J. Korean Math. Soc. **60** (2023), No. 1, pp. 33–69
https://doi.org/10.4134/JKMS.j210764
pISSN: 0304-9914 / eISSN: 2234-3008

WEAK HERZ-TYPE HARDY SPACES WITH VARIABLE EXPONENTS AND APPLICATIONS

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Abstract. Let $\alpha \in (0, \infty)$, $p \in (0, \infty)$ and $q(\cdot) : \mathbb{R}^n \to [1, \infty)$ satisfy the globally log-Hölder continuity condition. We introduce the weak Herz-type Hardy spaces with variable exponents via the radial grand maximal operator and to give its maximal characterizations, we establish a version of the boundedness of the Hardy-Littlewood maximal operator M and the Fefferman-Stein vector-valued inequality on the weak Herz spaces with variable exponents. We also obtain the atomic and the molecular decompositions of the weak Herz-type Hardy spaces with variable exponents. As an application of the atomic decomposition we provide various equivalent characterizations of our spaces by means of the Lusin area function, the Littlewood-Paley g-function and the Littlewood-Paley g_λ^*-function.

1. Introduction

The theory of function spaces with variable exponents has attracted a great interest in different fields of analysis and partial differential equations (see [1, 4, 8, 20, 27, 31, 33]). In 1991's, Kováčik and Rákosník [21] studied the variable Lebesgue spaces and later, they have been the subject of more intensive study, because of their intrinsic interest for applications into harmonic analysis, partial differential equations and variational integrals with nonstandard growth conditions (see [5, 7, 8, 18]). Lu, Yang and Hu [23] introduced the Herz type spaces and gave some applications, then later Izuki [16, 17] introduced the Herz space with variable exponents and established the boundedness of some sublinear operators on this space.

On the other hand, the theory of Hardy spaces with variable exponents have attracted a steadily increasing interest in harmonic analysis in recent years. In particular, Nakai and Sawano [24] introduced the variable Hardy spaces $H^{p(\cdot)}(\mathbb{R}^n)$ and established their atomic characterizations and their dual spaces, and also studied the boundedness of singular integral operators on

Received December 20, 2021; Revised October 7, 2022; Accepted November 4, 2022.
2020 *Mathematics Subject Classification.* Primary 42B30; Secondary 42B25, 42B35, 46E30.
Key words and phrases. Weak Herz-type Hardy spaces, variable exponent, maximal operator, atom, molecule, Littlewood-Paley function.

$H^{p(\cdot)}(\mathbb{R}^n)$. In [28] Sawano further extended the atomic characterization of $H^{p(\cdot)}(\mathbb{R}^n)$ and improved the corresponding results in [24], and gave out more applications, including the boundedness of several operators on $H^{p(\cdot)}(\mathbb{R}^n)$. In [6] the authors also introduced the variable Hardy spaces $H^{p(\cdot)}(\mathbb{R}^n)$ and established their equivalent characterizations by means of radial or non-tangential maximal functions or atoms, with the variable exponents $p(\cdot)$ satisfying some conditions slightly weaker than those used in [24]. Moreover, Zhuo et al. [34] established equivalent characterizations of $H^{p(\cdot)}(\mathbb{R}^n)$ via intrinsic square functions including intrinsic Lusin area function, the intrinsic Littlewood-Paley g-function or g_λ^*-function. Recently, Jiao et al. in [20] were mainly devoted to the study of the Hardy-Lorentz spaces with variable exponents $H^{p(\cdot),q}(\mathbb{R}^n)$.

The purpose of this article is to introduce and to investigate the weak Herz-type Hardy spaces with variable exponents on $\mathbb{R}^n$. It is well known that the classical weak Hardy spaces appear naturally in critical cases of the study on the boundedness of operators. Indeed the classical weak Hardy space $WH^1(\mathbb{R}^n)$ was originally introduced by Fefferman and Soria [10] to find out the biggest space from which Hilbert transform is bounded to the weak Lebesgue space $WL^1(\mathbb{R}^n)$. They also obtained the ∞-atomic characterization of $WH^1(\mathbb{R}^n)$ and the boundedness of some Calderón-Zygmund operators from $WH^1(\mathbb{R}^n)$ to $WL^1(\mathbb{R}^n)$. It is also well known that $H^p(\mathbb{R}^n)$ is a good substitute of the Lebesgue space $L^p(\mathbb{R}^n)$ with $p \in (0,1]$ in the study of the boundedness of operators and, moreover, when studying the boundedness of operators in the critical case, the weak Hardy spaces $WH^p(\mathbb{R}^n)$ naturally appear and prove to be a good substitute of Hardy spaces $H^p(\mathbb{R}^n)$ with $p \in (0,1]$.

Furthermore, Fefferman et al. [9] proved that the weak Hardy spaces naturally appear as the intermediate spaces in the real interpolation methods between the Hardy spaces, which is another main motivation to develop a real-variable theory of $WH^p(\mathbb{R}^n)$. He [14] and Grafakos and He [13] further investigated vector-valued weak Hardy spaces $H^{p,\infty}(\mathbb{R}^n, l^2)$ with $p \in (0,\infty)$. Recently, Liang et al. [22] introduced weak Musielak-Orlicz-Hardy spaces $WH^\varphi(\mathbb{R}^n)$ and various equivalent characterizations by means of maximal functions, atoms, molecules and Littlewood-Paley functions, and the boundedness of Calderón-Zygmund operators in the critical case were obtained. In [32] the authors introduced the variable weak Hardy spaces and gave some applications.

In this article, motivated by [22,32] we aim to introduce and investigate the weak Herz-type Hardy spaces with variable exponents and give some applications. These spaces are first defined via the radial grand maximal operator and then characterized by means of radial or non-tangential maximal operators. Via combining some ideas we borrowed from [3, 22, 32], we construct the atomic and the molecular decompositions of the weak Herz-type Hardy spaces with variable exponents. As an applications of the atomic decomposition, various equivalent characterizations by means of the Lusin area function, the Littlewood-Paley g-function and the Littlewood-Paley g_λ^*-function, are obtained. To all the above end, we proved a version of the boundedness of the Hardy-Littlewood maximal operator M and the Fefferman-Stein vector-valued inequality on the weak Herz spaces with variable exponents.

We end this introduction by describing the layout of this paper.

Section 2 is devoted to recalling some definitions and useful properties for our work.

In Section 3, we state some basic properties about the weak Herz-type spaces with variable exponents. We also define the weak Herz-type Hardy space with variable exponents $\mathbf{WHK}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)$ (or $\mathbf{WH\dot{K}}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)$) via the radial grand maximal function.

Section 4 is devoted to characterize the weak Herz-type Hardy spaces with variable exponents by means of several maximal operators, particulary, radial maximal operator, the non-tangential maximal operator, the non-tangential maximal operator corresponding to Poisson kernels and the discrete maximal operator. To this end, we first prove the boundedness of sublinear operators on the weak Herz spaces with variable exponents (see Theorem 4.1), and then we can deduce the boundedness of the Hardy-Littlewood maximal operator M on those spaces (see Corollary 4.2), moreover this result may be of independent interest. By using (Theorem 4.1 and Corollary 4.2) we establish the Fefferman-Stein vector-valued inequality of the Hardy-Littlewood maximal operator M on the weak Herz spaces with variable exponents (see Proposition 4.3), moreover this result will play a role in Section 7 when establishing the Littlewood-Paley function characterizations.

In Section 5, by borrowing some ideas from [3,22,32], we establish the atomic characterization of $\mathbf{WHK}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)$. Indeed, we first introduce the weak atomic Herz-type Hardy spaces $\mathbf{WHK}_{q(\cdot),s,at}^{\alpha,p}(\mathbb{R}^n)$ and then prove that $\mathbf{WHK}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)$ $\sim \mathbf{WHK}_{q(\cdot),s,at}^{\alpha,p}(\mathbb{R}^n)$ (see Theorem 5.1). To prove that $\mathbf{WHK}_{q(\cdot),s,at}^{\alpha,p}(\mathbb{R}^n) \subset$ $\mathbf{WHK}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)$, we mainly need to prove a key lemma result (see Lemma 5.2). To prove the converse, we adopt a strategy used in [22,32], originated from [3].

In Section 6, we establish the molecular characterization of $\mathbf{WHK}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)$. We first introduce the weak molecular Herz-type Hardy spaces $\mathbf{WHK}_{q(\cdot),s,mol}^{\alpha,p,\epsilon}$ $(\mathbb{R}^n)$ and then prove that $\mathbf{WHK}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n) \sim \mathbf{WHK}_{q(\cdot),s,mol}^{\alpha,p,\epsilon}(\mathbb{R}^n)$ (see Theorem 6.1). Since each $(\alpha, p, q(\cdot), \infty)$-atom is also an $(\alpha, p, q(\cdot), s, \epsilon)$-molecule, then to prove Theorem 6.1, it suffices to show that $\mathbf{WHK}_{q(\cdot),s,mol}^{\alpha,p,\epsilon}(\mathbb{R}^n) \subset$ $\mathbf{WHK}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)$.

Section 7 is devoted to establishing some square function characterizations of the weak Herz-type Hardy spaces with variable exponents, including characterizations via Lusin area function, the Littlewood-Paley g-function or the Littlewood-Paley g_λ^*-function, respectively, in Theorems 7.2, 7.3 and 7.4. Our main tool is the atomic decomposition (Theorem 5.1) and the Fefferman-Stein vector-valued inequality (Proposition 4.3).

As usually, throughout the paper, we denote by $\mathbb{N}$ and $\mathbb{Z}$ the set of non-negative integers and the set of integers, respectively. The symbol $A \lesssim B$ means $A \leq CB$ and the symbol $A \sim B$ means $A \lesssim B$ and $B \lesssim A$.

2. Preliminaries

In this section, we recall some definitions, properties and some lemmas used in this work.

A measurable function $q(\cdot) : \mathbb{R}^n \to (0, \infty)$ is called a variable exponent. For any variable exponent $q(\cdot)$, define

$$q_- := \operatorname{ess\,inf}\{q(x) : x \in \mathbb{R}^n\} \quad \text{and} \quad q_+ := \operatorname{ess\,sup}\{q(x) : x \in \mathbb{R}^n\}.$$

Denote by $\mathcal{P}(\mathbb{R}^n)$ the set of all variable exponents $q(\cdot)$ such that $1 < q_- \leq q_+ < \infty$.

For any measurable function f, define the operator $\varrho_{q(\cdot)}$ by

$$\varrho_{q(\cdot)}(f) = \int_{\mathbb{R}^n} |f(x)|^{q(x)} dx.$$

The variable Lebesgue space $L^{q(\cdot)}(\mathbb{R}^n)$ is defined to be the set of all measurable functions f on $\mathbb{R}^n$ such that $\varrho_{q(\cdot)}(f) < \infty$. Moreover, for any $f \in L^{q(\cdot)}(\mathbb{R}^n)$, its norm in this space is defined by

$$\|f\|_{L^{q(\cdot)}(\mathbb{R}^n)} := \inf \Big\{ \lambda \in (0, \infty) : \varrho_{q(\cdot)}(f/\lambda) \leq 1 \Big\}.$$

Similarly, $L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n)$ is the set of measurable functions f on $\mathbb{R}^n$ such that $f \in L^{q(\cdot)}(K)$ for every compact set $K \subset \mathbb{R}^n$ where

$$L^{q(\cdot)}(K) := \Big\{ f \text{ is measurable} : \varrho_{q(\cdot)}(f) = \int_K |f(x)|^{q(x)} dx < \infty \Big\}.$$

Recall that the Hardy-Littlewood maximal operator M is defined for any function $f \in L^1_{loc}(\mathbb{R}^n)$ by

$$M(f)(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy,$$

where the supremum is taken over all cubes Q of $\mathbb{R}^n$ containing x.

In what follows, we denote the set of all variable exponents $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, such that the Hardy-Littlewood maximal operator M is bounded on $L^{q(\cdot)}(\mathbb{R}^n)$, by $\mathfrak{B}(\mathbb{R}^n)$.

Remark 2.1. We recall that a variable exponent $q(\cdot)$ is said to satisfy the globally log-Hölder continuity condition if

$$|q(x) - q(y)| \leq \frac{C}{-\log(|x-y|)}, \; x, y \in \mathbb{R}^n, \; |x-y| \leq \frac{1}{2},$$

$$|q(x) - q(y)| \leq \frac{C}{\log(e + |x|)}, \; x, y \in \mathbb{R}^n, \; |y| \geq |x|.$$

It is worth noting that the set $C^{\log}(\mathbb{R}^n)$ of all variable exponents which satisfy the globally log-Hölder continuity condition is an important subset of $\mathfrak{B}(\mathbb{R}^n)$. We refer the reader to [5] and Nekvinda [25] for more details.

Next, we recall the definition of Herz spaces with variable exponent on $\mathbb{R}^n$. For this end, let $B_k := \{x \in \mathbb{R}^n : |x| \le 2^k\}$, $A_k = B_k \backslash B_{k-1}$ for $k \in \mathbb{Z}$. Denote by $\mathbb{Z}_+$ the set of positive integers, $\chi_k = \chi_{A_k}$ for $k \in \mathbb{Z}$, $\widetilde{\chi}_k = \chi_k$ for $k \in \mathbb{Z}_+$ and $\widetilde{\chi}_0 = \chi_{B_0}$.

Definition 1. Let $\alpha \in (0,\infty)$, $0 < p \le \infty$ and $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The non-homogeneous Herz space with variable exponent $\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined to be the set of all functions $f \in L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n)$ such that $\|f\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} < \infty$, where

$$\|f\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \left(\sum_{k=0}^{\infty} 2^{k\alpha p} \|f\widetilde{\chi}_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^{\frac{1}{p}}.$$

The homogeneous Herz space with variable exponent $\dot{\mathbf{K}}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined to be the set of all functions $f \in L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n\backslash\{0\})$ such that $\|f\|_{\dot{\mathbf{K}}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} < \infty$, where

$$\|f\|_{\dot{\mathbf{K}}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^{\frac{1}{p}}.$$

In what follows, we collect some useful lemmas for proving our results in the next sections.

Lemma 2.2 ([16])**.** *Let $q(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$. Then there exists $C > 0$ such that for all balls B in $\mathbb{R}^n$ and all measurable subsets $S \subset B$,*

$$\frac{\|\chi_S\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}} \le C\left(\frac{|S|}{|B|}\right)^{\delta_1}, \tag{2.1}$$

$$\frac{\|\chi_S\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q'(\cdot)}(\mathbb{R}^n)}} \le C\left(\frac{|S|}{|B|}\right)^{\delta_2}, \tag{2.2}$$

where $0 < \delta_1, \delta_2 < 1$ and for every $x \in \mathbb{R}^n : \frac{1}{q(x)} + \frac{1}{q'(x)} = 1$.

The next result can be easily checked by a simple computation using the p-convexity of the $\|\cdot\|_{L^{q(\cdot)}(\mathbb{R}^n)}$ norm.

Lemma 2.3. *For any $f \in \mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ and $s \in (0,\infty)$,*

$$\||f|^s\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \|f\|^s_{\mathbf{K}^{\frac{\alpha}{s},sp}_{sq(.)}(\mathbb{R}^n)}.$$

The same equality holds for the norm of the space $\dot{\mathbf{K}}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$.

The next lemma gives the boundedness of the Hardy-Littlewood maximal operator M on $\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ and $\dot{\mathbf{K}}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$. For the proof, we refer to [16, Theorem 4.1(i)].

Lemma 2.4. *Let $q(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $1 < q_- \le q_+ < \infty$, $0 < p < \infty$ and $-n\delta_1 < \alpha < n\delta_2$, where $0 < \delta_1, \delta_2 < 1$ are constants satisfying* (2.1) *and* (2.2)*. Then the Hardy-Littlewood maximal operator M is bounded on $\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ and $\dot{\mathbf{K}}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$.*

The Fefferman-Stein vector-valued inequality of M on $\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ and $\dot{\mathbf{K}}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is given by the next lemma. It was proved in [17, Remark 4.2].

Lemma 2.5. *Let $q(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $1 < q_- \le q_+ < \infty$, $0 < p < \infty$, $1 < r < \infty$ and $-n\delta_1 < \alpha < n\delta_2$, where $0 < \delta_1, \delta_2 < 1$ satisfying* (2.1) *and* (2.2)*. Then there exists a positive constant C such that, for all sequences $\{f_j\}_{j=1}^{\infty} \in \mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$,*

$$\Big\| \Big(\sum_{j=1}^{\infty} |M(f_j)|^r \Big)^{\frac{1}{r}} \Big\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \le C \Big\| \Big(\sum_{j=1}^{\infty} |f_j|^r \Big)^{\frac{1}{r}} \Big\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.$$

The same result holds for the space $\dot{\mathbf{K}}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$.

