain Ω. Then,

$$(x,y) \to (\xi_1, \xi_2) = (x + M(x,y), y + N(x,y))$$

is a local diffeomorphism [24, Lemma 4.4]. The the induced conformal metric on the minimal graph Σ in $\mathbb{R}^4$ is given by

$$g_{\Sigma} = \frac{\omega}{2 + \frac{E}{\omega} + \frac{G}{\omega}} \left(d\xi_1^{\,2} + d\xi_2^{\,2}\right).$$

The function $\mathcal{A}(x,y) + i\mathcal{B}(x,y)$ is holomorphic with respect to the conformal coordinates (ξ_1, ξ_2) if and only if the Cauchy–Riemann equations holds:

$$\begin{bmatrix} \frac{\partial}{\partial \xi_1}\left(\mathcal{A} \circ \Xi^{-1}\right) \\ \frac{\partial}{\partial \xi_2}\left(\mathcal{A} \circ \Xi^{-1}\right) \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial \xi_2}\left(\mathcal{B} \circ \Xi^{-1}\right) \\ -\frac{\partial}{\partial \xi_1}\left(\mathcal{B} \circ \Xi^{-1}\right) \end{bmatrix}.$$

It could be transformed to the desired system (4) via the chain rule. □

Remark 2.5. The Beltrami equations [2] associated to the metric

$$g_{\Sigma} = Edx^2 + 2Fdxdy + Gdy^2$$

is the system

$$\begin{bmatrix} \mathcal{B}_x \\ \mathcal{B}_y \end{bmatrix} = - \begin{bmatrix} \frac{E}{\omega} & \frac{F}{\omega} \\ \frac{F}{\omega} & \frac{G}{\omega} \end{bmatrix} \begin{bmatrix} \mathcal{A}_y \\ -\mathcal{A}_x \end{bmatrix}, \quad \text{where} \quad \omega = \sqrt{EG - F^2}.$$

3. Generalized Gauss map and Osserman system of the first order

To define the generalized Gauss map [6,10,22,24] of minimal surfaces in $\mathbb{R}^4$, we prepare the complex hyperquadric $\mathcal{Q}_2$ in the complex projective space $\mathbb{CP}^3$:

$$\mathcal{Q}_2 = \{[z_1 : z_2 : z_3 : z_4] \in \mathbb{CP}^3 \mid {z_1}^2 + {z_2}^2 + {z_3}^2 + {z_4}^2 = 0\}.$$

Definition (Generalized Gauss map of minimal surfaces in $\mathbb{R}^4$, [24, Section 2])**.** We consider a conformal harmonic immersion $X : \Sigma \to \mathbb{R}^4$, $\xi \mapsto X(\xi)$. The generalized Gauss map of Σ is the map $\mathcal{G} : \Sigma \to \mathcal{Q}_2 \subset \mathbb{CP}^3$ defined by

$$\mathcal{G}(\xi) = \left[\overline{\frac{\partial X}{\partial \xi}} \right] = \left[\frac{\partial X}{\partial \xi_1} + i \frac{\partial X}{\partial \xi_2} \right] \in \mathcal{Q}_2.$$

The conformality of the immersion X guarantees that the generalized Gauss map is a well-defined $\mathcal{Q}_2$-valued function. The harmonicity of the immersion X guarantees that the generalized Gauss map is anti-holomorphic.

Lemma 3.1 (Generalized Gauss map of two dimensional minimal graphs in $\mathbb{R}^4$)**.** *We consider the minimal graph Σ in $\mathbb{R}^4$*

$$\Sigma = \left\{ \begin{bmatrix} x \\ y \\ f(x,y) \\ g(x,y) \end{bmatrix} \in \mathbb{R}^4 \mid (x,y) \in \Omega \right\}.$$

The induced metric on Σ is $Edx^2 + 2Fdxdy + Gdy^2$. Let $\omega = \sqrt{EG - F^2}$. Its generalized Gauss map $\mathcal{G} : \Omega \to \mathcal{Q}_2 \subset \mathbb{CP}^3$ in terms of the coordinates (x, y) is

$$\begin{aligned} \mathcal{G}(x,y) &= [z_1 : z_2 : z_3 : z_4] \\ &= \left[\frac{G}{\omega} : i - \frac{F}{\omega} : \frac{G}{\omega} f_x + \left(i - \frac{F}{\omega} \right) f_y : \frac{G}{\omega} g_x + \left(i - \frac{F}{\omega} \right) g_y \right] \\ &= \left[1 - i\frac{F}{\omega} : i\frac{E}{\omega} : \left(1 - i\frac{F}{\omega} \right) f_x + i\frac{E}{\omega} f_y : \left(1 - i\frac{F}{\omega} \right) g_x + i\frac{E}{\omega} g_y \right]. \end{aligned}$$

Proof. For the details of the deduction of Lemma 3.1, we refer to [18, Proposition 6], which was inspired by the equality in [23, Lemma, p. 290]. □

Definition (Degenerate minimal surfaces in $\mathbb{R}^4$, [24, Section 2])**.** We say that a minimal surface Σ in $\mathbb{R}^4$ is degenerate if the image of its $\mathcal{Q}_2$-valued generalized Gauss map lies in a hyperplane of the complex projective space $\mathbb{CP}^3$.

Remark 3.2 (Degeneracy of entire two dimensional minimal graphs in arbitrary codimensions). Extending Bernstein's Theorem that the only entire minimal graphs in $\mathbb{R}^3$ are planes, Osserman [24, Chapter 5] showed that the generalized Gauss map of entire two dimensional minimal graphs in $\mathbb{R}^{n+2\geq 4}$ are degenerate. For a geometric illustration of generalized Gauss map of degenerate minimal surfaces, see [5, Figure 1]. As in [10, Theorem 4.7], degenerate minimal surfaces in $\mathbb{R}^4$ can be described by the Enneper-Weierstrass type representation formula.