The similar result of Lemma 2.5 was obtained in the variable Lebesgue space $L^{q(\cdot)}(\mathbb{R}^n)$ (see [4, Corollary 2.1]).

Lemma 2.6. *Let $q(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $1 < q_- \le q_+ < \infty$ and $1 < r < \infty$. Then there exists a positive constant C such that, for all sequences $\{f_j\}_{j=1}^{\infty}$, of measurable functions,*

$$\Big\| \Big(\sum_{j=1}^{\infty} |M(f_j)|^r \Big)^{\frac{1}{r}} \Big\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le C \Big\| \Big(\sum_{j=1}^{\infty} |f_j|^r \Big)^{\frac{1}{r}} \Big\|_{L^{q(\cdot)}(\mathbb{R}^n)}.$$

Remark 2.7. For all $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $0 < p < 1$ and $\alpha \in (0, \infty)$, we have

$$\begin{aligned} \|f\|_{L^{q(\cdot)}(\mathbb{R}^n)} &= \Big\| \sum_{k=0}^{\infty} \widetilde{\chi}_k f \Big\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\lesssim \sum_{k=0}^{\infty} \|\widetilde{\chi}_k f\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\lesssim \Big(\sum_{k=0}^{\infty} 2^{k\alpha p} \|\widetilde{\chi}_k f\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \Big)^{\frac{1}{p}} \\ &= \|f\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

The same holds for the space $\dot{\mathbf{K}}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$.

Remark 2.8. Let $q(\cdot) \in C^{log}(\mathbb{R}^n)$ with $1 < q_- \le q_+ < \infty$, $0 < p < \infty$ and $\alpha \in (0, \infty)$. By Lemma 2.5 and the fact that, for all cubes $B \subset \mathbb{R}^n$, $\beta \in [1, \infty)$ and $r \in (0, \min\{1, \frac{n\delta_2}{\alpha}\})$, $\chi_{\beta B} \le \beta^{\frac{n}{r}} [M(\chi_B)]^{\frac{1}{r}}$, we conclude that

$$\Big\| \sum_{j \in \mathbb{N}} \chi_{\beta B_j} \Big\|_{\mathbf{K}^{\alpha,p}_{q(.)}(\mathbb{R}^n)} \lesssim \beta^{\frac{n}{r}} \Big\| \sum_{j \in \mathbb{N}} \chi_{B_j} \Big\|_{\mathbf{K}^{\alpha,p}_{q(.)}(\mathbb{R}^n)}.$$

The dual spaces of the $\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ and $\dot{\mathbf{K}}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ are given in the next lemma. The reader is referred to [16, 17] for more details.

Lemma 2.9. *Let $\alpha \in \mathbb{R}$, $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $1 < p < \infty$. Then $\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ coincides with the dual space of $\mathbf{K}^{-\alpha,p'}_{q'(.)}(\mathbb{R}^n)$, where $\frac{1}{p} + \frac{1}{p'} = 1$ and for every $x \in \mathbb{R}^n : \frac{1}{q(x)} + \frac{1}{q'(x)} = 1$.*
Moreover,

$$\|f\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \sup\left\{\left|\int_{\mathbb{R}^n} f(x)g(x)dx\right| : \|g\|_{\mathbf{K}^{-\alpha,p'}_{q'(.)}(\mathbb{R}^n)} \leq 1\right\}.$$

The same duality is also true for $\dot{\mathbf{K}}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$.

The next lemma introduces the generalised Hölder inequality. It can be found in [19].

Lemma 2.10. *Let X be a Banach function space and X' denotes its associate space that means X' is the set of all complex-valued measurable functions f defined on $\mathbb{R}^n$ such that*

$$\|f\|_{X'} := \sup_g \left\{\left|\int_{\mathbb{R}^n} f(x)g(x)dx\right| : \|g\|_X \leq 1\right\} < \infty.$$

Then if $f \in X$ and $g \in X'$, we have

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \leq \|f\|_X \|g\|_{X'}.$$

3. Weak Herz-type Hardy spaces with variable exponents

Definition 2. Let $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $0 < p \leq \infty$ and $\alpha \in (0, \infty)$. The non-homogeneous weak Herz space with variable exponent $\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined to be the set of all measurable functions f such that

$$\|f\|_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} := \sup_{\beta \in (0,\infty)} \beta \|\chi_{\{x \in \mathbb{R}^n : |f(x)| > \beta\}}\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} < \infty.$$

The homogeneous weak Herz space with variable exponent $\mathbf{W\dot{K}}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined to be the set of all measurable functions f such that

$$\|f\|_{\mathbf{W\dot{K}}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} := \sup_{\beta \in (0,\infty)} \beta \|\chi_{\{x \in \mathbb{R}^n : |f(x)| > \beta\}}\|_{\dot{\mathbf{K}}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} < \infty.$$

Next, we give some properties of $\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$.

Lemma 3.1. *Let $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $0 < p \leq \infty$ and $\alpha \in (0, \infty)$. Then $\|\cdot\|_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}$ defines a quasi-norm on $\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$, namely,*

(i) $\|f\|_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = 0$ *if and only if* $f = 0$ *almost everywhere*;

(ii) *for all* $\lambda \in \mathbb{C}$ *and* $f \in \mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$, $\|\lambda f\|_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = |\lambda| \|f\|_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}$;

(iii) *for any* $f,\ g \in \mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$,

$$\|f+g\|^p_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \le 2^p\left[\|f\|^p_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} + \|g\|^p_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}\right].$$

The same properties holds for $\mathbf{W\dot{K}}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$.

Proof. Since (i) is obviously true, we only prove (ii) and (iii).

To prove (ii), without loss of generality, we may assume that $\lambda \neq 0$. By the definition of $\|\cdot\|_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}$, we have

$$\begin{aligned}
\|\lambda f\|_{\mathbf{WK}^{\beta,p}_{q(\cdot)}(\mathbb{R}^n)} &= \sup_{\beta\in(0,\infty)} \beta \|\chi_{\{x\in\mathbb{R}^n:|\lambda f(x)|>\beta\}}\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \\
&= |\lambda| \sup_{\beta\in(0,\infty)} \frac{\beta}{|\lambda|} \|\chi_{\{x\in\mathbb{R}^n:|f(x)|>\frac{\beta}{|\lambda|}\}}\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \\
&= |\lambda| \|f\|_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Then, (ii) holds true.

To prove (iii), for any $f,\ g \in \mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$, we have that

$$\begin{aligned}
&\|f+g\|^p_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \\
&= \sup_{\beta\in(0,\infty)} \beta^p \|\chi_{\{x\in\mathbb{R}^n:|f(x)+g(x)|>\beta\}}\|^p_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \\
&\le \sup_{\beta\in(0,\infty)} \beta^p \left[\|\chi_{\{x\in\mathbb{R}^n:|f(x)|>\frac{\beta}{2}\}}\|^p_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} + \|\chi_{\{x\in\mathbb{R}^n:|g(x)|>\frac{\beta}{2}\}}\|^p_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}\right] \\
&\le \sup_{\beta\in(0,\infty)} \beta^p \|\chi_{\{x\in\mathbb{R}^n:|f(x)|>\frac{\beta}{2}\}}\|^p_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} + \sup_{\beta\in(0,\infty)} \beta^p \|\chi_{\{x\in\mathbb{R}^n:|g(x)|>\frac{\beta}{2}\}}\|^p_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \\
&\le 2^p \left[\|f\|^p_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} + \|g\|^p_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}\right].
\end{aligned}$$

Then (iii) holds true. This finishes the proof. □

Remark 3.2. Let $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $0 < p \le \infty$ and $\alpha \in (0,\infty)$. Then by the Aoki-Rolewicz theorem (see [2,26] and [12, Exercise 1.4.6]), we find that there exists a positive constant $v \in (0,1)$ such that, for all $R \in \mathbb{N}$ and $\{f_j\}_{j=1}^R$,

$$\left\|\sum_{j=1}^R |f_j|\right\|^v_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \le 4\left[\sum_{j=1}^R \|f_j\|^v_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}\right].$$

The same holds for $\mathbf{W\dot{K}}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$.

Lemma 3.3. *Let* $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $0 < p \le \infty$ *and* $\alpha \in (0,\infty)$. *Then, for all* $f \in \mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ *and* $s \in (0,\infty)$, *we have*

$$\||f|^s\|_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \|f\|^s_{\mathbf{WK}^{\frac{\alpha}{s},sp}_{sq(\cdot)}(\mathbb{R}^n)}.$$

The same holds for $\mathbf{W\dot{K}}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$.

Proof. By Lemma 2.3, we find that

$$\begin{aligned}\||f|^s\|_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} &= \sup_{\beta\in(0,\infty)} \beta\|\chi_{\{x\in\mathbb{R}^n:|f(x)|^s>\beta\}}\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \\ &= \sup_{\gamma\in(0,\infty)} \gamma^s\|\chi_{\{x\in\mathbb{R}^n:|f(x)|>\gamma\}}\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \\ &= \sup_{\gamma\in(0,\infty)} \gamma^s\|\chi_{\{x\in\mathbb{R}^n:|f(x)|>\gamma\}}\|^s_{\mathbf{K}^{\frac{\alpha}{s},sp}_{sq(\cdot)}(\mathbb{R}^n)} \\ &= \|f\|^s_{\mathbf{WK}^{\frac{\alpha}{s},sp}_{sq(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

The proof is complete. □

Definition 3. Let $q(\cdot)\in\mathcal{P}(\mathbb{R}^n)$, $0<p<\infty$ and $\alpha\in(0,\infty)$.

(1) For each $N\in\mathbb{N}$, let

$$\mathcal{F}_N(\mathbb{R}^n) := \left\{\varphi\in\mathcal{S}(\mathbb{R}^n) : \sum_{\beta\in\mathbb{Z}^n_+,|\beta|\le N} \sup_{x\in\mathbb{R}^n}[(1+|x|)^N|D^\beta\varphi(x)|]\le 1\right\}.$$

(2) Let $f\in\mathcal{S}'(\mathbb{R}^n)$. Denote by $\mathcal{M}_N$ the grand maximal operator given by

$$\mathcal{M}_N f(x) = \sup_{t>0,\Psi\in\mathcal{F}_N} |t^{-n}\Psi(t^{-1}\cdot)*f(x)|.$$

(3) The non-homogeneous weak Herz-type Hardy space $\mathbf{WHK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$\mathbf{WHK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) := \{f\in\mathcal{S}'(\mathbb{R}^n) : \|f\|_{\mathbf{WHK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}<\infty\},$$

where

$$\|f\|_{\mathbf{WHK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} := \|\mathcal{M}_N f\|_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.$$

(4) The homogeneous weak Herz-type Hardy space $\mathbf{WH\dot{K}}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$\mathbf{WH\dot{K}}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) := \{f\in\mathcal{S}'(\mathbb{R}^n) : \|f\|_{\mathbf{WH\dot{K}}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}<\infty\},$$

where

$$\|f\|_{\mathbf{WH\dot{K}}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} := \|\mathcal{M}_N f\|_{\mathbf{W\dot{K}}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.$$

4. Maximal function characterizations

In this section, we give equivalent characterizations of the weak Herz-type Hardy spaces with variable exponents in terms of several maximal operators. To this end, we first prove the boundedness of the Hardy-Littlewood maximal operator M on the weak Herz spaces with variable exponents and further prove the Fefferman-Stein vector-valued inequality.

Theorem 4.1. *Let $q(\cdot)\in C^{\log}(\mathbb{R}^n)$ with $1<q_-\le q_+<\infty$, $0<p<1$ and $\alpha\in(0,\infty)$. If T is a sublinear operator and bounded on $WL^{q(\cdot)}(\mathbb{R}^n)$, then T is bounded on $\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$.*

Proof. To prove the claim, let $p_1 \in (1,\infty)$. Since T is bounded on $WL^{q(\cdot)}(\mathbb{R}^n)$, we have that

$$\begin{aligned}\|Tf\|^p_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} &= \sup_{\lambda\in(0,\infty)} \sum_{k=0}^{\infty} 2^{k\alpha p}\|Tf\widetilde{\chi}_k\|^p_{WL^{q(\cdot)}(\mathbb{R}^n)} \\ &\lesssim \sum_{k=0}^{\infty} 2^{k\alpha p(1-p_1)}\|Tf\|^p_{WL^{q(\cdot)}(\mathbb{R}^n)} \\ &\lesssim \sum_{k=0}^{\infty} 2^{k\alpha p(1-p_1)}\|f\|^p_{WL^{q(\cdot)}(\mathbb{R}^n)} \\ &\lesssim \sum_{j=0}^{\infty} \|f\widetilde{\chi}_j\|^p_{WL^{q(\cdot)}(\mathbb{R}^n)} \\ &\lesssim \sum_{j=0}^{\infty} 2^{j\alpha p}\|f\widetilde{\chi}_j\|^p_{WL^{q(\cdot)}(\mathbb{R}^n)} \\ &\lesssim \|f\|^p_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Thus

$$\|Tf\|_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.$$

The proof is complete. □

By Theorem 4.1 and the boundedness of M on $WL^{q(\cdot)}(\mathbb{R}^n)$ with $q_- > 1$ (see [32, Corollary 3.3]), we deduce the following boundedness of M on $\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$.

Corollary 4.2. *Let $q(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $1 < q_- \le q_+ < \infty$, $0 < p < 1$ and $\alpha \in (0,\infty)$. Then the Hardy-Littlewood maximal operator M is bounded on $\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$.*

Moreover, using Theorem 4.1, we obtain the following Fefferman-Stein vector-valued inequality of the Hardy-Littlewood maximal operator M on $\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$.

Proposition 4.3. *Let $q(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $1 < q_- \le q_+ < \infty$, $0 < p < 1$, $1 < r < \infty$ and $\alpha \in (0,\infty)$. Then there exists a positive constant C such that, for all sequences $\{f_j\}_{j=1}^{\infty}$ of measurable functions,*

$$\left\|\left(\sum_{j=1}^{\infty}|M(f_j)|^r\right)^{\frac{1}{r}}\right\|_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \le C\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\right\|_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.$$

Proof. Fix an arbitrary sequence of measurable functions $\{f_j\}_{j=1}^{\infty}$ and for any measurable function g, define

$$A(g)(x) := \left(\sum_{j=1}^{\infty}[M(g\eta_j)(x)]^r\right)^{\frac{1}{r}},$$

where $r \in (1, \infty)$, and

$$\eta_j := \frac{f_j}{\left(\sum_{j=1}^{\infty} |f_j|^r\right)^{\frac{1}{r}}} \text{ if } \left(\sum_{j=1}^{\infty} |f_j|^r\right)^{\frac{1}{r}} \neq 0, \text{ and } \eta_j \text{ otherwise.}$$

Then, by the Mikowski inequality, we find that, for any measurable functions g_1 and g_2,

$$\begin{aligned} A(g_1 + g_2)(x) &= \left(\sum_{j=1}^{\infty} [M([g_1 + g_2]\eta_j)(x)]^r\right)^{\frac{1}{r}} \\ &\leq \left(\sum_{j=1}^{\infty} [M(g_1\eta_j)(x) + M(g_2\eta_j)(x)]^r\right)^{\frac{1}{r}} \\ &\leq \left(\sum_{j=1}^{\infty} [M(g_1\eta_j)(x)]^r\right)^{\frac{1}{r}} + \left(\sum_{j=1}^{\infty} [M(g_2\eta_j)(x)]^r\right)^{\frac{1}{r}} \\ &= A(g_1)(x) + A(g_2)(x). \end{aligned}$$

Thus A is sublinear. Moreover, for a measurable function h and by applying Lemma 2.6, we have

$$\|A(h)\|_{L^{q(\cdot)}(\mathbb{R}^n)} \lesssim \|h\|_{L^{q(\cdot)}(\mathbb{R}^n)}.$$

Let $g := \left(\sum_{j=1}^{\infty} |f_j|^r\right)^{\frac{1}{r}}$. Then, by Theorem 4.1, we deduce that

$$\begin{aligned} \left\| \left(\sum_{j=1}^{\infty} |M(f_j)|^r\right)^{\frac{1}{r}} \right\|_{\mathbf{WK}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} &= \|A(g)\|_{\mathbf{WK}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} \\ &\lesssim \|g\|_{\mathbf{WK}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} \sim \left\| \left(\sum_{j=1}^{\infty} |f_j|^r\right)^{\frac{1}{r}} \right\|_{\mathbf{WK}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)}, \end{aligned}$$

which completes the proof. □

Definition 4. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \varphi(x)dx \neq 0$.

(1) We define the discrete maximal operator with respect to φ by

$$M_\varphi f(x) := \sup_{j \in \mathbb{Z}} |\varphi^j * f(x)|,$$

where

$$\varphi^j(x) = 2^{jn}\varphi(2^j x).$$

(2) Suppose that we are given an integer $L \gg 1$. We write

$$M_\varphi^* f(x) = M_{\varphi,L}^* f(x) := \sup_{j \in \mathbb{Z}} \sup_{y \in \mathbb{R}^n} \frac{|\varphi^j * f(y)|}{(1 + 4^j|x - y|^2)^L}.$$

Theorem 4.4. *Let $q(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $1 < q_- \le q_+ < \infty$, $0 < p < \infty$ and $\alpha \in (0,\infty)$. For all $f \in \mathcal{S}'(\mathbb{R}^n)$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \varphi(x)dx \neq 0$, we have*

$$\|f\|_{\mathbf{WHK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \sim \|M_\varphi f\|_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \sim \|M^*_\varphi f\|_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.$$

Proof. We need the following lemma.

Lemma 4.5. *For all $f \in \mathcal{S}'(\mathbb{R}^n)$ and $0 < \theta < 1$, we have*

$$M^*_\varphi f(x) \lesssim \Big(M[\sup_{k\in\mathbb{Z}} |\varphi^k * f|^\theta](x)\Big)^{\frac{1}{\theta}} = \Big(M[(M_\varphi f)^\theta](x)\Big)^{\frac{1}{\theta}}.$$

We fix $\theta \in (0, \min\{1, \frac{n\delta_2}{\alpha}\})$. Then by Lemmas 3.3 and 4.5 and Corollary 4.2, we have

$$\begin{aligned}\|M^*_\varphi f\|_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} &\lesssim \Big\|\Big(M[(M_\varphi f)^\theta]\Big)^{\frac{1}{\theta}}\Big\|_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \\ &\lesssim \Big\|M[(M_\varphi f)^\theta]\Big\|^{\frac{1}{\theta}}_{\mathbf{WK}^{\theta\alpha,\frac{p}{\theta}}_{\frac{q(\cdot)}{\theta}}(\mathbb{R}^n)} \\ &\lesssim \|M_\varphi f\|_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Then, from this and the definitions of M_φ, M^*_φ and $\mathcal{M}_N$, we have

$$\|M^*_\varphi f\|_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \sim \|M_\varphi f\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|_{\mathbf{WHK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.$$

Moreover, by [24, p. 3678], we know that for every $x \in \mathbb{R}^n$,

$$\mathcal{M}_N f(x) \lesssim M^*_\varphi f(x),$$

then

$$\|f\|_{\mathbf{WHK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \lesssim \|M^*_\varphi f\|_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.$$

The proof is complete. □

Definition 5. Let $f \in \mathcal{S}'(\mathbb{R}^n)$, $\psi \in \mathcal{S}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \psi(x)dx \neq 0$ and for $t > 0$, $\psi_t(x) := t^{-n}\psi(\frac{x}{t})$.

(1) The radial maximal operator of f is defined by

$$M_\psi f(x) := \sup_{t>0} |f(x) * \psi_t(x)|.$$

(2) The non-tangential maximal operator of f is defined by

$$M^*_{\psi,a} f(x) := \sup_{t>0, |y-x|<at} |f(x) * \psi_t(y)|, \ a \in (0,\infty).$$

(3) The grand maximal operator:

a) the grand radial maximal operator of f is defined by

$$\mathcal{M}_N f(x) := \sup_{\psi\in\mathcal{F}_N} M_\psi f(x)$$

b) the grand non-tangential maximal operator of f is defined by

$$\mathcal{M}^*_N f(x) := \sup_{\psi\in\mathcal{F}_N} M^*_{\psi,1} f(x).$$

(4) A distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ is called a bounded distribution if, $\forall \psi \in \mathcal{S}(\mathbb{R}^n)$, $f * \psi \in L^\infty(\mathbb{R}^n)$. For a bounded distribution f, its non-tangential maximal operator with respect to Poisson kernels $\{P_t\}_{t>0}$ is defined by setting, $\forall x \in \mathbb{R}^n$,

$$Nf(x) := \sup_{t>0, |y-x|<t} |f * P_t(y)|,$$

where

$$P_t(x) := \frac{\Gamma(\frac{[n+1]}{2})}{\pi^{\frac{n+1}{2}}} \frac{t}{(t^2 + |x|^2)^{\frac{(n+1)}{2}}},$$

$\forall x \in \mathbb{R}^n$, and $t > 0$.

Remark 4.6. Let $f \in \mathcal{S}'(\mathbb{R}^n)$.

(1) From the definitions of $\mathcal{M}_N f$ and $\mathcal{M}_N^* f$, and [34, Proposition 2.1], we have, $\forall x \in \mathbb{R}^n$,

$$\mathcal{M}_N f(x) \sim \mathcal{M}_N^* f(x).$$

(2) $\forall x \in \mathbb{R}^n$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$,

$$M_\psi f(x) \lesssim \mathcal{M}_N f(x).$$

(3) $\forall x \in \mathbb{R}^n, a \in (0, \infty)$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$,

$$\mathcal{M}_{\psi,a}^* f(x) \lesssim \mathcal{M}_N^* f(x).$$

Theorem 4.7. *Let $q(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $1 < q_- \le q_+ < \infty$, $0 < p < 1$, $\alpha \in (0, \infty)$ and $a \in (0, \infty)$. For all $f \in \mathcal{S}'(\mathbb{R}^n)$ and $N \in (\frac{n}{q_-} + n + 1, \infty)$, the following items are equivalent*:

(1) $f \in \mathbf{WHK}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)$;

(2) *there exists* $\psi \in \mathcal{S}(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} \psi(x)dx \neq 0$ *such that* $M_\psi f(x) \in \mathbf{WK}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)$;

(3) *there exists* $\psi \in \mathcal{S}(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} \psi(x)dx \neq 0$ *such that* $M_{\psi,a}^* f(x) \in \mathbf{WK}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)$;

(4) *f is a bounded distribution and* $Nf(x) \in \mathbf{WK}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)$.

Moreover

$$\|\mathcal{M}_N f\|_{\mathbf{WK}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} \sim \|M_\psi f\|_{\mathbf{WK}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} \sim \|M_{\psi,a}^* f\|_{\mathbf{WK}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} \sim \|Nf\|_{\mathbf{WK}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)}.$$

Proof. We prove (3) ⇒ (1). Assume that (3) holds true. Since

$$N \in (\max\{\frac{\alpha}{\delta_2}, \frac{n}{q_-}\} + n + 1, \infty),$$

it follows that there exists $T > \frac{n}{q_-}$ such that $N > T + n + 1$. From this and [6, (3.1)], we have, for all $x \in \mathbb{R}^n$,

$$\mathcal{M}_N f(x) \lesssim M_{\psi,T} f(x), \tag{4.1}$$

where

$$M_{\psi,T} f(x) = \sup_{t>0, y \in \mathbb{R}^n} |f * \psi_t(x-y)| \left(1 + \frac{|y|}{t}\right)^{-T} \quad \forall x \in \mathbb{R}^n.$$

On the other hand, by the proof of [12, Theorem 2.1.4(c)] and [6, (3.2)], we find that, for $L := \frac{n}{T}$ and all $x \in \mathbb{R}^n$,

$$[M_{\psi,T} f(x)]^L \lesssim M([M^*_{\psi,1} f]^L)(x).$$

Then, by the fact that $T > \frac{n}{q_-}$, Lemma 3.3 and Corollary 4.2, we deduce that

$$\begin{aligned}\|M_{\psi,T} f\|_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} &= \|(M_{\psi,T} f)^L\|^{\frac{1}{L}}_{\mathbf{WK}^{\alpha L,\frac{p}{L}}_{\frac{q(\cdot)}{L}}(\mathbb{R}^n)} \\ &\leq \|M([M^*_{\psi,1} f]^L)\|^{\frac{1}{L}}_{\mathbf{WK}^{\alpha L,\frac{p}{L}}_{\frac{q(\cdot)}{L}}(\mathbb{R}^n)} \\ &\leq \|(M^*_{\psi,1} f)^L\|^{\frac{1}{L}}_{\mathbf{WK}^{\alpha L,\frac{p}{L}}_{\frac{q(\cdot)}{L}}(\mathbb{R}^n)} \sim \|M^*_{\psi,1} f\|_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

From this and (4.1), we have

$$\|\mathcal{M}_N f\|_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \lesssim \|M^*_{\psi,1} f\|_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.$$

(1) ⇒ (3) It is true by Remark 4.6.

(1) ⇔ (2) It is included in Theorem 4.4.

(4) ⇒ (1) Suppose that (4) holds true. Then by [29, p. 99] there exists $\psi \in \mathcal{S}(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} \psi(x)dx \neq 0$ such that

$$M_\psi f(x) \lesssim N f(x) \in \mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$$

and thus

$$\|M_\psi f\|_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \lesssim \|N f\|_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.$$

(1) ⇒ (4) To show that $Nf \in \mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$, we use the fact [29, p. 98] that the Poisson kernel can be written as

$$P(x) = \sum_{k=0}^{\infty} 2^{-k} \psi^k_{2^k}(x),$$

where $\{\psi^k\}_{k\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}^n)$ have uniformly bounded semi-norms in $\mathcal{S}(\mathbb{R}^n)$. Fix x and y such that $|x - y| < t$. Then

$$|f * P_t(y)| \leq \sum_{k=0}^{\infty} 2^{-k} |f * \psi^k_{2^k t}(y)| \leq \sum_{k=0}^{\infty} 2^{-k} M^*_{\psi^k,1} f(x).$$

Taking the supremum over all such y and t we get, for all $x \in \mathbb{R}^n$

$$N f(x) \leq \sum_{k=0}^{\infty} 2^{-k} M^*_{\psi^k,1} f(x).$$

Since ψ^k are uniformly bounded, we have same Remark 4.6(2) holds for ψ^k and by using Remark 3.2, we have

$$\|N f\|^v_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \leq \sum_{k=0}^{\infty} 2^{-kv} \|M^*_{\psi^k,1} f\|^v_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}$$

$$\lesssim \|\mathcal{M}_N f\|^v_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} < \infty.$$

Then $Nf \in \mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$.

Next, we show that f is a bounded distribution. By Remark 4.6, we have that, for all $\psi \in \mathcal{S}(\mathbb{R}^n)$, $x \in \mathbb{R}^n$ and $y \in B(x,1)$,

$$|f * \psi(x)| \lesssim M^*_{\psi,1}(y) \lesssim \mathcal{M}^*_N f(y) \sim \mathcal{M}_N f(y).$$

Thus, for any $x \in \mathbb{R}^n$, we have

$$B(x,1) \subset \Omega_{f,x} := \{y \in \mathbb{R}^n : |f * \psi(x)| \lesssim \mathcal{M}_N f(y)\}.$$

By [32, (3.11)] and Remark 2.7, we conclude that

$$\begin{aligned} &\min\{|f * \psi(x)|^{q_-}, |f * \psi(x)|^{q_+}\} \\ \lesssim\ &\min\{|f * \psi(x)|^{q_-}, |f * \psi(x)|^{q_+}\} \max\{\|\Omega_{f,x}\|^{q_-}_{L^{q(\cdot)}(\mathbb{R}^n)}, \|\Omega_{f,x}\|^{q_+}_{L^{q(\cdot)}(\mathbb{R}^n)}\} \\ \lesssim\ &\min\{|f * \psi(x)|^{q_-}, |f * \psi(x)|^{q_+}\} \max\{\|\Omega_{f,x}\|^{q_-}_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}, \Omega_{f,x}\|^{q_+}_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}\} \\ \lesssim\ &\max\{\|\mathcal{M}_N f\|^{q_-}_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}, \|\mathcal{M}_N f\|^{q_+}_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}\} < \infty. \end{aligned}$$

Therefore, $f * \psi \in L^\infty(\mathbb{R}^n)$ and f is a bounded distribution. This ends the proof. □

5. Atomic characterizations

In this section, we establish the atomic characterizations of the weak Herz-type Hardy spaces with variable exponents. We begin with introducing the notion of $(\alpha, p, q(\cdot), s)$-atom.

Definition 6. Let $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $0 < p < \infty$, $\alpha \in (0,\infty)$, $s \in (1,\infty]$ and $d \in (\max\{\frac{n}{q_-} - n - 1, \frac{\alpha}{\delta_2} - n - 1\}, \infty) \cap \mathbb{Z}_+$. A measurable function a on $\mathbb{R}^n$ is called an $(\alpha, p, q(\cdot), s)$-atom if there exists a ball B such that

(1) *supp* $a \subset B$;
(2) $\|a\|_{L^s(\mathbb{R}^n)} \le |B|^{\frac{1}{s}} \|\chi_B\|^{-1}_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}$;
(3) $\int_{\mathbb{R}^n} a(x) x^\beta dx = 0$, for all $\beta \in \mathbb{Z}^n_+$ with $|\beta| \le d$.

Definition 7. Let $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $0 < p < \infty$, $\alpha \in (0,\infty)$, $s \in (1,\infty]$. The atomic weak Herz-type Hardy space with variable exponent $\mathbf{WHK}^{\alpha,p}_{q(\cdot),s,at}(\mathbb{R}^n)$ is defined as

$$\mathbf{WHK}^{\alpha,p}_{q(\cdot),s,at}(\mathbb{R}^n) := \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : f = \sum_{i\in\mathbb{Z}} \sum_{j\in\mathbb{N}} \lambda_{i,j} a_{i,j} \in \mathcal{S}'(\mathbb{R}^n) \right\},$$

where $\{a_{i,j}\}_{i\in\mathbb{Z}, j\in\mathbb{N}}$ is a sequence of $(\alpha, p, q(\cdot), s)$-atoms, associated with balls $\{B_{i,j}\}_{i\in\mathbb{Z}, j\in\mathbb{N}}$, satisfying that there exists a positive constant $c \in (0,1]$, such that, for all $x \in \mathbb{R}^n$ and $i \in \mathbb{Z}$, $\sum_{j\in\mathbb{N}} \chi_{cB_{i,j}(x)} \le A$ with A being a positive constant independent of x and i and, for all $i \in \mathbb{Z}$ and $j \in \mathbb{N}$, $\lambda_{i,j} := \tilde{A} 2^i \|\chi_{B_{i,j}}\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}$ with $\tilde{A}$ being a positive constant independent of i and j.

Moreover, define

$$\|f\|_{\mathbf{WHK}^{\alpha,p}_{q(\cdot),s,at}(\mathbb{R}^n)} := \inf\left[\sup_{i\in\mathbb{Z}}\left\|\left(\sum_{j\in\mathbb{N}}\left[\frac{\lambda_{i,j}\chi_{B_{i,j}}}{\|\chi_{B_{i,j}}\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}}\right]^b\right)^{\frac{1}{b}}\right\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}\right],$$

where $b\in(0,1)$ and the infimum is taken over all decompositions of f as above.

Theorem 5.1. *Let $q(\cdot)\in C^{\log}(\mathbb{R}^n)$ with $1<q_-\le q_+<\infty$, $0<p<1$, $\alpha\in(0,\infty)$, $s\in(\max\{q_+,\frac{bn\delta_1}{n\delta_1-\alpha b}\},\infty]$ and $b\in(0,p)$. Then*

$$\|f\|_{\mathbf{WHK}^{\alpha,p}_{q(\cdot),s,at}(\mathbb{R}^n)} \sim \|f\|_{\mathbf{WHK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.$$

To prove Theorem 5.1, we have to prove the following useful technical lemma.