Definition (Osserman system for minimal graphs in $\mathbb{R}^4$)**.** Let Σ be the graph in $\mathbb{R}^4$ of the pair $(f(x,y), g(x,y))$ of height functions defined on the domain Ω:

$$\Sigma = \left\{ \Phi(x,y) = \begin{bmatrix} x \\ y \\ f(x,y) \\ g(x,y) \end{bmatrix} \in \mathbb{R}^4 \,\middle|\, (x,y) \in \Omega \right\}.$$

The induced metric g_Σ and the area element on the surface Σ are given by

$$g_\Sigma = Edx^2 + 2Fdxdy + Gdy^2, \quad dA_\Sigma = \omega\, dx\, dy, \quad \omega = \sqrt{EG - F^2},$$

where the coefficients of the first fundamental form are determined by

$$E = \Phi_x \cdot \Phi_x = 1 + {f_x}^2 + {g_x}^2,\ F = \Phi_x \cdot \Phi_y = f_x f_y + g_x g_y,\ G = \Phi_y \cdot \Phi_y = 1 + {f_y}^2 + {g_y}^2.$$

Given a constant $\mu \in \mathbb{R} - \{0\}$, we introduce

$$\begin{bmatrix} f_x \\ f_y \end{bmatrix} = \mu \begin{bmatrix} \frac{E}{\omega} & \frac{F}{\omega} \\ \frac{F}{\omega} & \frac{G}{\omega} \end{bmatrix} \begin{bmatrix} g_y \\ -g_x \end{bmatrix}, \tag{5}$$

or equivalently,

$$\begin{bmatrix} g_x \\ g_y \end{bmatrix} = -\frac{1}{\mu} \begin{bmatrix} \frac{E}{\omega} & \frac{F}{\omega} \\ \frac{F}{\omega} & \frac{G}{\omega} \end{bmatrix} \begin{bmatrix} f_y \\ -f_x \end{bmatrix}, \tag{6}$$

which will be called the Osserman system with the coefficient $\mu \in \mathbb{R} - \{0\}$.

Remark 3.3. To prove the equivalence of two systems (5) and (6), one may use

$$\begin{bmatrix} \frac{E}{\omega} & \frac{F}{\omega} \\ \frac{F}{\omega} & \frac{G}{\omega} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{G}{\omega} & -\frac{F}{\omega} \\ -\frac{F}{\omega} & \frac{E}{\omega} \end{bmatrix}.$$

Theorem 3.4 (Minimality and degeneracy of Osserman minimal graphs in $\mathbb{R}^4$)**.** *If the pair $(f(x,y), g(x,y))$ satisfies the Osserman system* (5) *with $\mu \in \mathbb{R} - \{0\}$, then the graph*

$$\Sigma = \left\{ \begin{bmatrix} x \\ y \\ f(x,y) \\ g(x,y) \end{bmatrix} \in \mathbb{R}^4 \,\middle|\, (x,y) \in \Omega \right\}$$

is minimal in $\mathbb{R}^4$. Moreover, its generalized Gauss map lies on the hyperplane $z_3 + i\mu z_4 = 0$ of the complex projective space $\mathbb{CP}^3$.

Proof. To show the minimality of the graph Σ, we employ Proposition 2.1. Indeed, we use the equalities (6) to obtain

$$\begin{aligned}\triangle_\Sigma f &= \frac{1}{\omega}\left[\frac{\partial}{\partial x}\left(\frac{G}{\omega}f_x - \frac{F}{\omega}f_y\right) + \frac{\partial}{\partial y}\left(-\frac{F}{\omega}f_x + \frac{E}{\omega}f_y\right)\right]\\ &= \frac{1}{\omega}\left[\frac{\partial}{\partial x}(\mu g_y) + \frac{\partial}{\partial y}(-\mu g_x)\right] = 0,\end{aligned}$$

and use the equalities in (5) to obtain

$$\begin{aligned}\triangle_\Sigma g &= \frac{1}{\omega}\left[\frac{\partial}{\partial x}\left(\frac{G}{\omega}g_x - \frac{F}{\omega}g_y\right) + \frac{\partial}{\partial y}\left(-\frac{F}{\omega}g_x + \frac{E}{\omega}g_y\right)\right]\\ &= \frac{1}{\omega}\left[\frac{\partial}{\partial x}\left(-\frac{1}{\mu}f_y\right) + \frac{\partial}{\partial y}\left(\frac{1}{\mu}f_x\right)\right] = 0.\end{aligned}$$

To prove the degeneracy of the minimal graph Σ, we exploit Lemma 3.1. It follows from the Osserman systems (5) and (6) that

$$(f_y, g_y) = \left(\mu\left(\frac{F}{\omega}g_y - \frac{G}{\omega}g_x\right), \frac{1}{\mu}\left(\frac{G}{\omega}f_x - \frac{F}{\omega}f_y\right)\right),$$

which can be complexified to

$$\frac{G}{\omega}f_x + \left(i - \frac{F}{\omega}\right)f_y = -i\mu\left(\frac{G}{\omega}g_x + \left(i - \frac{F}{\omega}\right)g_y\right).$$

We conclude that the generalized Gauss map $\mathcal{G} : \Omega \to \mathcal{Q}_2 \subset \mathbb{CP}^3$, which can be explicitly given in terms of the coordinates (x, y),

$$\begin{aligned}\mathcal{G}(x,y) &= [z_1 : z_2 : z_3 : z_4]\\ &= \left[\frac{G}{\omega} : i - \frac{F}{\omega} : \frac{G}{\omega}f_x + \left(i - \frac{F}{\omega}\right)f_y : \frac{G}{\omega}g_x + \left(i - \frac{F}{\omega}\right)g_y\right]\end{aligned}$$

lies on the hyperplane $z_3 = -i\mu\, z_4$. □

4. Applications of Lagrange potentials on minimal graphs in $\mathbb{R}^3$

It would be not easy to construct explicit examples of non-holomorphic minimal graphs in $\mathbb{R}^4$ by directly solving the minimal surface system of the second order. We solve the Osserman system of the first order to construct explicit examples of two dimensional minimal graphs in $\mathbb{R}^4$.