Lemma 5.2. *Let $q(\cdot)\in C^{\log}(\mathbb{R}^n)$ with $1<q_-\le q_+<\infty$, $s>\max\{q_+,\frac{bn\delta_1}{n\delta_1-\alpha b}\}$, $\alpha\in(0,\infty)$, $p\in(0,\infty)$ and $b\in(0,\min\{1,p\})$. Then there exists a positive constant C such that, for all sequences $\{Q_j\}_{j\in\mathbb{N}}$ of cubes, $\{\lambda_j\}_{j\in\mathbb{N}}\subset\mathbb{C}$ and functions $\{a_j\}_{j\in\mathbb{N}}$ satisfying, for all $j\in\mathbb{N}$, $\operatorname{supp} a_j\subset Q_j$ and $\|a_j\|_{L^s(\mathbb{R}^n)}\le |Q_j|^{\frac{1}{s}}$, then*

$$\left\|\left(\sum_{j=1}^{\infty}|\lambda_j a_j|^b\right)^{\frac{1}{b}}\right\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \le C\left\|\left(\sum_{j=1}^{\infty}|\lambda_j \chi_{Q_j}|^b\right)^{\frac{1}{b}}\right\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.$$

Proof. Let $\{\lambda_j\}_{j\in\mathbb{N}}\subset\mathbb{C}$ and $\{a_j\}_{j\in\mathbb{N}}$ be a sequence of functions satisfying for any $j\in\mathbb{N}$, $\operatorname{supp} a_j\subseteq Q_j$ where Q_j is a cube of $\mathbb{R}^n$. Then, by Lemmas 2.3 and 2.9, we deduce that there exists a function $g\in\mathbf{K}^{-b\alpha,(\frac{p}{b})'}_{(\frac{q(\cdot)}{b})'}(\mathbb{R}^n)$ with $\|g\|_{\mathbf{K}^{-b\alpha,(\frac{p}{b})'}_{(\frac{q(\cdot)}{b})'}(\mathbb{R}^n)}\le 1$ such that

$$\left\|\left(\sum_{j=1}^{\infty}|\lambda_j a_j|^b\right)^{\frac{1}{b}}\right\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \left\|\sum_{j=1}^{\infty}|\lambda_j a_j|^b\right\|^{\frac{1}{b}}_{\mathbf{K}^{b\alpha,\frac{p}{b}}_{\frac{q(\cdot)}{b}}(\mathbb{R}^n)}$$

$$\lesssim\left(\int_{\mathbb{R}^n}\sum_{j=1}^{\infty}|\lambda_j a_j(x)|^b|g(x)|dx\right)^{\frac{1}{b}}. \tag{5.1}$$

Then, by Lemma 2.10, we get

$$\begin{aligned}&\int_{\mathbb{R}^n}\sum_{j=1}^{\infty}|\lambda_j a_j(x)|^b|g(x)|dx\\ &=\sum_{j=1}^{\infty}|\lambda_j|^b\int_{\mathbb{R}^n}|a_j(x)|^b|g(x)|dx\\ &\le\sum_{j=1}^{\infty}|\lambda_j|^b\|a_j\|^b_{L^s}\|g\|_{L^{(\frac{s}{b})'}(Q_j)}\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{j=1}^{\infty} |\lambda_j|^b |Q_j|^{\frac{b}{s}} \|g\|_{L^{(\frac{s}{b})'}(Q_j)} \\
&\le \sum_{j=1}^{\infty} \int_{\mathbb{R}^n} |\lambda_j|^b \chi_{Q_j}(x) \Big[M(|g|^{(\frac{s}{b})'})(x)\Big]^{\frac{1}{(\frac{s}{b})'}} dx \\
&\le \Big\| \sum_{j=1}^{\infty} |\lambda_j \chi_{Q_j}|^b \Big\|_{\mathbf{K}^{b\alpha,\frac{p}{b}}_{\frac{q(\cdot)}{b}}(\mathbb{R}^n)} \Big\| \Big[M(|g|^{(\frac{s}{b})'})\Big]^{\frac{1}{(\frac{s}{b})'}} \Big\|_{\mathbf{K}^{-b\alpha,(\frac{p}{b})'}_{(\frac{q(\cdot)}{b})'}(\mathbb{R}^n)}.
\end{aligned}
$$

Thus, by (5.1), Lemmas 2.3 and 2.4, we obtain

$$
\Big\| \Big(\sum_{j=1}^{\infty} |\lambda_j a_j|^b \Big)^{\frac{1}{b}} \Big\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \lesssim \Big\| \Big(\sum_{j=1}^{\infty} |\lambda_j \chi_{Q_j}|^b \Big)^{\frac{1}{b}} \Big\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.
$$

The proof is complete. □

Now, we turn to the proof of Theorem 5.1.

Proof. **Step 1**: In this step we show that $\mathbf{WHK}^{\alpha,p}_{q(\cdot),s,at}(\mathbb{R}^n) \subset \mathbf{WHK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$.

To prove that $f \in \mathbf{WHK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$, it suffices to show that

$$
\sup_{\beta\in(0,\infty)} \beta \|\chi_{\{x\in\mathbb{R}^n : |f^*(x)|>\beta\}}\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|_{\mathbf{WHK}^{\alpha,p}_{q(\cdot),s,at}(\mathbb{R}^n)}.
$$

To simplify the notation, let $f^* =: \mathcal{M}_N(f)$ with N as Theorem 4.7. For any given $\beta \in (0,\infty)$, we choose $i_0 \in \mathbb{Z}$ such that $2^{i_0} \le \beta < 2^{i_0+1}$ and write

$$
f = \sum_{i=-\infty}^{i_0-1} \sum_{j\in\mathbb{N}} \lambda_{i,j} a_{i,j} + \sum_{i=i_0}^{\infty} \sum_{j\in\mathbb{N}} \lambda_{i,j} a_{i,j} = f_1 + f_2.
$$

Moreover, it holds true that

$$
\begin{aligned}
\|\chi_{\{x\in\mathbb{R}^n : f^*(x)>\beta\}}\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} &\lesssim \|\chi_{\{x\in\mathbb{R}^n : f_1^*(x)>\frac{\beta}{2}\}}\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \\
&\quad + \|\chi_{\{x\in A_{i_0} : f_2^*(x)>\frac{\beta}{2}\}}\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \\
&\quad + \|\chi_{\{x\in (A_{i_0})^c : f_2^*(x)>\frac{\beta}{2}\}}\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \\
&= I_1 + I_2 + I_3,
\end{aligned}
$$

where $A_{i_0} = \cup_{i=i_0}^{\infty} \cup_{j\in\mathbb{N}} (2B_{i,j})$.

For I_1, it is easy to see that

$$
\begin{aligned}
I_1 &\lesssim \|\chi_{\{x\in\mathbb{R}^n : \sum_{i=-\infty}^{i_0-1}\sum_{j\in\mathbb{N}} \lambda_{i,j}(a_{i,j})^*(x)\chi_{2B_{i,j}}(x)>\frac{\beta}{4}\}}\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \\
&\quad + \|\chi_{\{x\in\mathbb{R}^n : \sum_{i=-\infty}^{i_0-1}\sum_{j\in\mathbb{N}} \lambda_{i,j}(a_{i,j})^*(x)\chi_{(2B_{i,j})^c}(x)>\frac{\beta}{4}\}}\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \\
&= I_{1,1} + I_{1,2}.
\end{aligned}
$$

To estimate $I_{1,1}$, for any $b\in(0,p)$, let $\tilde{q}\in(1,\min\{\frac{s}{\max\{q_+,\frac{bn\delta_1}{n\delta_1-\alpha b}\}},\frac{1}{b}\})$ and $a\in(0,1-\frac{1}{\tilde{q}})$. Then by the Hölder inequality, we find that for all $x\in\mathbb{R}^n$,

$$\begin{aligned}&\sum_{i=-\infty}^{i_0-1}\sum_{j\in\mathbb{N}}\lambda_{i,j}(a_{i,j})^*(x)\chi_{2B_{i,j}}(x)\\ &\leq\bigg(\sum_{i=-\infty}^{i_0-1}2^{ia\tilde{q}'}\bigg)^{\frac{1}{\tilde{q}'}}\bigg\{\sum_{i=-\infty}^{i_0-1}2^{-ia\tilde{q}}\Big[\sum_{j\in\mathbb{N}}\lambda_{i,j}(a_{i,j})^*(x)\chi_{2B_{i,j}}(x)\Big]^{\tilde{q}}\bigg\}^{\frac{1}{\tilde{q}}}\\ &=\frac{2^{i_0a}}{(2^{a\tilde{q}'}-1)^{\frac{1}{\tilde{q}'}}}\bigg\{\sum_{i=-\infty}^{i_0-1}2^{-ia\tilde{q}}\Big[\sum_{j\in\mathbb{N}}\lambda_{i,j}(a_{i,j})^*(x)\chi_{2B_{i,j}}(x)\Big]^{\tilde{q}}\bigg\}^{\frac{1}{\tilde{q}}},\end{aligned}$$

where $\tilde{q}'$ denotes the conjugate exponent of $\tilde{q}$, namely, $\frac{1}{\tilde{q}}+\frac{1}{\tilde{q}'}=1$. From this, the facts that $\tilde{q}b<1$ and $f^*(x)\lesssim Mf(x)$ for all $x\in\mathbb{R}^n$, we deduce that

$$\begin{aligned}I_{1,1}^p&\lesssim\Big\|\chi_{\{x\in\mathbb{R}^n:\frac{2^{i_0a}}{(2^{a\tilde{q}'}-1)^{\frac{1}{\tilde{q}'}}}[\sum_{i=-\infty}^{i_0-1}2^{-ia\tilde{q}}\{\sum_{j\in\mathbb{N}}\lambda_{i,j}(a_{i,j})^*\chi_{2B_{i,j}}(x)\}^{\tilde{q}}]^{\frac{1}{\tilde{q}}}>2^{i_0-2}\}}\Big\|^p_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}\\ &\lesssim2^{-i_0\tilde{q}(1-a)p}\Big\|\sum_{i=-\infty}^{i_0-1}2^{-ia\tilde{q}}\Big[\sum_{j\in\mathbb{N}}\lambda_{i,j}(a_{i,j})^*\chi_{2B_{i,j}}\Big]^{\tilde{q}}\Big\|^p_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}\\ &\lesssim2^{-i_0\tilde{q}(1-a)p}\sum_{i=-\infty}^{i_0-1}2^{-ia\tilde{q}p}\Big\|\Big[\sum_{j\in\mathbb{N}}\lambda_{i,j}M(a_{i,j})\chi_{2B_{i,j}}\Big]^{\tilde{q}}\Big\|^p_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}\\ &\lesssim2^{-i_0\tilde{q}(1-a)p}\sum_{i=-\infty}^{i_0-1}2^{(1-a)i\tilde{q}p}\Big\|\Big\{\sum_{j\in\mathbb{N}}\Big[\|\chi_{B_{i,j}}\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}M(a_{i,j})\chi_{2B_{i,j}}\Big]^{\tilde{q}b}\Big\}^{\frac{1}{b}}\Big\|^p_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Moreover, by the boundedness of M on L^r $(1<r<\infty)$, we have

$$\begin{aligned}\Big\|\Big[\|\chi_{B_{i,j}}\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}M(a_{i,j})\chi_{2B_{i,j}}\Big]^{\tilde{q}}\Big\|_{L^{\frac{s}{\tilde{q}}}(\mathbb{R}^n)}&\leq\|\chi_{B_{i,j}}\|^{\tilde{q}}_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}\|M(a_{i,j})\chi_{2B_{i,j}}\|^{\tilde{q}}_{L^s(\mathbb{R}^n)}\\ &\lesssim|\chi_{2B_{i,j}}|^{\frac{\tilde{q}}{s}}.\end{aligned}$$

Then, by Lemma 5.2 and Remark 2.8

$$\begin{aligned}I_{1,1}^p&\lesssim2^{-i_0\tilde{q}(1-a)p}\sum_{i=-\infty}^{i_0-1}2^{(1-a)i\tilde{q}p}\Big\|\Big(\sum_{j\in\mathbb{N}}\chi_{2B_{i,j}}\Big)^{\frac{1}{b}}\Big\|^p_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}\\ &\lesssim2^{-i_0\tilde{q}(1-a)p}\sum_{i=-\infty}^{i_0-1}2^{[(1-a)\tilde{q}-1]ip}\sup_{i\in\mathbb{Z}}2^{ip}\Big\|\Big(\sum_{j\in\mathbb{N}}\chi_{B_{i,j}}\Big)^{\frac{1}{b}}\Big\|^p_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}\\ &\lesssim\beta^{-p}\|f\|^p_{\mathbf{WHK}^{\alpha,p}_{q(\cdot),s,at}(\mathbb{R}^n)},\end{aligned}$$

which implies that

$$\beta I_{1,1}\lesssim\|f\|_{\mathbf{WHK}^{\alpha,p}_{q(\cdot),s,at}(\mathbb{R}^n)}.\tag{5.2}$$

For $I_{1,2}$, from [32, (4.9)], by similar argument we find this key estimate

$$(a_{i,j})^*(x) \lesssim \|\chi_{B_{i,j}}\|^{-1}_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} [M(\chi_{B^{i,j}})(x)]^{\frac{n+d+1}{n}}, \; x \in (2B_{i,j})^c. \tag{5.3}$$

From this, the Hölder inequality, Lemma 2.3, Lemma 2.5 and the fact that d as in Definition 6, we find for any $b \in (0,1)$, $q_1 \in (\max\{1, \frac{n}{(n+d+1)b}, \frac{\alpha}{\delta_2(n+d+1)}\}, \frac{1}{b})$ and $a \in (0, 1-\frac{1}{q_1})$,

$$\begin{aligned}
I^p_{1,2} &\lesssim \left\| \chi_{\{x\in\mathbb{R}^n: \frac{2^{i_0 a}}{(2^{aq_1'}-1)^{\frac{1}{q_1'}}} [\sum_{i=-\infty}^{i_0-1} 2^{-iaq_1} \{\sum_{j\in\mathbb{N}} \lambda_{i,j}(a_{i,j})^* \chi_{(2B_{i,j})^c}(x)\}^{q_1}]^{\frac{1}{q_1}} > 2^{i_0-2}\}} \right\|^p_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \\
&\lesssim 2^{-i_0 q_1(1-a)p} \left\| \sum_{i=-\infty}^{i_0-1} 2^{-iaq_1} \left[\sum_{j\in\mathbb{N}} \lambda_{i,j}(a_{i,j})^* \chi_{(2B_{i,j})^c} \right]^{q_1} \right\|^p_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \\
&\lesssim 2^{-i_0 q_1(1-a)p} \sum_{i=-\infty}^{i_0-1} 2^{(1-a)iq_1 p} \left\| \left\{ \sum_{j\in\mathbb{N}} [M(\chi_{B_{i,j}})]^{\frac{(n+d+1)q_1 b}{n}} \right\}^{\frac{n}{(n+d+1)q_1 b}} \right\|^{\frac{q_1(n+d+1)p}{n}}_{\mathbf{K}^{\frac{\alpha n}{q_1(n+d+1)}, \frac{pq_1(n+d+1)}{n}}_{\frac{q(\cdot)q_1(n+d+1)}{n}}(\mathbb{R}^n)} \\
&\lesssim 2^{-i_0 q_1(1-a)p} \sum_{i=-\infty}^{i_0-1} 2^{[(1-a)q_1-1]ip} 2^{ip} \left\| \left\{ \sum_{j\in\mathbb{N}} \chi_{B_{i,j}} \right\}^{\frac{1}{b}} \right\|^p_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \\
&\lesssim \beta^{-p} \sup_{i\in\mathbb{Z}} 2^{ip} \left\| \left\{ \sum_{j\in\mathbb{N}} \chi_{B_{i,j}} \right\}^{\frac{1}{b}} \right\|^p_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \\
&\lesssim \beta^{-p} \|f\|^p_{\mathbf{WHK}^{\alpha,p}_{q(\cdot),s,at}(\mathbb{R}^n)},
\end{aligned}$$

which implies that

$$\beta I_{1,2} \lesssim \|f\|_{\mathbf{WHK}^{\alpha,p}_{q(\cdot),s,at}(\mathbb{R}^n)}. \tag{5.4}$$

Then, by (5.4) and (5.2), we have

$$\beta I_1 \lesssim \|f\|_{\mathbf{WHK}^{\alpha,p}_{q(\cdot),s,at}(\mathbb{R}^n)}. \tag{5.5}$$

For I_2, by Remark 2.8, we have

$$\begin{aligned}
I^p_2 \le \|\chi_{A_{i_0}}\|^p_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} &\le \left\| \sum_{i=i_0}^{\infty} \sum_{j\in\mathbb{N}} \chi_{2B_{i,j}} \right\|^p_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \\
&\lesssim \sum_{i=i_0}^{\infty} \left\| \sum_{j\in\mathbb{N}} \chi_{2B_{i,j}} \right\|^p_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \\
&\lesssim \sum_{i=i_0}^{\infty} 2^{-ip} \sup_{i\in\mathbb{Z}} 2^{ip} \left\| \sum_{j\in\mathbb{N}} \chi_{B_{i,j}} \right\|^p_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \\
&\lesssim \beta^{-p} \|f\|^p_{\mathbf{WHK}^{\alpha,p}_{q(\cdot),s,at}(\mathbb{R}^n)},
\end{aligned}$$

which implies that

$$\beta I_2 \lesssim \|f\|_{\mathbf{WHK}^{\alpha,p}_{q(\cdot),s,at}(\mathbb{R}^n)}. \tag{5.6}$$

For I_3, by the fact that d as in Definition 6, we choose

$$r_2 \in (\max\{\frac{n}{n+d+1}, \frac{\alpha}{\delta_2(n+d+1)}\}, 1).$$

Then by Lemma 2.5 and (5.3), we have

$$\begin{aligned}
I_3 &\lesssim \left\| \chi_{\{x\in(A_{i_0})^c:\sum_{i=i_0}^{\infty}\sum_{j\in\mathbb{N}}\lambda_{i,j}(a_{i,j})^*(x)>\frac{\beta}{2}\}} \right\|^p_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \\
&\lesssim \beta^{-r_2p} \left\| \sum_{i=i_0}^{\infty}\sum_{j\in\mathbb{N}} [\lambda_{i,j}(a_{i,j})^*]^{r_2}\chi_{(A_{i_0})^c} \right\|^p_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \\
&\lesssim \beta^{-r_2p} \sum_{i=i_0}^{\infty} 2^{ir_2p} \left\| \sum_{j\in\mathbb{N}} [\|\chi_{B_{i,j}}\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}(a_{i,j})^*]^{r_2}\chi_{(A_{i_0})^c} \right\|^p_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \\
&\lesssim \beta^{-r_2p} \sum_{i=i_0}^{\infty} 2^{ir_2p} \left\| \left\{ \sum_{j\in\mathbb{N}} [M(\chi_{B_{i,j}})]^{\frac{r_2(n+d+1)}{n}} \right\}^{\frac{n}{r_2(n+d+1)}} \right\|^{\frac{r_2(n+d+1)p}{n}}_{\mathbf{K}^{\frac{\alpha n}{r_2(n+d+1)}, \frac{r_2(n+d+1)p}{n}}_{\frac{r_2(n+d+1)q(\cdot)}{n}}(\mathbb{R}^n)} \\
&\lesssim \beta^{-r_2p} \sum_{i=i_0}^{\infty} 2^{ir_2p} 2^{-ip} \sup_{i\in\mathbb{Z}} 2^{ip} \left\| \sum_{j\in\mathbb{N}} \chi_{B_{i,j}} \right\|^p_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \\
&\lesssim \beta^{-p}\|f\|^p_{\mathbf{WHK}^{\alpha,p}_{q(\cdot),s,at}(\mathbb{R}^n)},
\end{aligned}$$

that is

$$\beta I_3 \lesssim \|f\|_{\mathbf{WHK}^{\alpha,p}_{q(\cdot),s,at}(\mathbb{R}^n)}. \tag{5.7}$$

Combining (5.5), (5.6) and (5.7), we obtain

$$\begin{aligned}
\|f\|_{\mathbf{WHK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} &= \sup_{\beta\in(0,\infty)} \beta\|\chi_{\{x\in\mathbb{R}^n:|f^*(x)|>\beta\}}\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \\
&\lesssim \sup_{\beta\in(0,\infty)} \beta(I_1+I_2+I_3) \lesssim \|f\|_{\mathbf{WHK}^{\alpha,p}_{q(\cdot),s,at}(\mathbb{R}^n)},
\end{aligned}$$

which implies $f \in \mathbf{WHK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$.

Step 2: In this step we show that $\mathbf{WHK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) \subset \mathbf{WHK}^{\alpha,p}_{q(\cdot),s,at}(\mathbb{R}^n)$. To prove the claim, it suffices to show that $\mathbf{WHK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) \subset \mathbf{WHK}^{\alpha,p}_{q(\cdot),\infty,at}(\mathbb{R}^n)$, due to the obvious fact that each $(\alpha,p,q(\cdot),\infty)$-atom is also an $(\alpha,p,q(\cdot),s)$-atom for any $s\in(1,\infty)$.

We need the following lemma, which was obtained in [3, p. 219] (see also [32, Lemma 4.6]).

Lemma 5.3. *Let $\psi\in\mathcal{S}(\mathbb{R}^n)$ be such that supp $\psi\subset B(\vec{0}_n,1)$ and $\int\psi(x)dx=0$. Then there exists a function $\phi\in\mathcal{S}(\mathbb{R}^n)$ such that $\hat{\phi}$ has compact support away from the origin and, for all $x\in\mathbb{R}^n\setminus\{\vec{0}_n\}$,*

$$\int_0^\infty \hat{\psi}(tx)\hat{\phi}(tx)\frac{dt}{t} = 1.$$

Recall that, for any $d \in \mathbb{Z}_+$, $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, a locally integrable function f on $\mathbb{R}^n$ is said to belong the Campanato space $\mathcal{L}_{1,q(\cdot),d}(\mathbb{R}^n)$ if

$$\|f\|_{\mathcal{L}_{1,q(\cdot),d}(\mathbb{R}^n)} := \sup_{Q\subset\mathbb{R}^n} \frac{1}{\|\chi_Q\|_{L^{q(\cdot)}}} \int_Q |f(x) - \mathcal{P}_Q^d f(x)|dx < \infty,$$

where $\mathcal{P}_Q^d$ denotes the unique polynomial P having degree at most d and satisfies that, for any polynomial R on $\mathbb{R}^n$ with order at most d, $\int_Q [f(x) - P(x)]R(x)dx = 0$.

Now, let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be such that *supp* $\psi \subset B(\vec{0}_n, 1)$, $\int \psi(x)x^\beta dx = 0$ for all $\beta \in \mathbb{Z}_+^n$ with $|\beta| \le d$, then by Lemma 5.3 there exists $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that $\hat{\phi}$ has compact support away from the origin and, for all $x \in \mathbb{R}^n \setminus \{\vec{0}_n\}$,

$$\int_0^\infty \hat{\psi}(tx)\hat{\phi}(tx)\frac{dt}{t} = 1.$$

Define a function η on $\mathbb{R}^n$ by setting, for all $x \in \mathbb{R}^n \setminus \{\vec{0}_n\}$,

$$\hat{\eta}(x) = \int_1^\infty \hat{\psi}(tx)\hat{\phi}(tx)\frac{dt}{t}$$

and $\hat{\eta}(\vec{0}_n) = 1$. Then, by [3, p. 219], we know that η is infinitely differentiable, has compact support and equals to 1 near the origin.

Let $x_0 = (2, \ldots, 2) \in \mathbb{R}^n$ and $f \in \mathbf{WHK}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)$. Following [3], for all $x \in \mathbb{R}^n$, set

$$\tilde{\phi}(x) = \phi(x - x_0),\ \tilde{\psi}(x) = \psi(x + x_0),$$
$$F(x,t) = f * \tilde{\phi}_t(x) \text{ and } G(x,t) = f * \eta_t(x).$$

Then by [3, p. 220], we have

$$f(\cdot) = \int_0^\infty \int_{\mathbb{R}^n} F(y,t)\tilde{\psi}(\cdot - y)\frac{dydt}{t} \text{ in } \mathcal{S}'(\mathbb{R}^n).$$

For all $x \in \mathbb{R}^n$, let

$$M_\nabla f(x) = \sup_{t\in(0,\infty),|y-x|\le 3(|x_0|+1)t} (|F(y,t)| + |G(y,t)|).$$

By Remark 4.6, we have $M_\nabla f \in \mathbf{WK}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)$ and

$$\|M_\nabla f\|_{\mathbf{WK}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} \lesssim \|f\|_{\mathbf{WHK}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)}. \tag{5.8}$$

For all $i \in \mathbb{Z}$, set $\Omega_i = \{x \in \mathbb{R}^n : M_\nabla f(x) > 2^i\}$, $M_\nabla f$ is lower semi continuous (implies Ω_i is open). Since Ω_i is a proper open subset of $\mathbb{R}^n$, by the Whitney decomposition we know that there exists a sequence $\{Q_{i,j}\}_{j\in\mathbb{N}}$ of cubes such that for all $i \in \mathbb{Z}$,

(i) $\cup_{j\in\mathbb{N}} Q_{i,j} = \Omega_i$ and $\{Q_{i,j}\}_j$ have disjoint interiors;
(ii) for all $j \in \mathbb{N}$, $\sqrt{n}l_{Q_{i,j}} \le d(Q_{i,j}, \Omega_i^c) \le 4\sqrt{n}l_{Q_{i,j}}$, where $l_{Q_{i,j}}$ denotes the length of the cube $Q_{i,j}$ and $d(Q_{i,j}, \Omega_i^c) = \inf\{|x-y| : x \in Q_{i,j}, y \in \Omega_i^c\}$;

(iii) for any $j, k \in \mathbb{N}$, if the boundaries of two cubes $Q_{i,j}$ and $Q_{i,k}$ touch, then $\frac{1}{4} \le \frac{l_{Q_{i,j}}}{l_{Q_{i,k}}} \le 4$;

(iv) for a given $j \in \mathbb{N}$, there exist at most $12n$ different cubes $Q_{i,k}$ that touch $Q_{i,j}$.

Now for any $\epsilon \in (0,\infty)$, $j \in \mathbb{N}$, $i \in \mathbb{Z}$ and $x \in \mathbb{R}^n$, let

$$d(x,\Omega_i^c) = \inf\{|x-y| : y \in \Omega_i^c\};$$

$$\tilde{\Omega}_i = \{(x,t) \in \mathbb{R}^{n+1}_+ : 0 < 2t(|x_0|+1) < d(x,\Omega_i^c)\};$$

$$\tilde{Q}_{i,j} = \{(x,t) \in \mathbb{R}^{n+1}_+ : x \in Q_{i,j}, (x,t) \in \tilde{\Omega}_i \setminus \tilde{\Omega}_{i+1}\}$$

and

$$b^{\epsilon}_{i,j} = \int_{\epsilon}^{\infty}\int_{\mathbb{R}^n} \chi_{\tilde{Q}_{i,j}}(y,t)F(y,t)\tilde{\psi}_t(x-y)\frac{dydt}{t}.$$

By the same argument used in [3, pp. 221–222] (see also [20, p. 16]), we conclude that there exist positive constants c_1 and $c_2 > 0$ such that for all $\epsilon \in (0,\infty)$, $i \in \mathbb{Z}$ and $j \in \mathbb{N}$, *supp* $b^{\epsilon}_{i,j} \subset c_1 Q_{i,j}$, $\|b^{\epsilon}_{i,j}\|_{L^\infty(\mathbb{R}^n)} \le c_2 2^i$, $\int_{\mathbb{R}^n} b^{\epsilon}_{i,j}(x)x^\beta dx = 0$ for all $\beta \in \mathbb{Z}^n_+$ satisfying $|\beta| \le d$ and

$$f = \lim_{\epsilon\to 0} \sum_{i\in\mathbb{Z}, j\in\mathbb{N}} b^{\epsilon}_{i,j} \text{ in } \mathcal{S}'(\mathbb{R}^n).$$

Moreover, by similar argument that used in [20, p. 16] (see also [32, p. 2855]), we find that there exist $\{b_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}} \subset L^\infty(\mathbb{R}^n)$ and a sequence $\{\epsilon_k\}_{k\in\mathbb{N}} \subset (0,\infty)$ such that $\epsilon_k \to 0$ as $k \to \infty$ and for any $g \in L^1(\mathbb{R}^n)$,

$$\lim_{k\to\infty} \langle b^{\epsilon_k}_{i,j}, g\rangle = \langle b_{i,j}, g\rangle,$$

supp $b_{i,j} \subset c_1 Q_{i,j}$, $\|b_{i,j}\|_{L^\infty(\mathbb{R}^n)} \le c_2 2^i$. For all $\beta \in \mathbb{Z}^n_+$ satisfying $|\beta| \le d$,

$$\int_{\mathbb{R}^n} b_{i,j}(x)x^\beta dx = \langle b_{i,j}, x^\beta \chi_{c_1 Q_{i,j}}\rangle = \lim_{k\to\infty}\int_{\mathbb{R}^n} b^{\epsilon_k}_{i,j}(x)x^\alpha dx = 0.$$

Next we show that

$$\lim_{k\to\infty}\sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{N}} b^{\epsilon_k}_{i,j} = \sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{N}} b_{i,j} \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n). \tag{5.9}$$

Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. By the estimate in [32, pp. 2855–2856], Remark 2.7 and (5.8), for $k, N \in \mathbb{N}$, we get

$$\begin{aligned}
&\sum_{|i|>N}\sum_{j\in\mathbb{N}}[|\langle b^{\epsilon_k}_{i,j},\varphi\rangle| + |\langle b_{i,j},\varphi\rangle|] \\
&\lesssim 2^{-N}\|\varphi\|_{L^1(\mathbb{R}^n)} + \|\varphi\|_{\mathcal{L}_{1,\frac{q(\cdot)}{r},d}}[\sup_{i\in\mathbb{Z}} 2^i\|\chi_{\Omega_i}\|_{L^{q(\cdot)}(\mathbb{R}^n)}]^r \sum_{i=N+1}^{\infty} 2^{-i(r-1)} \\
&\lesssim 2^{-N}\|\varphi\|_{L^1(\mathbb{R}^n)} + \|\varphi\|_{\mathcal{L}_{1,\frac{q(\cdot)}{r},d}}[\sup_{i\in\mathbb{Z}} 2^i\|\chi_{\Omega_i}\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}]^r \sum_{i=N+1}^{\infty} 2^{-i(r-1)}
\end{aligned}$$

$$\lesssim 2^{-N}\|\varphi\|_{L^1(\mathbb{R}^n)} + \|\varphi\|_{\mathcal{L}_{1,\frac{q(\cdot)}{r},d}}[\|M_\nabla f\|_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}]^r 2^{-N(r-1)}$$
$$\lesssim 2^{-N}\|\varphi\|_{L^1(\mathbb{R}^n)} + 2^{-N(r-1)}\|\varphi\|_{\mathcal{L}_{1,\frac{q(\cdot)}{r},d}}\|f\|^r_{\mathbf{WHK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)},$$

which tends to 0 as $N \to \infty$, where r is chosen such that $r > \max\{q_+, 1\}$. Above we used (5.8) and the fact that, for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $\|\varphi\|\mathcal{L}_{1,\frac{q(\cdot)}{r},d}$ is finite (see [34, Lemma 2.8]).

Similarly, we have

$$\sum_{|i|\le N}\sum_{j\in\mathbb{N}}[|\langle b^{\epsilon_k}_{i,j},\varphi\rangle| + |\langle b_{i,j},\varphi\rangle|] < \infty.$$

Then using the same argument as in [22, p. 651], we get (5.9).

For $i \in \mathbb{Z}$ and $j \in \mathbb{N}$, let $B_{i,j}$ be the ball having the same center as $Q_{i,j}$ with radius $5\sqrt{n}c_1 l_{Q_{i,j}}$,

$$a_{i,j} = \frac{b_{i,j}}{c_2 2^i\|\chi_{B_{i,j}}\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}} \text{ and } \lambda_{i,j} = c_2 2^i\|\chi_{B_{i,j}}\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.$$

Then $a_{i,j}$ is an $(\alpha, p, q(\cdot), \infty)$-atom associated to $B_{i,j}$ and

$$f = \sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{N}}\lambda_{i,j}a_{i,j} \text{ in } \mathcal{S}'(\mathbb{R}^n).$$

Moreover, by Remark 2.8 and (5.8), we find

$$\begin{aligned}
\|f\|_{\mathbf{WHK}^{\alpha,p}_{q(\cdot),\infty,at}(\mathbb{R}^n)} &\lesssim \sup_{i\in\mathbb{Z}} 2^i\left\|\left(\sum_{j\in\mathbb{N}}\chi_{B_{i,j}}\right)^{\frac{1}{b}}\right\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}\\
&\lesssim \sup_{i\in\mathbb{Z}} 2^i\left\|\left(\sum_{j\in\mathbb{N}}\chi_{Q_{i,j}}\right)^{\frac{1}{b}}\right\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}\\
&\lesssim \sup_{i\in\mathbb{Z}} 2^i\|\chi_{\Omega_i}\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}\\
&\lesssim \|M_\nabla f\|_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}\\
&\lesssim \|f\|_{\mathbf{WHK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)},
\end{aligned}$$

which completes the proof. □

Remark 5.4. The space $\mathbf{WH\dot{K}}^{\alpha,p}_{q(\cdot),s,at}(\mathbb{R}^n)$ is defined by the same way as in Definition 7 via replacing the norm of $\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ by the norm of $\dot{\mathbf{K}}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ and all the results in this section are also valid on this space.