Lemma 4.1 (Existence of Lagrange potentials on minimal graphs in $\mathbb{R}^3$)**.** *Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain. We consider the two dimensional graph*

$$\Sigma = \left\{ \begin{bmatrix} x \\ y \\ p(x,y) \end{bmatrix} \in \mathbb{R}^3 \,\middle|\, (x,y) \in \Omega \right\}$$

of the $\mathcal{C}^2$ height function $p : \Omega \to \mathbb{R}$. Then, the following two statements are equivalent:

(a) *The graph Σ is a minimal surface in $\mathbb{R}^3$.*

(b) *There exists a function $q : \Omega \to \mathbb{R}$ satisfying the Lagrange system*

$$\begin{bmatrix} q_y \\ \\ -q_x \end{bmatrix} = \begin{bmatrix} \frac{p_x}{\sqrt{1+{p_x}^2+{p_y}^2}} \\ \frac{p_y}{\sqrt{1+{p_x}^2+{p_y}^2}} \end{bmatrix} \tag{7}$$

and the gradient estimate

$${q_x}^2 + {q_y}^2 < 1. \tag{8}$$

Proof. The graph Σ is minimal in $\mathbb{R}^3$ if and only if the function $p(x, y)$ satisfies

$$0 = \frac{\partial}{\partial x}\left(\frac{p_x}{\sqrt{1+{p_x}^2+{p_y}^2}}\right) + \frac{\partial}{\partial y}\left(\frac{p_y}{\sqrt{1+{p_x}^2+{p_y}^2}}\right), \ (x, y) \in \Omega, \tag{9}$$

which indicates that the following one form is closed:

$$\omega = -\frac{p_y}{\sqrt{1+{p_x}^2+{p_y}^2}}dx + \frac{p_x}{\sqrt{1+{p_x}^2+{p_y}^2}}dy. \tag{10}$$

Since Ω is simply connected, by Poincaré Lemma, the one form ω is exact. So, we can find a potential function $q : \Omega \to \mathbb{R}$ such that

$$-\frac{p_y}{\sqrt{1+{p_x}^2+{p_y}^2}}dx + \frac{p_x}{\sqrt{1+{p_x}^2+{p_y}^2}}dy = dq = q_x dx + q_y dy.$$

The inequality (8) follows from the equality $1 - {q_x}^2 - {q_y}^2 = \frac{1}{1+{p_x}^2+{p_y}^2}$. □

Remark 4.2 (Lagrange potentials and conjugate surfaces of minimal graphs in $\mathbb{R}^3$). The exactness of the one form ω in (10) on the minimal graph is discovered by Lagrange [14], who deduced the minimal surface equation (9). When we have the Cauchy-Riemann equations

$$\begin{bmatrix} (x_k^*)_x \\ \\ (x_k^*)_y \end{bmatrix} = \begin{bmatrix} \frac{p_x p_y}{\sqrt{1+{p_x}^2+{p_y}^2}} & -\frac{1+{p_x}^2}{\sqrt{1+{p_x}^2+{p_y}^2}} \\ \frac{1+{p_y}^2}{\sqrt{1+{p_x}^2+{p_y}^2}} & -\frac{p_x p_y}{\sqrt{1+{p_x}^2+{p_y}^2}} \end{bmatrix} \begin{bmatrix} (x_k)_x \\ \\ (x_k)_y \end{bmatrix}, \tag{11}$$

on the minimal graph

$$\Sigma = \left\{ \begin{bmatrix} x_1(x,y) \\ x_2(x,y) \\ x_3(x,y) \end{bmatrix} = \begin{bmatrix} x \\ y \\ p(x,y) \end{bmatrix} \in \mathbb{R}^3 \,\middle|\, (x, y) \in \Omega \right\},$$

the function $x_k + i x_k^*$ is holomorphic on Σ for each $k \in \{1, 2, 3\}$. It is straightforward to check that this observation is a particular case of Proposition 2.4 with the pair $(f(x, y), g(x, y)) = (p(x, y), 0)$.

(a) The conjugate surface

$$\Sigma^* = \left\{ \begin{bmatrix} (x_1^*(x,y) \\ x_2^*(x,y) \\ x_3^*(x,y) \end{bmatrix} \in \mathbb{R}^3 \,\middle|\, (x, y) \in \Omega \right\}$$

is a minimal surface locally isometric to the minimal surface Σ.

(b) Taking $k = 3$ in the Cauchy-Riemann equations (11) yields the Lagrange system (7), which reduces to

$$\begin{bmatrix} q_x \\ q_y \end{bmatrix} = \begin{bmatrix} \frac{p_x p_y}{\sqrt{1+{p_x}^2+{p_y}^2}} & -\frac{1+{p_x}^2}{\sqrt{1+{p_x}^2+{p_y}^2}} \\ \frac{1+{p_y}^2}{\sqrt{1+{p_x}^2+{p_y}^2}} & -\frac{p_x p_y}{\sqrt{1+{p_x}^2+{p_y}^2}} \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix}.$$

The function $p+iq$ is holomorphic on Σ with respect to the classical conformal coordinates constructed in Proposition 2.4.

(c) Combining the Lagrange system (7) and the gradient estimation (8) yields

$$\frac{\partial}{\partial x}\left(\frac{q_x}{\sqrt{1-{q_x}^2-{q_y}^2}}\right) + \frac{\partial}{\partial y}\left(\frac{q_y}{\sqrt{1-{q_x}^2-{q_y}^2}}\right) = 0, \tag{12}$$

which reduces to

$$\left(1-{q_y}^2\right) q_{xx} + 2q_x q_y q_{xy} + \left(1-{q_x}^2\right) q_{yy} = 0. \tag{13}$$

As a historical remark, the dual equation (13) is reported in 1855 by Catalan [4, Equation (C), p. 1020], where he discovered minimal surfaces generated by a one parameter family of parabolas. Calabi [3] observed that (8) and the dual equation (12) indicates that the graph $z = q(x, y)$ is a maximal surface (spacelike surface with zero mean curvature) in Lorentz-Minkowski space $\mathbb{L}^3 = \left(\mathbb{R}^3, dx^2 + dy^2 - dz^2\right)$.