6. Molecular characterizations

In this section, we establish the molecular characterizations of the weak Herz-type Hardy spaces with variable exponents. We begin with introducing the notion of $(\alpha, p, q(\cdot), s, \epsilon)$-molecule.

Definition 8. Let $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $0 < p < \infty$, $\alpha \in (0,\infty)$, $s \in (1,\infty]$, $d \in (\max\{\frac{n}{q_-} - n - 1, \frac{\alpha}{\delta_2} - n - 1\}, \infty) \cap \mathbb{Z}_+$, and $\epsilon \in (0,\infty)$. A measurable function m is called an $(\alpha, p, q(\cdot), s, \epsilon)$-molecule associated with some ball $B \subset \mathbb{R}^n$ if

(i) for each $j \in \mathbb{N}$, $\|m\|_{L^s(U_j(B))} \leq 2^{-j\epsilon}|U_j(B)|^{\frac{1}{s}}\|\chi_B\|^{-1}_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}$, where $U_0(B) := B$ and, for all $j \in \mathbb{N}$, $U_j(B) := (2^j B) \setminus (2^{j-1}B)$;

(ii) $\int_{\mathbb{R}^n} m(x)x^\beta dx = 0$ for all $\beta \in \mathbb{Z}^n_+$ with $|\beta| \leq d$.

Definition 9. Let $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $0 < p < \infty$, $\alpha \in (0,\infty)$, $s \in (1,\infty]$ and $\epsilon \in (0,\infty)$. The molecular weak Herz-type Hardy space with variable exponent $\mathbf{WHK}^{\alpha,p,\epsilon}_{q(\cdot),s,mol}(\mathbb{R}^n)$ is defined as

$$\mathbf{WHK}^{\alpha,p,\epsilon}_{q(\cdot),s,mol}(\mathbb{R}^n) := \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : f = \sum_{i\in\mathbb{Z}} \sum_{j\in\mathbb{N}} \lambda_{i,j} m_{i,j} \in \mathcal{S}'(\mathbb{R}^n) \right\},$$

where $\{m_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}$ is a sequence of $(\alpha, p, q(\cdot), s, \epsilon)$-molecules, associated with balls $\{B_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}$, satisfying that there exists a positive constant $c \in (0,1]$, such that, for all $x \in \mathbb{R}^n$ and $i \in \mathbb{Z}$, $\sum_{j\in\mathbb{N}} \chi_{cB_{i,j}(x)} \leq A$ with A being a positive constant independent of x and i and, for all $i \in \mathbb{Z}$ and $j \in \mathbb{N}$, $\lambda_{i,j} := \tilde{A}2^i\|\chi_{B_{i,j}}\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}$ with $\tilde{A}$ being a positive constant independent of i and j.

Moreover, define

$$\|f\|_{\mathbf{WHK}^{\alpha,p,\epsilon}_{q(\cdot),s,mol}(\mathbb{R}^n)} := \inf \left[\sup_{i\in\mathbb{Z}} \left\| \left(\sum_{j\in\mathbb{N}} \left[\frac{\lambda_{i,j}\chi_{B_{i,j}}}{\|\chi_{B_{i,j}}\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}} \right]^b \right)^{\frac{1}{b}} \right\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \right],$$

where $b \in (0,1)$ and the infimum is taken over all decompositions of f as above.

Theorem 6.1. *Let $q(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $1 < q_- \leq q_+ < \infty$, $0 < p < 1$, $\alpha \in (0,\infty)$, $s \in (\max\{q_+, \frac{bn\delta_1}{n\delta_1 - \alpha b}\}, \infty]$ and $\epsilon \in (n + d + 1, \infty)$. Then*

$$\|f\|_{\mathbf{WHK}^{\alpha,p,\epsilon}_{q(\cdot),s,mol}(\mathbb{R}^n)} \sim \|f\|_{\mathbf{WHK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.$$

Proof. Notice that an $(\alpha, p, q(\cdot), \infty)$-atom is also an $(\alpha, p, q(\cdot), s, \epsilon)$-molecule. Then by Theorem 5.1, we have

$$\mathbf{WHK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) \subset \mathbf{WHK}^{\alpha,p}_{q(\cdot),\infty,at}(\mathbb{R}^n) \subset \mathbf{WHK}^{\alpha,p,\epsilon}_{q(\cdot),s,mol}(\mathbb{R}^n).$$

Therefore, it suffices to show $\mathbf{WHK}^{\alpha,p,\epsilon}_{q(\cdot),s,mol}(\mathbb{R}^n) \subset \mathbf{WHK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$.
Let m be any fixed $(\alpha, p, q(\cdot), s, \epsilon)$-molecule associated with a ball $B := B(x_{B,r_B})$. We now claim that m is an infinite linear combination of $(\alpha, p, q(\cdot), s)$-atoms. To this end, for all $k \in \mathbb{Z}_+$, let $m_k := m\chi_{U_k(B)}$ with $U_k(B)$ as in Definition 8, and $\mathcal{P}_k$ be the linear vector space generated by the set $\{x^\gamma \chi_{U_k(B)}\}_{|\gamma|\leq d}$ of polynomials with d as in Definition 8. It is well known (see, for example, [30]) that, for any given $k \in \mathbb{Z}_+$, there exists a unique polynomial $P_k \in \mathcal{P}_k$ such that, for all multi-indices β with $|\beta| \leq d$,

$$\int_{\mathbb{R}^n} x^\beta [m_k(x) - P_k(x)]dx = 0, \tag{6.1}$$

where P_k is given by the following formula

$$P_k := \sum_{\beta\in\mathbb{Z}_+^n, |\beta|\le d} \left[\int_{\mathbb{R}^n} \frac{1}{|U_k(B)|} x^\beta m_k(x)dx\right] Q_{\beta,k} \tag{6.2}$$

and $Q_{\beta,k}$ is the unique polynomial in $\mathcal{P}_k$ satisfying that, for all multi-indices β with $|\beta| \le d$ and the Kronecker delta $\delta_{\gamma,\beta}$,

$$\int_{\mathbb{R}^n} x^\gamma Q_{\beta,k}(x)dx = |U_k(B)|\delta_{\gamma,\beta},$$

where, when $\gamma = \beta$, $\delta_{\gamma,\beta} := 1$ and, when $\gamma \ne \beta$, $\delta_{\gamma,\beta} := 0$.

It was proved in [30, p. 83] that, for all $k \in \mathbb{Z}_+$,

$$\sup_{x\in U_k(B)} |P_k(x)| \lesssim \frac{1}{|U_k(B)|}\|m_k\|_{L^1(\mathbb{R}^n)}.$$

From this and the Hölder inequality, we deduce that, for all $k \in \mathbb{Z}_+$,

$$\begin{aligned}\|m_k - P_k\|_{L^s(U_k(B))} &\le \|m_k\|_{L^s(U_k(B))} + \|P_k\|_{L^s(U_k(B))}\\ &\le \tilde{C}\|m_k\|_{L^s(U_k(B))}\\ &\le \tilde{C}2^{-k\epsilon}|2^kB|^{\frac{1}{s}}\|\chi_B\|^{-1}_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)},\end{aligned} \tag{6.3}$$

where $\tilde{C}$ is a positive constant independent of m, B and k. For all $k \in \mathbb{Z}_+$, let

$$\mu_k := \frac{\tilde{C}2^{-k\epsilon}\|\chi_{2^kB}\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}}$$

and

$$a_k := \frac{2^{k\epsilon}\|\chi_B\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}(m_k - P_k)}{\tilde{C}\|\chi_{2^kB}\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}}.$$

This, combined with (6.1), (6.3) and the fact that $\operatorname{supp}(m_k - P_k) \subset U_k(B)$, implies that, for each $k \in \mathbb{Z}_+$, a_k is an $(\alpha, p, q(\cdot), s)$-atom and

$$\sum_{k=0}^\infty (m_k - P_k) = \sum_{k=0}^\infty \mu_k a_k \tag{6.4}$$

is an infinite linear combination of $(\alpha, p, q(\cdot), s)$-atoms.

Moreover, for any $j \in \mathbb{Z}_+$ and $l \in \mathbb{Z}_+^n$, let

$$N_l^j := \sum_{k=j}^\infty \int_{U_k(B)} m_k(x)x^l dx.$$

Then, for any $l \in \mathbb{Z}_+^n$ with $|l| \le d$, it holds that

$$N_l^0 := \sum_{k=0}^\infty \int_{U_k(B)} m_k(x)x^l dx = \int_{\mathbb{R}^n} m(x)x^l dx = 0.$$

From this and (6.2), we deduce that

$$\begin{aligned}\sum_{k=0}^{\infty} P_k &= \sum_{l\in\mathbb{Z}_+^n,|l|\le d}\sum_{k=0}^{\infty} |U_k(B)|^{-1}Q_{l,k}\int_{\mathbb{R}^n} m_k(x)x^l dx\\ &= \sum_{l\in\mathbb{Z}_+^n,|l|\le d}\sum_{k=0}^{\infty} N_l^{k+1}[|U_{k+1}(B)|^{-1}Q_{l,k+1}\chi_{U_{k+1}(B)}(x) - |U_k(B)|^{-1}Q_{l,k}\chi_{U_k(B)}(x)]\\ &=: \sum_{l\in\mathbb{Z}_+^n,|l|\le d}\sum_{k=0}^{\infty} b_l^k.\end{aligned}$$

By an argument similar to that used in the proof of [15, (4.35)], we deduce that, for any $k \in \mathbb{Z}_+$ and $l \in \mathbb{Z}_+^n$ with $|l| \le d$,

$$\|b_l^k\|_{L^\infty(\mathbb{R}^n)} \lesssim 2^{-k\epsilon}\|\chi_B\|^{-1}_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \quad \text{and} \quad \operatorname{supp} b_l^k \subset 2^{k+1}B; \tag{6.5}$$

moreover, for any $\gamma \in \mathbb{Z}_+^n$ with $|\gamma| \le d$, $\int_{\mathbb{R}^n} b_l^k(x)x^\gamma dx = 0$. For all $k \in \mathbb{Z}_+$ and $l \in \mathbb{Z}_+^n$ with $|l| \le d$, let

$$\mu_l^k := 2^{-k\epsilon}\frac{\|\chi_{2^{k+1}B}\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}}$$

and

$$a_l^k := 2^{k\epsilon}b_l^k\frac{\|\chi_B\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_{2^{k+1}B}\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}}.$$

Then, for any $k \in \mathbb{Z}_+$ and $l \in \mathbb{Z}_+^n$ with $|l| \le d$, by (6.5) and the definition of a_l^k, we conclude that a_l^k is an $(\alpha, p, q(\cdot), \infty)$-atom supported on $2^{k+1}B$ up to a positive constant multiple. Therefore,

$$\sum_{k=0}^{\infty} P_k = \sum_{l\in\mathbb{Z}_+^n,|l|\le d}\sum_{k=0}^{\infty} \mu_l^k a_l^k \tag{6.6}$$

is an infinite linear combination of $(\alpha, p, q(\cdot), \infty)$-atoms.

Combining (6.4) and (6.6), we find that

$$m = \sum_{k=0}^{\infty} m_k = \sum_{k=0}^{\infty}(m_k - P_k) + \sum_{k=0}^{\infty} P_k = \sum_{k=0}^{\infty}\mu_k a_k + \sum_{l\in\mathbb{Z}_+^n,|l|\le d}\sum_{k=0}^{\infty}\mu_l^k a_l^k. \tag{6.7}$$

This shows that an $(\alpha, p, q(\cdot), s, \epsilon)$-molecule can be divided into an infinite linear combination of $(\alpha, p, q(\cdot), s)$-atoms.

To prove $f \in \mathbf{WHK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$, it suffices to show that, for any $\beta \in (0,\infty)$

$$\beta\|\chi_{\{x\in\mathbb{R}^n:|f^*(x)|>\beta\}}\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|_{\mathbf{WHK}^{\alpha,p,\epsilon}_{q(\cdot),s,mol}(\mathbb{R}^n)}. \tag{6.8}$$

To simplify the notation, let $f^* =: \mathcal{M}_N(f)$ with N as Theorem 4.7.

For any given $\beta \in (0,\infty)$, we choose $i_0 \in \mathbb{Z}$ such that $2^{i_0} \le \beta < 2^{i_0+1}$ and write

$$f = \sum_{i=-\infty}^{i_0-1} \sum_{j\in\mathbb{N}} \lambda_{i,j} m_{i,j} + \sum_{i=i_0}^{\infty} \sum_{j\in\mathbb{N}} \lambda_{i,j} m_{i,j} = f_1 + f_2.$$

Moreover, it holds true that

$$\begin{aligned} \|\chi_{\{x\in\mathbb{R}^n:|f^*(x)|>\beta\}}\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} &\lesssim \|\chi_{\{x\in\mathbb{R}^n:|f_1^*(x)|>\frac{\beta}{2}\}}\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \\ &\quad + \|\chi_{\{x\in\mathbb{R}^n:|f_2^*(x)|>\frac{\beta}{2}\}}\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \\ &= I_1 + I_2. \end{aligned}$$

We first estimate I_1. To this end, we need another estimate for $(m_{i,j})^*$. From (6.7), we deduce that, for all $i \in \mathbb{Z}$ and $j \in \mathbb{N}$, there exists a sequences of multiples of $(\alpha, p, q(\cdot), s)$-atoms, $\{a^l_{i,j}\}_{l\in\mathbb{Z}_+}$, associated with balls $\{2^{l+1}B_{i,j}\}_{l\in\mathbb{Z}_+}$ such that

$$\|a^l_{i,j}\|_{L^s(\mathbb{R}^n)} \lesssim \frac{2^{-l\epsilon}|2^{l+1}B_{i,j}|^{\frac{1}{s}}}{\|\chi_{B_{i,j}}\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}}$$

and $m_{i,j} = \sum_{l\in\mathbb{Z}_+} a^l_{i,j}$ almost everywhere in $\mathbb{R}^n$. Then, for all $i \in \mathbb{Z}\cap(-\infty, i_0-1]$ and $j \in \mathbb{N}$, we have

$$\begin{aligned} (m_{i,j})^* \le \sum_{l\in\mathbb{Z}_+} (a^l_{i,j})^* &= \sum_{l\in\mathbb{Z}_+} \sum_{k\in\mathbb{Z}_+} (a^l_{i,j})^* \chi_{U_k(2^l B_{i,j})} \\ &=: \sum_{l\in\mathbb{Z}_+} \sum_{k=0}^{2} J_{l,k} + \sum_{l\in\mathbb{Z}_+} \sum_{k=3}^{\infty} J_{l,k}, \end{aligned} \tag{6.9}$$

where $U_k(2^l B_{i,j})$ is defined as Definition 8 with B replaced by $2^l B_{i,j}$. Thus, it follows that

$$\begin{aligned} &\|\chi_{\{x\in\mathbb{R}^n:|f_1^*(x)|>\frac{\beta}{2}\}}\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \\ &\le \|\chi_{\{x\in\mathbb{R}^n:\sum_{i=-\infty}^{i_0-1}\sum_{j\in\mathbb{N}}\lambda_{i,j}(m_{i,j})^*(x)>\frac{\beta}{2}\}}\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \\ &\lesssim \|\chi_{\{x\in\mathbb{R}^n:\sum_{i=-\infty}^{i_0-1}\sum_{j\in\mathbb{N}}\sum_{l\in\mathbb{Z}_+}\sum_{k=0}^{2}\lambda_{i,j}J_{l,k}>\frac{\beta}{4}\}}\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \\ &\quad + \|\chi_{\{x\in\mathbb{R}^n:\sum_{i=-\infty}^{i_0-1}\sum_{j\in\mathbb{N}}\sum_{l\in\mathbb{Z}_+}\sum_{k=3}^{\infty}\lambda_{i,j}J_{l,k}>\frac{\beta}{4}\}}\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \\ &= I_{1,1} + I_{1,2}. \end{aligned}$$

For $I_{1,1}$, by an argument similar to that used in the proof of (5.2), we deduce that

$$\beta I_{1,1} \lesssim \|f\|_{\mathbf{WHK}^{\alpha,p,\epsilon}_{q(\cdot),s,mol}(\mathbb{R}^n)}. \tag{6.10}$$