(d) Taking $k = 1$ and $k = 2$ in the Cauchy-Riemann equations (11) yields two identities

$$\frac{\partial}{\partial y}\left(\frac{p_x p_y}{\sqrt{1+{p_x}^2+{p_y}^2}}\right) = \frac{\partial}{\partial x}\left(\frac{1+{p_y}^2}{\sqrt{1+{p_x}^2+{p_y}^2}}\right),$$

and

$$\frac{\partial}{\partial y}\left(\frac{1+{p_x}^2}{\sqrt{1+{p_x}^2+{p_y}^2}}\right) = \frac{\partial}{\partial x}\left(\frac{p_x p_y}{\sqrt{1+{p_x}^2+{p_y}^2}}\right).$$

Following previous notations, these two equalities can be rewritten as

$$\frac{\partial}{\partial y}\left(\frac{F}{\omega}\right) = \frac{\partial}{\partial x}\left(\frac{G}{\omega}\right) \quad \text{and} \quad \frac{\partial}{\partial y}\left(\frac{E}{\omega}\right) = \frac{\partial}{\partial x}\left(\frac{F}{\omega}\right).$$

Theorem 4.3 (Degenerate minimal graphs in $\mathbb{R}^4$ derived from minimal graphs in $\mathbb{R}^3$)**.** *Let Σ_0 be the minimal graph*

$$\Sigma_0 = \left\{ \begin{bmatrix} x \\ y \\ p(x,y) \end{bmatrix} \in \mathbb{R}^3 \,\middle|\, (x,y) \in \Omega \right\}$$

of the $\mathcal{C}^2$ *height function* $p : \Omega \to \mathbb{R}$ *defined on a domain* $\Omega \subset \mathbb{R}^2$*. Let* $q : \Omega \to \mathbb{R}$ *be the Lagrange potential of the function* p*, which solves the Lagrange system*

$$\begin{bmatrix} q_y \\ -q_x \end{bmatrix} = \begin{bmatrix} \frac{p_x}{\sqrt{1+{p_x}^2+{p_y}^2}} \\ \frac{p_y}{\sqrt{1+{p_x}^2+{p_y}^2}} \end{bmatrix}. \tag{14}$$

For a constant $\lambda \in \mathbb{R} - \{0\}$*, we associate the two dimensional graph in* $\mathbb{R}^4$*:*

$$\Sigma_\lambda = \left\{ \begin{bmatrix} x \\ y \\ f(x,y) \\ g(x,y) \end{bmatrix} = \begin{bmatrix} x \\ y \\ (\cosh\lambda)\ p(x,y) \\ (\sinh\lambda)\ q(x,y) \end{bmatrix} \in \mathbb{R}^4 \ \middle|\ (x,y) \in \Omega \right\}.$$

Then, the graph Σ_λ *is minimal in* $\mathbb{R}^4$*. Also, we obtain the conformal invariance of the conformally changed induced metric*

$$\frac{1}{\sqrt{\det\left(g_{\Sigma_\lambda}\right)}}\, g_{\Sigma_\lambda} = \frac{1}{\sqrt{\det\left(g_{\Sigma_0}\right)}}\, g_{\Sigma_0}. \tag{15}$$

Proof. We want to show that the pair $(f,g) = (\,(\cosh\lambda)\,p,\ (\sinh\lambda)\,q\,)$ satisfies the Osserman system (5) in Theorem 3.4 with the coefficient $\mu = \coth\lambda$:

$$\begin{bmatrix} f_x \\ f_y \end{bmatrix} = \coth\lambda \begin{bmatrix} \frac{E}{\omega} & \frac{F}{\omega} \\ \frac{F}{\omega} & \frac{G}{\omega} \end{bmatrix} \begin{bmatrix} g_y \\ -g_x \end{bmatrix}, \quad \text{where} \quad \omega = \sqrt{EG - F^2}. \tag{16}$$

Taking $W = \sqrt{1 + {p_x}^2 + {p_y}^2} \geq 1$ and using the system (14), we deduce

$$(q_x, q_y) = \left(-\frac{p_y}{W}, \frac{p_x}{W}\right) \quad \text{and} \quad {q_x}^2 + {q_y}^2 = \frac{W^2 - 1}{W^2}. \tag{17}$$

We recall the definition $(f,g) = (\,(\cosh\lambda)\,p,\ (\sinh\lambda)\,q\,)$ and deduce

$$f_x g_y - f_y g_x = \cosh\lambda \sinh\lambda \frac{W^2 - 1}{W}.$$

We use the definition $\omega = \sqrt{EG - F^2}$ to obtain

$$\begin{aligned} \omega^2 &= \left(1 + {f_x}^2 + {g_x}^2\right)\left(1 + {f_y}^2 + {g_y}^2\right) - \left(f_x f_y + g_x g_y\right)^2 \\ &= 1 + \left({f_x}^2 + {f_y}^2\right) + \left({g_x}^2 + {g_y}^2\right) + \left(f_x g_y - f_y g_x\right)^2 \\ &= 1 + \cosh^2\lambda\left(W^2 - 1\right) + \sinh^2\lambda + \left(\cosh\lambda\sinh\lambda\frac{W^2-1}{W}\right)^2 \\ &= \left[\left(\cosh^2\lambda\right)W - \frac{\sinh^2\lambda}{W}\right]^2. \end{aligned}$$

However, it follows from $W \geq 1$ that

$$\left(\cosh^2\lambda\right)W - \frac{\sinh^2\lambda}{W} - 1 = \left(\cosh^2\lambda\right)(W - 1) + \frac{\sinh^2\lambda}{W}(W - 1) \geq 0,$$

which implies that

$$\left(\cosh^2\lambda\right)W-\frac{\sinh^2\lambda}{W}\geq 1>0.$$

We conclude that

$$\omega=\left(\cosh^2\lambda\right)W-\frac{\sinh^2\lambda}{W}. \tag{18}$$

We use (18) and (17) to deduce the first row equality in (16):

$$\begin{aligned}
\frac{E}{\omega}g_y-\frac{F}{\omega}g_x &= \frac{1}{\omega}\left[\left(1+{f_x}^2+{g_x}^2\right)g_y-(f_xf_y+g_xg_y)g_x\right]\\
&= \frac{1}{\omega}\left[\left(1+{f_x}^2\right)g_y-f_xf_yg_x\right]\\
&= \frac{1}{\omega}\left[\left(1+\left(\cosh^2\lambda\right){p_x}^2\right)(\sinh\lambda)\ q_y-\left(\cosh^2\lambda\right)p_xp_y\left(\sinh\lambda\right)\ q_x\right]\\
&= \frac{\sinh\lambda}{\omega}\cdot\frac{p_x}{W}\cdot\left[1+\left(\cosh^2\lambda\right)\left({p_x}^2+{p_y}^2\right)\right]\\
&= \frac{\sinh\lambda}{\omega}\cdot\frac{p_x}{W}\cdot\left[-\sinh^2\lambda+\left(\cosh^2\lambda\right)W^2\right]\\
&= (\sinh\lambda)\,p_x\frac{1}{\omega}\left[\left(\cosh^2\lambda\right)W-\frac{\sinh^2\lambda}{W}\right]\\
&= (\sinh\lambda)\,p_x=\frac{f_x}{\coth\lambda}.
\end{aligned}$$

We omit a similar verification of the second row equality in (16). Finally, one can use the equalities

$$\left(\frac{E}{\omega},\frac{F}{\omega},\frac{G}{\omega}\right)=\left(\frac{1+{p_x}^2}{W},\frac{p_xp_y}{W},\frac{1+{p_y}^2}{W}\right)$$

to check the conformal invariance (15). □

Remark 4.4 (Holomorphic null curves lifted from degenerate minimal graphs in $\mathbb{R}^4$). In Theorem 4.3, if the initial minimal graph Σ_0 in $\mathbb{R}^3$ is induced by the holomorphic null curve

$$\phi=(\phi_1(\zeta),\phi_3(\zeta),\phi_3(\zeta))$$

in $\mathbb{C}^3$ with ${\phi_1}^2+{\phi_2}^2+{\phi_3}^2=0$ and a local conformal coordinate ζ on Σ_0, the minimal graph Σ_λ in $\mathbb{R}^4$ is induced by

$$\phi_\lambda=(\phi_1(\zeta),\phi_2(\zeta),(\cosh\lambda)\,\phi_3(\zeta),(-i\sinh\lambda)\,\phi_3(\zeta))$$

with the conformal coordinate ζ on Σ_λ. The identity $\cosh^2\lambda-\sinh^2\lambda=1$ implies the nullity of the induced holomorphic curve ϕ_λ in $\mathbb{C}^4$:

$${\phi_1}^2+{\phi_2}^2+\left[(\cosh\lambda)\,\phi_3\right]^2+\left[(-i\sinh\lambda)\,\phi_3\right]^2={\phi_1}^2+{\phi_2}^2+{\phi_3}^2=0.$$

For a survey of various deformations of holomorphic null curves in $\mathbb{C}^n$ lifted from minimal surfaces in $\mathbb{R}^n$, we invite readers to refer to [19, Section 2].

We apply Theorem 4.3 to classical minimal graphs in $\mathbb{R}^3$ to find explicit examples of old and new minimal graphs in $\mathbb{R}^4$.

Example 4.5 (Minimal surfaces in $\mathbb{R}^4$ foliated by hyperbolas or lines)**.** We consider the fundamental piece of the helicoid

$$\Sigma_0 = \left\{ \begin{bmatrix} x \\ y \\ p(x,y) \end{bmatrix} = \begin{bmatrix} x \\ y \\ x \tan y \end{bmatrix} \in \mathbb{R}^3 \,\middle|\, (x,y) \in \Omega = \mathbb{R} \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \right\}.$$

Solving the induced Lagrange system (7) in Lemma 4.1

$$\begin{bmatrix} q_y \\ \\ -q_x \end{bmatrix} = \begin{bmatrix} \frac{p_x}{\sqrt{1+{p_x}^2+{p_y}^2}} \\ \frac{p_y}{\sqrt{1+{p_x}^2+{p_y}^2}} \end{bmatrix} = \begin{bmatrix} \frac{\cos y \sin y}{\sqrt{\cos^2 y + x^2}} \\ \frac{x}{\sqrt{\cos^2 y + x^2}} \end{bmatrix},$$

we obtain $q(x,y) = -\sqrt{\cos^2 y + x^2}$, up to an additive constant. Let $\lambda \in \mathbb{R}$ be a constant. Theorem 4.3 yields the two dimensional minimal graph Σ_λ^- in $\mathbb{R}^4$:

$$\Sigma_\lambda^- = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x \\ y \\ (\cosh\lambda)\, x \tan y \\ \sinh\lambda \sqrt{\cos^2 y + x^2} \end{bmatrix} \in \mathbb{R}^4 \,\middle|\, (x,y) \in \Omega \right\}.$$

(a) When $\lambda = 0$, the graph Σ_λ^- recovers the helicoid in $\mathbb{R}^3$ foliated by lines.