On the other hand, by an argument similar to that used in the proof of [32, (5.17)], we deduce that, for any $i \in \mathbb{Z}$, $j \in \mathbb{N}$, $l \in \mathbb{Z}_+$, $k \in [3,\infty)\cap\mathbb{Z}_+$,

$x \in U_k(2^l B_{i,j})$ and $y \in 2^{l+1} B_{i,j}$,

$$J_{l,k} \lesssim \frac{2^{-l\epsilon - k(n+d+1)}}{\|\chi_{B_{i,j}}\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}} \chi_{U_k(2^l B_{i,j})}(x), \tag{6.11}$$

which, combined with (6.9), Remark 2.8, the fact that d as in Definition 8 and via choosing $p < r < 1$ and $r_1 \in (\frac{n}{n+d+1}, \min\{1, \frac{n\delta_2}{\alpha r}\})$, implies that

$$\begin{aligned}
\beta^p I_{1,2}^p &\lesssim \beta^{(1-\frac{1}{r})p} \Big\| \sum_{i=-\infty}^{i_0-1} \sum_{j\in\mathbb{N}} \sum_{l\in\mathbb{Z}_+} \sum_{k=3}^{\infty} 2^i 2^{-l\epsilon} 2^{-k(n+d+1)} \chi_{U_k(2^l B_{i,j})} \Big\|^{\frac{p}{r}}_{\mathbf{K}^{\alpha r, \frac{p}{r}}_{\frac{q(\cdot)}{r}}(\mathbb{R}^n)} \\
&\lesssim \beta^{(1-\frac{1}{r})p} \sum_{l\in\mathbb{Z}_+} \sum_{k=3}^{\infty} 2^{-l\epsilon\frac{p}{r}} 2^{-k(n+d+1)\frac{p}{r}} \sum_{i=-\infty}^{i_0-1} 2^{i\frac{p}{r}} \Big\| \sum_{j\in\mathbb{N}} \chi_{U_k(2^l B_{i,j})} \Big\|^{\frac{p}{r}}_{\mathbf{K}^{\alpha r, \frac{p}{r}}_{\frac{q(\cdot)}{r}}(\mathbb{R}^n)} \\
&\lesssim \beta^{(1-\frac{1}{r})p} \sum_{l\in\mathbb{Z}_+} \sum_{k=3}^{\infty} 2^{-l\epsilon\frac{p}{r}} 2^{-k(n+d+1)\frac{p}{r}} 2^{\frac{n(k+l)}{r_1}\frac{p}{r}} \sum_{i=-\infty}^{i_0-1} 2^{i\frac{p}{r}} \Big\| \sum_{j\in\mathbb{N}} \chi_{B_{i,j}} \Big\|^{\frac{p}{r}}_{\mathbf{K}^{\alpha r, \frac{p}{r}}_{\frac{q(\cdot)}{r}}(\mathbb{R}^n)} \\
&\lesssim \beta^{(1-\frac{1}{r})p} \sup_{i\in\mathbb{Z}} 2^{ip} \Big\| \sum_{j\in\mathbb{N}} \chi_{B_{i,j}} \Big\|^{\frac{p}{r}}_{\mathbf{K}^{\alpha r, \frac{p}{r}}_{\frac{q(\cdot)}{r}}(\mathbb{R}^n)} \sum_{i=-\infty}^{i_0-1} 2^{ip(\frac{1}{r}-1)} \\
&\lesssim \beta^{(1-\frac{1}{r})p} \sup_{i\in\mathbb{Z}} 2^{ip} \Big\| \sum_{j\in\mathbb{N}} \chi_{B_{i,j}} \Big\|^{\frac{p}{r}}_{\mathbf{K}^{\alpha r, \frac{p}{r}}_{\frac{q(\cdot)}{r}}(\mathbb{R}^n)} \sim \|f\|^p_{\mathbf{WHK}^{\alpha,p,\epsilon}_{q(\cdot),s,mol}(\mathbb{R}^n)},
\end{aligned}$$

that is

$$\beta I_{1,2} \lesssim \|f\|_{\mathbf{WHK}^{\alpha,p,\epsilon}_{q(\cdot),s,mol}(\mathbb{R}^n)}. \tag{6.12}$$

Combining (6.10) and (6.12), we deduce that

$$\beta I_1 \lesssim \|f\|_{\mathbf{WHK}^{\alpha,p,\epsilon}_{q(\cdot),s,mol}(\mathbb{R}^n)}. \tag{6.13}$$

We next estimate I_2. By (6.9), we know that

$$\begin{aligned}
I_2 &\lesssim \|\chi_{\{x\in\mathbb{R}^n : \sum_{i=i_0}^{\infty} \sum_{j\in\mathbb{N}} \sum_{l\in\mathbb{Z}_+} \sum_{k=0}^{2} \lambda_{i,j} J_{l,k} > \frac{\beta}{4}\}}\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \\
&\quad + \|\chi_{\{x\in\mathbb{R}^n : \sum_{i=i_0}^{\infty} \sum_{j\in\mathbb{N}} \sum_{l\in\mathbb{Z}_+} \sum_{k=3}^{\infty} \lambda_{i,j} J_{l,k} > \frac{\beta}{4}\}}\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \\
&= I_{2,1} + I_{2,2}.
\end{aligned}$$

For any $b \in (0,p)$, let $\tilde{q} \in (1, \min\{\frac{s}{\max\{q_+, \frac{bn\delta_1}{n\delta_1 - \alpha b}\}}, \frac{1}{b}\})$ and $a \in (1 - \frac{1}{q}, \infty)$. Then by similar argument to that used in the proof of (6.10), we obtain

$$\begin{aligned}
I_{2,1}^p &\lesssim 2^{-i_0\tilde{q}(1-a)p} \sum_{i=-\infty}^{i_0-1} 2^{[(1-a)\tilde{q}-1]ip} \sup_{i\in\mathbb{Z}} 2^{ip} \Big\| \Big(\sum_{j\in\mathbb{N}} \chi_{B_{i,j}} \Big)^{\frac{1}{b}} \Big\|^p_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \\
&\lesssim \beta^{-p} \|f\|^p_{\mathbf{WHK}^{\alpha,p}_{q(\cdot),s,mol}(\mathbb{R}^n)},
\end{aligned}$$

that is

$$\beta I_{2,1} \lesssim \|f\|_{\mathbf{WHK}^{\alpha,p,\epsilon}_{q(\cdot),s,mol}(\mathbb{R}^n)}. \tag{6.14}$$

On the other hand, let $1 < q < \frac{1}{p}$ and $a \in (1 - \frac{1}{q}, \infty)$. By the Hölder inequality, we find that, for all $x \in \mathbb{R}^n$,

$$\begin{aligned}\sum_{i=0}^{\infty}\sum_{j\in\mathbb{N}}\sum_{l\in\mathbb{Z}_+}\sum_{k=3}^{\infty}\lambda_{i,j}J_{l,k} &\le \left(\sum_{i=0}^{\infty}2^{iaq'}\right)^{\frac{1}{q'}}\left\{\sum_{i=0}^{\infty}2^{-iaq}\left[\sum_{j\in\mathbb{N}}\sum_{l\in\mathbb{Z}_+}\sum_{k=3}^{\infty}\lambda_{i,j}J_{l,k}\right]^q\right\}^{\frac{1}{q}}\\ &= \frac{2^{i_0a}}{(2^{aq'}-1)^{\frac{1}{q'}}}\left\{\sum_{i=0}^{\infty}2^{-iaq}\left[\sum_{j\in\mathbb{N}}\sum_{l\in\mathbb{Z}_+}\sum_{k=3}^{\infty}\lambda_{i,j}J_{l,k}\right]^q\right\}^{\frac{1}{q}}.\end{aligned}$$

By the fact that d as in Definition 8, we choose $r \in (\frac{n}{n+d+1}, \min\{1, \frac{n\delta_2 q}{\alpha}\})$. Then by (6.11) and Remark 2.8, we find that

$$\begin{aligned}I_{2,2}^p &\lesssim \left\|\chi_{\{x\in\mathbb{R}^n:\frac{2^{i_0a}}{(2^{aq'}-1)^{\frac{1}{q'}}}[\sum_{i=i_0}^{\infty}2^{-iaq}\{\sum_{j\in\mathbb{N}}\sum_{l\in\mathbb{Z}_+}\sum_{k=3}^{\infty}\lambda_{i,j}J_{l,k}\}^q]^{\frac{1}{q}}>2^{i_0-2}\}}\right\|^p_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}\\
&\lesssim 2^{-i_0q(1-a)p}\left\|\sum_{i=i_0}^{\infty}2^{-iaq}\left[\sum_{j\in\mathbb{N}}\sum_{l\in\mathbb{Z}_+}\sum_{k=3}^{\infty}\lambda_{i,j}J_{l,k}\right]^q\right\|^p_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}\\
&\lesssim 2^{-i_0q(1-a)p}\left\|\sum_{i=i_0}^{\infty}2^{i(1-a)}\sum_{l\in\mathbb{Z}_+}\sum_{k=3}^{\infty}2^{-l\epsilon}2^{-k(n+d+1)}\sum_{j\in\mathbb{N}}\chi_{U_k(2^lB_{i,j})}\right\|^{pq}_{\mathbf{K}^{\frac{\alpha}{q},pq}_{q(\cdot)q}(\mathbb{R}^n)}\\
&\lesssim 2^{-i_0q(1-a)p}\sum_{i=i_0}^{\infty}2^{i(1-a)pq}\sum_{l\in\mathbb{Z}_+}\sum_{k=3}^{\infty}2^{-l\epsilon pq}2^{-k(n+d+1)pq}\left\|\sum_{j\in\mathbb{N}}\chi_{U_k(2^lB_{i,j})}\right\|^{pq}_{\mathbf{K}^{\frac{\alpha}{q},pq}_{q(\cdot)q}(\mathbb{R}^n)}\\
&\lesssim 2^{-i_0q(1-a)p}\sum_{i=i_0}^{\infty}2^{i(1-a)pq}\sum_{l\in\mathbb{Z}_+}\sum_{k=3}^{\infty}2^{-l\epsilon pq}2^{-k(n+d+1)pq}2^{\frac{n(k+l)}{r}pq}\left\|\sum_{j\in\mathbb{N}}\chi_{B_{i,j}}\right\|^{pq}_{\mathbf{K}^{\frac{\alpha}{q},pq}_{q(\cdot)q}(\mathbb{R}^n)}\\
&\lesssim 2^{-i_0q(1-a)p}\sum_{i=i_0}^{\infty}2^{i(1-a)pq}\left\|\sum_{j\in\mathbb{N}}\chi_{B_{i,j}}\right\|^{pq}_{\mathbf{K}^{\frac{\alpha}{q},pq}_{q(\cdot)q}(\mathbb{R}^n)}\\
&\lesssim 2^{-i_0q(1-a)p}\sum_{i=i_0}^{\infty}2^{i[(1-a)pq-p]}\sup_{i\in\mathbb{Z}}2^{ip}\left\|\sum_{j\in\mathbb{N}}\chi_{B_{i,j}}\right\|^{p}_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}\\
&\lesssim \beta^{-p}\|f\|^p_{\mathbf{WHK}^{\alpha,p}_{q(\cdot),s,at}(\mathbb{R}^n)},\end{aligned}$$

that is

$$\beta I_{2,2} \lesssim \|f\|_{\mathbf{WHK}^{\alpha,p,\epsilon}_{q(\cdot),s,mol}(\mathbb{R}^n)}. \tag{6.15}$$

Combining (6.14) and (6.15), implies that $\beta I_2 \lesssim \|f\|_{\mathbf{WHK}^{\alpha,p,\epsilon}_{q(\cdot),s,mol}(\mathbb{R}^n)}$. This, together with (6.13), shows that (6.8) holds true and this finishes the proof. □

Remark 6.2. The space $\mathbf{WH\dot{K}}^{\alpha,p,\epsilon}_{q(\cdot),s,mol}(\mathbb{R}^n)$ is defined by the same way as in Definition 9 via replacing the norm of $\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ by the norm of $\dot{\mathbf{K}}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ and all the results in this section are also valid on this space.

7. Littlewood-Paley function characterizations

In this section, as an application of the atomic decomposition, we provide several equivalent characterizations of the weak Herz-type Hardy spaces with variable exponents via the Lusin area function, the Littlewood-Paley g-function and the Littlewood-Paley g^*_λ-function.

Let d be as in Definition 6 and $\phi \in \mathcal{S}(\mathbb{R}^n)$ be a radial function satisfying

$$supp\phi \subset \{x \in \mathbb{R}^n : |x| \leq 1\}, \tag{7.1}$$

$$\int_{\mathbb{R}^n} \phi(x) x^\beta dx = 0 \quad \text{for all } \beta \in \mathbb{Z}^n_+ \text{ with } |\beta| \leq d \tag{7.2}$$

and

$$\int_0^\infty |\hat{\phi}(\varepsilon t)|^2 \frac{dt}{t} = 1 \quad \text{for all } \varepsilon \in \mathbb{R}^n \setminus \{0\}. \tag{7.3}$$

For all $f \in \mathcal{S}'(\mathbb{R}^n)$, the Littlewood-Paley g-function, the Lusin area function and the Littlewood-Paley g^*_λ-function with $\lambda \in (0, \infty)$ are defined, respectively, by setting, for all $x \in \mathbb{R}^n$,

$$g(f)(x) = \left(\int_0^\infty |f * \phi_t(x)|^2 \frac{dt}{t} \right)^{\frac{1}{2}},$$

$$S(f)(x) = \left(\int_{\Gamma(x)} |f * \phi_t(x)|^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}}$$

and

$$g^*_\lambda(f)(x) = \left(\int_0^\infty \int_{\mathbb{R}^n} (\frac{t}{t + |x-y|})^{\lambda n} |f * \phi_t(y)|^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}},$$

where, for any $x \in \mathbb{R}^n$, $\Gamma(x) = \{(y,t) \in \mathbb{R}^n \times (0,\infty) : |y - x| < t\}$ and, for any $t \in (0, \infty)$, $\phi_t(\cdot) = t^{-n}\phi(\cdot t^{-1})$.

For all $t, a \in (0, \infty)$ and $x \in \mathbb{R}^n$, let

$$(\phi^*_t f)_a(x) = \sup_{y \in \mathbb{R}^n} \frac{|\phi_t * f(x+y)|}{(1 + \frac{|y|}{t})^a}.$$

Then, we define

$$g_{a,*}(f)(x) = \left(\int_0^\infty [(\phi^*_t f)_a(x)]^2 \frac{dt}{t} \right)^{\frac{1}{2}}.$$

Recall that $f \in \mathcal{S}'(\mathbb{R}^n)$ is said to vanish weakly at infinity if, for every $\phi \in \mathcal{S}(\mathbb{R}^n)$, $f * \phi_t \to 0$ in $\mathcal{S}'(\mathbb{R}^n)$ as $t \to \infty$ (see [11, p. 50]).