(b) Let $\lambda \neq 0$. Fix $y_0 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. The intersection $\mathcal{C}_{y_0}$ of the surface Σ_λ^- and the hyperplane $x_2 = y = y_0$ is a hyperbola. Indeed, letting the new orthogonal coordinates $(\widetilde{x_1}, \widetilde{x_3})$ in the x_1x_3-plane defined by

$$\widetilde{x_1} + i\widetilde{x_3} = \frac{\cosh\lambda \sin y_0 + i \cos y_0}{\sqrt{\cosh^2\lambda \sin^2 y_0 + \cos^2 y_0}} \, (x_1 + i x_3)\,,$$

we can check that the level curve $\mathcal{C}_{y_0}$ in the $\widetilde{x_1}x_2\widetilde{x_3}x_4$-space lies on $x_2 = y_0$, $\widetilde{x_1} = 0$, and

$$\left(\frac{x_4}{\sinh\lambda \cos y_0} \right)^2 - \left(\frac{\widetilde{x_3}}{\sqrt{\cosh^2\lambda \sin^2 y_0 + \cos^2 y_0}} \right)^2 = 1.$$

Under the coordinate transformation $(x,y) = (\sinh\mathcal{U} \cos\mathcal{V}, \mathcal{V}) \to (\mathcal{U}, \mathcal{V})$, we obtain the conformal harmonic patch for the minimal surface Σ_λ^- in $\mathbb{R}^4$:

$$F_\theta^-(\mathcal{U},\mathcal{V}) = \begin{bmatrix} \sinh\mathcal{U} \cos\mathcal{V} \\ \mathcal{V} \\ \cosh\lambda\, \sinh\mathcal{U} \sin\mathcal{V} \\ \sinh\lambda\, \cosh\mathcal{U} \cos\mathcal{V} \end{bmatrix}. \tag{19}$$

The graph Σ_λ^- belongs to the family of minimal surfaces discovered by the author [19, Example 6.1]. It was originally discovered by an application of the so called parabolic rotations of holomorphic null curves in $\mathbb{C}^3 \subset \mathbb{C}^4$ lifted from helicoids in $\mathbb{R}^3$.

Example 4.6 (Hoffman-Osserman's minimal surfaces in $\mathbb{R}^4$)**.** Over the domain

$$\Omega = \left\{(x,y) \in \mathbb{R}^2 \mid x^2 \leq \cosh^2 y\right\},$$

we consider a half of the catenoid

$$\Sigma_0 = \left\{ \begin{bmatrix} x \\ y \\ p(x,y) \end{bmatrix} = \begin{bmatrix} x \\ y \\ \sqrt{-x^2 + \cosh^2 y} \end{bmatrix} \in \mathbb{R}^3 \mid (x,y) \in \Omega \right\}.$$

The pair $(p(x,y), q(x,y)) = \left(\sqrt{-x^2+\cosh^2 y}\,,\, x \tanh y\right)$ solves the Lagrange system (7) in Lemma 4.1:

$$\begin{bmatrix} q_y \\ \\ -q_x \end{bmatrix} = \begin{bmatrix} \frac{p_x}{\sqrt{1+{p_x}^2+{p_y}^2}} \\ \frac{p_y}{\sqrt{1+{p_x}^2+{p_y}^2}} \end{bmatrix} = \begin{bmatrix} \frac{-x}{\cosh^2 y} \\ \frac{\sinh y}{\cosh y} \end{bmatrix}.$$

Let λ be a constant. Theorem 4.3 yields the minimal graph Σ_λ^+ in $\mathbb{R}^4$:

$$\Sigma_\lambda^+ = \left\{ \begin{bmatrix} x \\ y \\ f(x,y) \\ g(x,y) \end{bmatrix} = \begin{bmatrix} x \\ y \\ \cosh\lambda\sqrt{-x^2+\cosh^2 y} \\ (\sinh\lambda)\, x \tanh y \end{bmatrix} \in \mathbb{R}^4 \mid (x,y) \in \Omega \right\}.$$

Under the coordinate transformation $(x,y) = (\cosh\mathcal{U}\cos\mathcal{V}, \mathcal{U}) \to (\mathcal{U},\mathcal{V})$, we obtain the conformal harmonic patch for the minimal surface Σ_λ^+ in $\mathbb{R}^4$:

$$F_\lambda^+(\mathcal{U},\mathcal{V}) = \begin{bmatrix} \cosh\mathcal{U}\cos\mathcal{V} \\ \mathcal{U} \\ \cosh\lambda\cosh\mathcal{U}\sin\mathcal{V} \\ \sinh\lambda\sinh\mathcal{U}\cos\mathcal{V} \end{bmatrix}. \tag{20}$$

This recovers Osserman-Hoffman's minimal annuli in $\mathbb{R}^4$ ([10, Proposition 6.6 and Remark 1] and [19, Example 6.2 and Theorem 6.3]). The conformal harmonic patches (19) in Example 4.5 and (20) in Example 4.6 represent *conjugate* minimal surfaces in $\mathbb{R}^4$. For the notion of associate family of locally isometric minimal surfaces in $\mathbb{R}^{n+2\geq 3}$, we invite interested readers to refer to [16].

Example 4.7 (Doubly periodic minimal graphs in $\mathbb{R}^4$ derived from Scherk's doubly periodic graph in $\mathbb{R}^3$)**.** Over the open square

$$\Omega = \left\{(x,y) \in \mathbb{R}^2 \mid x, y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right\},$$

we define the fundamental piece of the doubly periodic graph in $\mathbb{R}^3$:

$$\left\{ \begin{bmatrix} x \\ y \\ \ln\left(\frac{\cos x}{\cos y}\right) \end{bmatrix} \in \mathbb{R}^3 \,\middle|\, (x,y) \in \Omega \right\}, \tag{21}$$

which was originally discovered by Scherk [27, p. 196]. Let $\lambda \in \mathbb{R}$ be a constant. Theorem 4.3 yields the minimal graph in $\mathbb{R}^4$:

$$\left\{ \begin{bmatrix} x \\ y \\ (\cosh\lambda)\ln\left(\frac{\cos x}{\cos y}\right) \\ (\sinh\lambda)\arcsin\left(\sin x \sin y\right) \end{bmatrix} \in \mathbb{R}^4 \;\middle|\; (x,y) \in \Omega \right\}.$$

Remark 4.8 (Jenkins-Serrin type minimal graphs). Inspired by the existence of Scherk's first surfaces, Jenkins and Serrin [11] offers a powerful analytic method to extend Scherk's construction. The fundamental piece of Scherk's first surface can be obtained as a Jenkins-Serrin graph by solving the Dirichlet problem for the minimal surface equation over a square and taking boundary values plus infinity on two opposite sides and minus infinity on the other two opposite sides.

Example 4.9 (Doubly periodic minimal graphs in $\mathbb{R}^4$ derived from sheared Scherk's doubly periodic graph in $\mathbb{R}^3$)**.** Scherk [27, p. 187] showed that his surface (21) in Example 4.7 belongs to a one parameter family of minimal graphs. Following [21, p. 70], for an angle constant $2\alpha \in (0,\pi)$ and a dilation constant $\rho > 0$, we define Scherk's doubly periodic minimal graph $\Sigma_\rho^{2\alpha}$ by

$$z = p(x,y) = \frac{1}{\rho}\ln\left[\frac{\cos\left(\frac{\rho}{2}\left[\frac{x}{\cos\alpha} - \frac{y}{\sin\alpha}\right]\right)}{\cos\left(\frac{\rho}{2}\left[\frac{x}{\cos\alpha} + \frac{y}{\sin\alpha}\right]\right)}\right], \tag{22}$$

where its domain is an infinite chess board-like net of rhomboids $\Omega = \cup_{i,j\in\mathbb{Z}}\mathcal{R}_{ij}$. Here, we define the rhomboid domain $\mathcal{R}_{ij}$ with the length $\frac{\pi}{\rho}$ as follows

$$\mathcal{R}_{ij} = \left\{ (x,y)\in\mathbb{R}^2 \;\middle|\; \left|\frac{x}{\cos\alpha} - \frac{y}{\sin\alpha} - \frac{4i}{\rho}\pi\right| < \frac{\pi}{\rho},\ \left|\frac{x}{\cos\alpha} + \frac{y}{\sin\alpha} - \frac{4j}{\rho}\pi\right| < \frac{\pi}{\rho} \right\}.$$

Taking $\alpha = \frac{\pi}{4}$ and $\rho = 2$ in (22), the graph $\Sigma_2^{\frac{\pi}{2}}$ is congruent to the minimal graph (21) in Example 4.7, after the $\frac{\pi}{4}$-rotation. Let $\lambda \in \mathbb{R}$ be a constant. Theorem 4.3 yields the minimal graph in $\mathbb{R}^4$:

$$\left\{ \begin{bmatrix} x \\ y \\ (\cosh\lambda)\,p(x,y) \\ (\sinh\lambda)\,q(x,y) \end{bmatrix} \in \mathbb{R}^4 \;\middle|\; (x,y) \in \Omega \right\},$$

where we take the Lagrange potential

$$q(x,y) = \frac{1}{\rho}\arccos\left[\cos^2\alpha\cos\left(\frac{\rho x}{\cos\alpha}\right) - \sin^2\alpha\cos\left(\frac{\rho y}{\sin\alpha}\right)\right].$$

Example 4.10 (Minimal graphs in $\mathbb{R}^4$ derived from Scherk's saddle tower in $\mathbb{R}^3$)**.** Over the domain $\Omega = \{(x,y) \in \mathbb{R}^2 \mid -1 < \sinh x \, \sinh y < 1\}$, we consider a fundamental piece of the singly periodic multi-valued graph in $\mathbb{R}^3$:

$$\left\{ \begin{bmatrix} x \\ y \\ \arcsin(\sinh x \sinh y) \end{bmatrix} \in \mathbb{R}^3 \,\middle|\, (x,y) \in \Omega \right\},$$

which was originally discovered by Scherk [27, p. 198]. Let $\lambda \in \mathbb{R}$ be a constant. Theorem 4.3 yields the minimal graph in $\mathbb{R}^4$:

$$\left\{ \begin{bmatrix} x \\ y \\ (\cosh\lambda) \arcsin(\sinh x \sinh y) \\ (\sinh\lambda) \ln\left(\frac{\cosh x}{\cosh y}\right) \end{bmatrix} \in \mathbb{R}^4 \,\middle|\, (x,y) \in \Omega \right\}.$$

Remark 4.11 (Scherk's saddle tower in $\mathbb{R}^3$ and its influences). Geometrically, Scherk's saddle tower is a smooth minimal desingularization of two perpendicular vertical planes. Scherk's saddle tower plays a fundamental role in the modern theory of desingularizations and gluing construction for surfaces with constant mean curvature and solitons to various curvature flows. Karcher [13] discovered embedded minimal surfaces in $\mathbb{R}^3$ derived from Scherk's examples, and Pacard [25] showed the existence of $(N-2)$-periodic embedded minimal hypersurfaces in $\mathbb{R}^{N\geq 4}$ with four hyperplanar ends.

Example 4.12 (Minimal graphs in $\mathbb{R}^4$ derived from Scherk's generalized tower in $\mathbb{R}^3$)**.** As in [25, Section 1] and [21, p. 74], we take the fundamental piece of the singly periodic multi-valued graph in $\mathbb{R}^3$:

$$z = p(x,y) = \frac{1}{\rho} \arccos\left[\cos^2\alpha \cosh\left(\frac{\rho x}{\cos\alpha}\right) - \sin^2\alpha \cosh\left(\frac{\rho y}{\sin\alpha}\right)\right].$$

Let $\lambda \in \mathbb{R}$ be a constant. Theorem 4.3 yields the minimal graph in $\mathbb{R}^4$:

$$\left\{ \begin{bmatrix} x \\ y \\ (\cosh\lambda)\, p(x,y) \\ (\sinh\lambda)\, q(x,y) \end{bmatrix} \in \mathbb{R}^4 \,\middle|\, (x,y) \in \Omega \right\},$$

where we take the Lagrange potential

$$q(x,y) = \frac{1}{\rho} \ln\left[\frac{\cosh\left(\frac{\rho}{2}\left[\frac{x}{\cos\alpha} - \frac{y}{\sin\alpha}\right]\right)}{\cosh\left(\frac{\rho}{2}\left[\frac{x}{\cos\alpha} + \frac{y}{\sin\alpha}\right]\right)}\right].$$

5. Minimal graphs in $\mathbb{R}^3$ and special Lagrangian graphs in $\mathbb{C}^3$

Fu [8], Jost-Xin [12], Tsui-Wang [28], Yuan [29] established Bernstein type results for entire special Lagrangian graphs in even dimensional Euclidean space. Here, we construct non-entire special Lagrangian graphs in $\mathbb{C}^3$.