Lemma 7.1. *Let* $q(\cdot) \in C^{\log}(\mathbb{R}^n)$ *with* $1 < q_- \leq q_+ < \infty$, $0 < p < 1$ *and* $\alpha \in (0, \infty)$. *If* $f \in \mathbf{WHK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$, *then* f *vanishes weakly at infinity.*

Proof. Let $f \in \mathbf{WHK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$. For $\phi \in \mathcal{S}(\mathbb{R}^n)$, $x \in \mathbb{R}^n$, $t \in (0,\infty)$ and $N \in (\frac{n}{q_-} + n + 1, \infty)$, it follows from [34, p. 1553], Remark 4.6 and Remark 2.7 that

$$\begin{aligned}&\min\{|f * \phi_t(x)|^{q_-}, |f * \phi_t(x)|^{q_+}\}\\ &\lesssim \frac{1}{B(x,t)} \max\{\|\mathcal{M}_N f\|^{q_-}_{L^{q(\cdot)}(\mathbb{R}^n)}, \|\mathcal{M}_N f\|^{q_+}_{L^{q(\cdot)}(\mathbb{R}^n)}\}\\ &\lesssim \frac{1}{B(x,t)} \max\{\|\mathcal{M}_N f\|^{q_-}_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}, \|\mathcal{M}_N f\|^{q_+}_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}\} \to 0\end{aligned}$$

as $t \to \infty$, which implies that f vanishes weakly at infinity. This finishes the proof. □

Theorem 7.2. *Let $q(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $1 < q_- \le q_+ < \infty$, $0 < p < 1$ and $\alpha \in (0,\infty)$. Then $f \in \mathbf{WHK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'(\mathbb{R}^n)$, f vanishes weakly at infinity and $S(f) \in \mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$. Moreover, for all $f \in \mathbf{WHK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$,*

$$\|S(f)\|_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \sim \|f\|_{\mathbf{WHK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.$$

Proof. For $f \in \mathcal{S}'(\mathbb{R}^n)$ such that f vanishes weakly at infinity and $S(f) \in \mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$, we need to prove that $f \in \mathbf{WHK}^{\alpha,p}_{q(\cdot),s,at}(\mathbb{R}^n)$ for some s and d as in Theorem 5.1 and

$$\|f\|_{\mathbf{WHK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \sim \|f\|_{\mathbf{WHK}^{\alpha,p}_{q(\cdot),s,at}(\mathbb{R}^n)} \lesssim \|S(f)\|_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.$$

Denote by $\mathcal{Q}$ the set of all dyadic cubes in $\mathbb{R}^n$. For any $i \in \mathbb{Z}$, let

$$\Omega_i = \{x \in \mathbb{R}^n : S(f)(x) > 2^i\}$$

and

$$\mathcal{Q}_i = \left\{Q \in \mathcal{Q} : |Q \cap \Omega_i| \ge \frac{|Q|}{2} \text{ and } |Q \cap \Omega_{i+1}| < \frac{|Q|}{2}\right\}.$$

For all $i \in \mathbb{Z}$, we use $\{Q_{i,j}\}_j$ to denote the maximal dyadic cubes in $\mathcal{Q}_i$, namely, there does not exist $Q \in \mathcal{Q}_i$ such that $Q_{i,j} \subsetneqq Q$. For any $Q \in \mathcal{Q}$, let l_Q denote its side length and

$$Q^+ = \{(y,t) \in \mathbb{R}^{n+1}_+, y \in Q, \sqrt{n} l_Q < t \le 2\sqrt{n} l_Q\}$$

and, for all $i \in \mathbb{Z}$ and j, let

$$B_{i,j} = \cup_{Q \in \mathcal{Q}_i, Q \subset Q_{i,j}} Q^+.$$

Here we point out that Q^+ for different $Q \in \mathcal{Q}_i$ and $Q \subset Q_{i,j}$ are mutually disjoint. For $i \in \mathbb{Z}$, j and $x \in \mathbb{R}^n$, set

$$\lambda_{i,j} = 2^i \|\chi_{4\sqrt{n}Q_{i,j}}\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}$$

and

$$a_{i,j}(x) = \frac{1}{\lambda_{i,j}} \sum_{Q \in \mathcal{Q}_i, Q \subset Q_{i,j}} \int_{Q^+} f * \phi_t(y) \phi_t(x-y) \frac{dydt}{t},$$

where ϕ is as in (7.1), (7.2) and (7.3). It follows from [22, Theorem 4.5] and [20, (8.5)] that

$$f=\sum_{i,j}\lambda_{i,j}a_{i,j} \text{ in } \mathcal{S}'(\mathbb{R}^n).$$

By [32, Theorem 6.1], we find that for any $i\in\mathbb{Z}$ and j,

$$suppa_{i,j}\subset \tilde{Q}_{i,j}:=4\sqrt{n}Q_{i,j};$$

$$\int_{\mathbb{R}^n} a_{i,j}(x)x^{\beta}dx=0,\ |\beta|\leq d;$$

and

$$\|a_{i,j}\|_{L^s(\mathbb{R}^n)}\lesssim \frac{|\tilde{Q}_{i,j}|^{\frac{1}{s}}}{\|\chi_{\tilde{Q}_{i,j}}\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}}.$$

This implies that $a_{i,j}$ is an $(\alpha,p,q(\cdot),s)$-atom up to a harmless constant multiple. Moreover, by Remark 2.8, $|Q_{i,j}\cap\Omega_i|\geq \frac{|Q_{i,j}|}{2}$ and the fact that $\{Q_{i,j}\}_j$ have disjoint interiors, we find that, for any $i\in\mathbb{Z}$,

$$\begin{aligned}
\left\|\left(\sum_j\left(\frac{\lambda_{i,j}\chi_{\tilde{Q}_{i,j}}}{\|\chi_{\tilde{Q}_{i,j}}\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}}\right)^b\right)^{\frac{1}{b}}\right\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} &\lesssim 2^i\left\|\left(\sum_j\chi_{\tilde{Q}_{i,j}}\right)^{\frac{1}{b}}\right\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}\\
&\lesssim 2^i\left\|\left(\sum_j\chi_{Q_{i,j}}\right)^{\frac{1}{b}}\right\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}\\
&\lesssim 2^i\left\|\left(\sum_j\chi_{Q_{i,j}\cap\Omega_i}\right)^{\frac{1}{b}}\right\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}\\
&\lesssim 2^i\|\chi_{\Omega_i}\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}\\
&\lesssim \|S(f)\|_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Conversely, take $f\in\mathbf{WHK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$. Obviously, by Lemma 7.1, we know that f vanishes weakly at infinity. Due to Theorem 5.1, we can decompose f as follows

$$f=\sum_{i=-\infty}^{i_0-1}\sum_{j\in\mathbb{N}}\lambda_{i,j}a_{i,j}+\sum_{i=i_0}^{\infty}\sum_{j\in\mathbb{N}}\lambda_{i,j}a_{i,j}=f_1+f_2,$$

where $\{\lambda_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}$ and $\{a_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}$ are as in Theorem 5.1. Thus, we obtain

$$\begin{aligned}
\|\chi_{\{x\in\mathbb{R}^n:S(f)(x)>\beta\}}\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} &\lesssim \|\chi_{\{x\in\mathbb{R}^n:S(f_1)(x)>\frac{\beta}{2}\}}\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}\\
&\quad+\|\chi_{\{x\in A_{i_0}:S(f_2)(x)>\frac{\beta}{2}\}}\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}\\
&\quad+\|\chi_{\{x\in (A_{i_0})^c:S(f_2)(x)>\frac{\beta}{2}\}}\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}\\
&=I_1+I_2+I_3,
\end{aligned}$$

where $A_{i_0}:=\cup_{i=i_0}^{\infty}\cup_{j\in\mathbb{N}}(4B_{i,j})$ and $\{B_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}$ are the balls as in Theorem 5.1.

For I_1, it is easy to see that

$$\begin{aligned} I_1 &\lesssim \|\chi_{\{x\in\mathbb{R}^n:\sum_{i=-\infty}^{i_0-1}\sum_{j\in\mathbb{N}}\lambda_{i,j}S(a_{i,j})(x)\chi_{4B_{i,j}}(x)>\frac{\beta}{4}\}}\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \\ &\quad + \|\chi_{\{x\in\mathbb{R}^n:\sum_{i=-\infty}^{i_0-1}\sum_{j\in\mathbb{N}}\lambda_{i,j}S(a_{i,j})(x)\chi_{(4B_{i,j})^c}(x)>\frac{\beta}{4}\}}\|_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \\ &= I_{1,1} + I_{1,2}. \end{aligned}$$

For $I_{1,1}$, by the boundedness of S on $L^r(\mathbb{R}^n)$ $(1 < r < \infty)$, Lemma 5.2, Remark 2.8 and an argument similar to that used in the proof of (5.2), we conclude that

$$\beta I_{1,1} \lesssim \|f\|_{\mathbf{WHK}^{\alpha,p}_{q(\cdot),s,at}(\mathbb{R}^n)}. \tag{7.4}$$

On the other hand, by an argument similar to that used in the proof of [32, (6.15)], we deduce that, for any $i \in \mathbb{Z}, j \in \mathbb{N}$, and $x \in (4B_{i,j})^c$,

$$|S(a_{i,j})(x)| \lesssim (M\chi_{B_{i,j}}(x))^{\frac{n+d+1}{n}} \|\chi_{B_{i,j}}\|^{-1}_{\mathbf{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.$$

From this, the Hölder inequality, Lemma 2.3, Lemma 2.5 and an argument similar to that used in the proof of (5.4), we deduce that $\beta I_{1,2} \lesssim \|f\|_{\mathbf{WHK}^{\alpha,p}_{q(\cdot),s,at}(\mathbb{R}^n)}$. Combining this with (7.4), we conclude that

$$\beta I_1 \lesssim \|f\|_{\mathbf{WHK}^{\alpha,p}_{q(\cdot),s,at}(\mathbb{R}^n)}. \tag{7.5}$$

By an argument similar to that used in the proof of (5.6), we also find that

$$\beta I_2 \lesssim \|f\|_{\mathbf{WHK}^{\alpha,p}_{q(\cdot),s,at}(\mathbb{R}^n)}. \tag{7.6}$$

Finally, for I_3, by Lemma 2.3, Lemma 2.5 and an argument similar to that used in the proof of (5.7), we deduce that

$$\beta I_3 \lesssim \|f\|_{\mathbf{WHK}^{\alpha,p}_{q(\cdot),s,at}(\mathbb{R}^n)}. \tag{7.7}$$

Combining (7.5), (7.6) and (7.7), we conclude that

$$\|S(f)\|_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|_{\mathbf{WHK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}, \tag{7.8}$$

which completes the proof. □

Theorem 7.3. *Let $q(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $1 < q_- \le q_+ < \infty$, $0 < p < 1$ and $\alpha \in (0, \infty)$. Then $f \in \mathbf{WHK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'(\mathbb{R}^n)$, f vanishes weakly at infinity and $g(f) \in \mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$. Moreover, for all $f \in \mathbf{WHK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$,*

$$\|g(f)\|_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \sim \|f\|_{\mathbf{WHK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.$$

Proof. Let us prove first that $\|g(f)\|_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \sim \|g_{a,*}(f)\|_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}$.

By definition we easily see that $g(f) \lesssim g_{a,*}(f)$, then

$$\|g(f)\|_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \lesssim \|g_{a,*}(f)\|_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.$$

Conversely, by choosing $a \in (\frac{n}{\min\{2,q_-\}}, \infty)$, it follows that there exists $r \in (0, \min\{2, q_-\})$ such that $a \in (\frac{n}{r}, \infty)$. Choose N_0 sufficiently large, by the estimate in [20, 32], we find that

$$g_{a,*}(f)(x) \leq \left\{ \sum_{j\in\mathbb{Z}} \left[\sum_{k=0} 2^{-kN_0 r} 2^{(k+j)n} \int_{\mathbb{R}^n} \frac{[\int_1^2 |(\phi_{2^{-(k+j)}})_t * f(y)|^2 \frac{dt}{t}]^{\frac{r}{2}}}{(1+2^j|x-y|)^{ar}} dy \right]^{\frac{2}{r}} \right\}^{\frac{1}{2}}.$$

This, together with the Minkowski series inequality, Remark 3.2 and Lemma 3.3, implies that

$$\begin{aligned}
&\|g_{a,*}(f)\|^{vr}_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \\
&\lesssim \left\| \sum_k 2^{-k(N_0 r - n)} \left(\sum_j 2^{j\frac{2n}{r}} \left[\int_{\mathbb{R}^n} \frac{[\int_1^2 |(\phi_{2^{-(k+j)}})_t * f(y)|^2 \frac{dt}{t}]^{\frac{r}{2}}}{(1+2^j|\cdot-y|)^{ar}} dy \right]^{\frac{2}{r}} \right)^{\frac{r}{2}} \right\|^{v}_{\mathbf{WK}^{\alpha r,\frac{p}{r}}_{\frac{q(\cdot)}{r}}(\mathbb{R}^n)} \\
&\lesssim \sum_k 2^{-kv(N_0 r - n)} \left\| \left(\sum_j 2^{j\frac{2n}{r}} \left[\int_{\mathbb{R}^n} \frac{[\int_1^2 |(\phi_{2^{-(k+j)}})_t * f(y)|^2 \frac{dt}{t}]^{\frac{r}{2}}}{(1+2^j|\cdot-y|)^{ar}} dy \right]^{\frac{2}{r}} \right)^{\frac{r}{2}} \right\|^{v}_{\mathbf{WK}^{\alpha r,\frac{p}{r}}_{\frac{q(\cdot)}{r}}(\mathbb{R}^n)} \\
&\lesssim \sum_k 2^{-kv(N_0 r - n)} \left\| \left\{ \sum_j 2^{j\frac{2n}{r}} \left(\sum_i 2^{-iar} \times \int_{|\cdot-y|\sim 2^{i-j}} \left[\int_1^2 |(\phi_{2^{-(k+j)}})_t * f(y)|^2 \frac{dt}{t} \right]^{\frac{r}{2}} dy \right)^{\frac{2}{r}} \right\}^{\frac{1}{2}} \right\|^{vr}_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)},
\end{aligned}$$

where $|\cdot-y| \sim 2^{i-j}$ means that $|x-y| < 2^{-j}$ if $i = 0$, or $2^{i-j-1} \leq |x-y| < 2^{i-j}$ if $i \in \mathbb{N}$. Applying Minkowski's inequality, Remark 3.2 and Proposition 4.3, we find

$$\begin{aligned}
&\|g_{a,*}(f)\|^{vr}_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \\
&\lesssim \sum_{k=0}^{\infty} 2^{-kv(N_0 r - n)} \sum_{i=0}^{\infty} 2^{(-iar+in)v} \left\| \left\{ \sum_j \left[M\left(\left[\int_1^2 |(\phi_{2^{-(k+j)}})_t * f(y)|^2 \frac{dt}{t} \right]^{\frac{r}{2}} \right) \right]^{\frac{2}{r}} \right\}^{\frac{r}{2}} \right\|^{v}_{\mathbf{WK}^{\alpha r,\frac{p}{r}}_{\frac{q(\cdot)}{r}}(\mathbb{R}^n)} \\
&\lesssim \sum_{k=0}^{\infty} 2^{-kv(N_0 r - n)} \sum_{i=0}^{\infty} 2^{(-iar+in)v} \left\| \left\{ \sum_j \left[\int_1^2 |(\phi_{2^{-(k+j)}})_t * f(y)|^2 \frac{dt}{t} \right] \right\}^{\frac{1}{2}} \right\|^{vr}_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \\
&\lesssim \|g(f)\|^{vr}_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Then

$$\|g(f)\|_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \sim \|g_{a,*}(f)\|_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}. \tag{7.9}$$

Now, let $f \in \mathcal{S}'(\mathbb{R}^n)$ vanishes weakly at infinity. It is easy see that, for any $a \in (0, \infty)$ and $x \in \mathbb{R}^n$, $S(f)(x) \lesssim g_{a,*}(f)(x)$ (see [34, p. 1557]), then by Theorem 7.2 and (7.9), we have

$$\|f\|_{\mathbf{WHK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \|S(f)\|_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \lesssim \|g_{a,*}(f)\|_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \lesssim \|g(f)\|_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.$$

Conversely, by an argument similar to that used in the proof of (7.8), we find that

$$\|g(f)\|_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|_{\mathbf{WHK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.$$

This finishes the proof. □

Theorem 7.4. *Let $q(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $1 < q_- \le q_+ < \infty$, $0 < p < 1$, $\alpha \in (0,\infty)$ and $\lambda \in (1 + \frac{2}{\min\{2,q_-\}}, \infty)$. Then $f \in \mathbf{WHK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'(\mathbb{R}^n)$, f vanishes weakly at infinity and $g^*_\lambda(f) \in \mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$. Moreover, for all $f \in \mathbf{WHK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$,*

$$\|g^*_\lambda(f)\|_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \sim \|f\|_{\mathbf{WHK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.$$

Proof. It is easy to see that, for all $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\lambda \in (1,\infty)$ and $x \in \mathbb{R}^n$, $S(f)(x) \lesssim g^*_\lambda(f)(x)$. By this and Theorem 7.2, we have

$$\|f\|_{\mathbf{WHK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \lesssim \|g^*_\lambda(f)\|_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.$$

Conversely, take $f \in \mathbf{WHK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$. It follows from Lemma 7.1 that f vanishes weakly at infinity. By the fact that $\lambda \in (1 + \frac{2}{\min\{2,q_-\}}, \infty)$, we see that there exists $a \in (\frac{n}{\min\{2,q_-\}}, \infty)$ such that, $\lambda \in (1 + \frac{2a}{n}, \infty)$. By this and the proof of [32, Theorem 6.3] (see also [20, Theorem 8.3]), we have $g^*_\lambda(f)(x) \lesssim g_{a,*}(f)(x)$. Then by Theorem 7.3 and (7.9), we obtain

$$\|g^*_\lambda(f)\|_{\mathbf{WK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|_{\mathbf{WHK}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.$$

The proof is complete. □

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