Proposition 5.1 (Special Lagrangian equation in $\mathbb{C}^3$, [9, Theorem 2. 3 and (4.8)]). *Let $\mathcal{S}$ be the gradient graph of the $\mathbb{R}$-valued function $F(x,y,z)$ in $\mathbb{R}^3$. Then, the 3-fold $\mathcal{S}$ in $\mathbb{R}^6$ admits an orientation making it into a special Lagrangian graph in $\mathbb{C}^3$ under the complexification*

$$(x_1 + iy_1, x_2 + iy_2, x_3 + iy_3) = (x_1, x_2, x_3, y_1, y_2, y_3)$$

when the function $F(x,y,z)$ satisfies the special Lagrangian equation

$$det \begin{bmatrix} F_{xx} & F_{xy} & F_{xz} \\ F_{yx} & F_{yy} & F_{yz} \\ F_{zx} & F_{zy} & F_{zz} \end{bmatrix} = tr \begin{bmatrix} F_{xx} & F_{xy} & F_{xz} \\ F_{yx} & F_{yy} & F_{yz} \\ F_{zx} & F_{zy} & F_{zz} \end{bmatrix}. \tag{23}$$

Remark 5.2. In [9, III.4.B. Degenerate projections and harmonic gradients], Harvey and Lawson investigated the interesting special case when the solution of the equation (23) is affine with respect to the coordinate z. The function $F(x,y,z) = p(x,y) + z\, q(x,y)$ satisfies the special Lagrangian equation (23) if and only if the pair $(p(x,y), q(x,y))$ solves the system

$$\begin{cases} \left(1 + {p_y}^2\right) p_{xx} - 2p_x p_y p_{xy} + \left(1 + {p_x}^2\right) p_{yy} = 0, \\ \left(1 + {p_y}^2\right) q_{xx} - 2p_x p_y q_{xy} + \left(1 + {p_x}^2\right) q_{yy} = 0. \end{cases}$$

The first equation means that the graph of $p(x,y)$ is a minimal surface in $\mathbb{R}^3$. The second equation means that $q(x,y)$ is harmonic on the graph of $p(x,y)$.

Combining the harmonicity of the Lagrange potentials of height functions of the minimal graph in $\mathbb{R}^3$ and the Harvey-Lawson reduction [9, Theorem 4.9], we immediately deduce the following result.

Corollary 5.3 (Lagrange potential construction of special Lagrangian graphs in $\mathbb{R}^6 = \mathbb{C}^3$). *Let*

$$\Sigma = \left\{ \begin{bmatrix} x \\ y \\ p(x,y) \end{bmatrix} \in \mathbb{R}^3 \mid (x,y) \in \Omega \right\}$$

be the minimal graph of the height function $p(x,y) : \Omega \to \mathbb{R}$ on the domain $\Omega \subset \mathbb{R}^2$. Let $q : \Omega \to \mathbb{R}$ be the Lagrange potential of the function p such that

$$\begin{bmatrix} q_y \\ \\ -q_x \end{bmatrix} = \begin{bmatrix} \frac{p_x}{\sqrt{1+{p_x}^2+{p_y}^2}} \\ \frac{p_y}{\sqrt{1+{p_x}^2+{p_y}^2}} \end{bmatrix}.$$

For any constant $\lambda \in \mathbb{R}$, we obtain the special Lagrangian graph in $\mathbb{C}^3$:

$$\Sigma_\lambda = \left\{ \begin{bmatrix} x \\ y \\ z \\ p_x + \lambda z q_x \\ p_y + \lambda z q_y \\ \lambda q \end{bmatrix} \in \mathbb{R}^6 \;\middle|\; (x,y) \in \Omega,\ z \in \mathbb{R} \right\}.$$

Proof. By the item (b) in Remark 4.2, the function $p + iq$ is holomorphic on the minimal graph Σ. Since p and q are harmonic on the minimal graph Σ, by Remark 2.2, we obtain the system of equations

$$\begin{cases} \left(1+{p_y}^2\right) p_{xx} - 2p_x p_y p_{xy} + \left(1+{p_x}^2\right) p_{yy} = 0, \\ \left(1+{p_y}^2\right) (\lambda q)_{xx} - 2p_x p_y (\lambda q)_{xy} + \left(1+{p_x}^2\right) (\lambda q)_{yy} = 0. \end{cases} \tag{24}$$

Applying the Harvey-Lawson reduction [9, Theorem 4.9] to the system (24) yields that the gradient graph of the function $F(x, y, z) = p(x, y) + \lambda z\, q(x, y)$ becomes a special Lagrangian 3-fold in $\mathbb{C}^3$. □

Example 5.4 (Doubly periodic special Lagrangian graph in $\mathbb{C}^3$)**.** Let λ be a constant. We apply Corollary 5.3 to the fundamental piece of Scherk's doubly periodic graph on the domain Ω in Example 4.7 to have the one parameter family of special Lagrangian graph Σ_λ in $\mathbb{C}^3$:

$$\Sigma_\lambda = \left\{ \left. \begin{bmatrix} x \\ y \\ z \\ -\frac{\sin x}{\cos x} + \lambda z \frac{\sin x \cos y}{\sqrt{1-\sin^2 x \, \sin^2 y}} \\ \frac{\sin y}{\cos y} + \lambda z \frac{\cos x \sin y}{\sqrt{1-\sin^2 x \, \sin^2 y}} \\ \lambda \arcsin\left(\sin x \sin y\right) \end{bmatrix} \in \mathbb{R}^6 \,\right|\, (x, y) \in \Omega,\ z \in \mathbb{R} \right\}.$$

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Hojoo Lee
Department of Mathematics and Institute of Pure and Applied Mathematics
Jeonbuk National University
Jeonju 54896, Korea
Email address: compactkoala@gmail.com, kiarostami@jbnu.ac.kr