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LENS SPACES ADMITTING MINIMAL SYMPLECTIC FILLINGS WITH THE SECOND BETTI NUMBER ONE

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Abstract. We classify lens spaces with the Milnor fillable contact structure that admit minimal symplectic fillings whose second Betti numbers are one.

1. Introduction

A lens space $L(n,a)$ has its standard contact structure ξ_{st} called the *Milnor fillable contact structure*. A *symplectic filling* of $L(n,a)$ is a symplectic 4-manifold (W,ω) with the boundary $\partial W = L(n,a)$ satisfying the compatibility condition $\omega = d\xi_{\mathrm{st}}$.

Lisca [5] classifies minimal symplectic fillings of $L(n,a)$ equipped with the standard contact structure up to deformations and symplectomorphisms. As a result, lens spaces $L(n,a)$ admitting symplectic fillings W with $b_2(W) = 0$ are completely classified. A lens space $L(n,a)$ has a symplectic filling W with $b_2(W) = 0$ if and only if $n = p^2$ and $a = pq - 1$ for some positive integers p, q satisfying $q < p$ and $(p,q) = 1$. Furthermore for such $L(n,a)$ there is only one symplectic filling W with $b_2(W) = 0$ (up to deformations and symplectomorphisms).

In this paper we investigate the next case, that is, lens spaces admitting minimal symplectic fillings with the second Betti number one. We classify lens spaces $L(n,a)$ that admit minimal symplectic fillings W with $b_2(W) = 1$:

Theorem (Theorem 5.5)**.** *A lens space $L(n,a)$ admits a minimal symplectic filling with $b_2 = 1$ if and only if either* (1) $(n,a) = (2,1)$; *or* (2) $(n,a) = (2m^2, 2ma - 1)$ *for some integers m, a with $0 < a < m$ and $(m,a) = 1$; or* (3) *the Hirzebruch-Jung continued fraction of n/a is one of the Hirzebruch-Jung fractions in Proposition 5.1.*

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We then show that a lens space cannot have too many minimal symplectic fillings with $b_2 = 1$.

Theorem (Theorem 5.6)**.** *A lens space $L(n,a)$ may have at most two different minimal symplectic fillings with $b_2 = 1$ up to deformations and symplectomorphisms.*

For these results, we apply the relation between minimal symplectic fillings of the lens space $L(n,a)$ (classified by Lisca [5]) and Milnor fibers of the cyclic quotient surface singularities $\frac{1}{n}(1,a)$ (described by special partial resolutions; cf. [8]).

2. Symplectic fillings of lens spaces

Lisca [5] classifies minimal symplectic fillings of lens spaces up to deformations and symplectomorphisms. Lisca [5] proves that there is a one-to-one correspondence between the set of minimal symplectic fillings of $L(n,a)$ and the set $K(n/n-a)$ of sequence of integers $\underline{k} = (k_1, \ldots, k_e)$ defined as follows: Let $[a_1, \ldots, a_e]$ be the *Hirzebruch-Jung continued fraction* of $n/n-a$, that is,

$$\frac{n}{n-a} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\cdots - \cfrac{1}{a_e}}} =: [a_1, \ldots, a_e],$$

where a_i's are integers with $a_i \geq 2$. Then the set $K(n/n-a)$ is defined by

$$K(n/n-a) = \{\underline{k} = (k_1, \ldots, k_e) \in \mathrm{adm}(\mathbb{N}^e) \mid [k_1, \ldots, k_e] = 0 \text{ and } 0 < k_i \leq a_i, \forall i\},$$

where we denote by $\mathrm{adm}(\mathbb{N}^e)$ the set of all sequences $(k_1, \ldots, k_e) \in \mathbb{N}^e$ such that the matrix $M(k_1, \ldots, k_e)$ defined by $M_{i,i} = k_i$, $M_{i,j} = -1$ if $|i-j| = 1$, and $M_{i,j} = 0$ otherwise is positive semi-definite of rank at least $e-1$.

Example 2.1. Let $n = 19$ and $a = 7$. Then $19/(19-7) = [2,3,2,3]$. So

$$K(n/n-a) = \{(1,2,2,1), (1,3,1,2), (2,2,1,3)\}.$$

Hence there are three minimal symplectic fillings of $L(19,7)$ up to deformations and symplectomorphisms.

Indeed Lisca [5] constructs a smooth 4-manifold $W_{n,a}(\underline{k})$ with $L(n,a)$ as its boundary for each $\underline{k} \in K(n/n-a)$ as follows: Let $N(\underline{k})$ be the closed oriented 3-manifold given by surgery on S^3 along the framed link in Figure 1. Note that $N(\underline{k})$ is orientation-preserving diffeomorphic to $S^1 \times S^2$. Then $W_{n,a}(\underline{k})$ is defined by the smooth 4-manifold with boundary obtained by attaching 2-handles to $S^1 \times D^3$ along the framed link as in Figure 2. By Lisca [5, Theorem 1.1], each minimal symplectic filling of $L(n,a)$ is orientation preserving diffeomorphic to $W_{n,a}(\underline{k})$ for some $\underline{k} \in K(n/n-a)$. Notice that the attaching circle of each 2-handle of $W_{n,a}(\underline{k})$ is homologically non-trivial in $S^1 \times S^2$. Therefore:

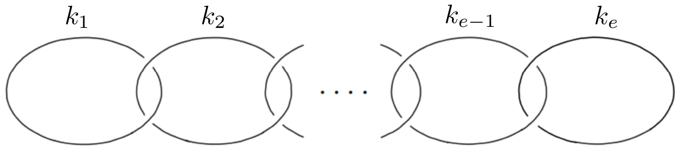


FIGURE 1. The manifold $N(\underline{k})$ (Lisca [5, Figure 1])

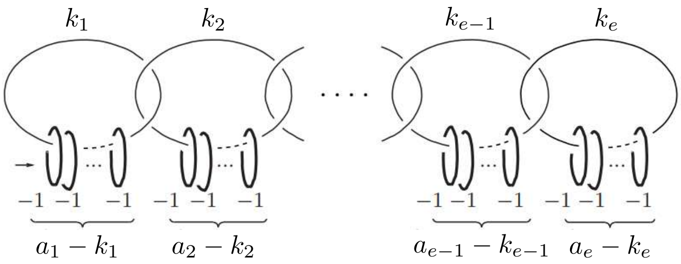


FIGURE 2. A symplectic filling $W_{n,a}(\underline{k})$ (Lisca [5, Figure 2])

Lemma 2.2. *The second Betti number $b_2(W_{n,a}(\underline{k}))$ is given by*

$$b_2(W_{n,a}(\underline{k})) = -1 + \sum_{i=1}^{e}(a_i - k_i).$$

As an easy consequence:

Corollary 2.3. *The second Betti number $b_2(W_{n,a}(\underline{k})) = 1$ if and only if either (1) there are two different indices α, β such that $a_\alpha - k_\alpha = a_\beta - k_\beta = 1$ and $a_i - k_i = 0$ for all $i \neq \alpha, \beta$ or (2) there is one index γ such that $a_\gamma - k_\gamma = 2$ and $a_i - k_i = 0$ for all $i \neq \gamma$.*

Example 2.4 (Continued from Example 2.1)**.** The second Betti numbers of the minimal symplectic fillings of $L(19,7)$ are as follows:

$$b_2(W_{19,7}(1,2,2,1)) = 3, \quad b_2(W_{19,7}(1,3,1,2)) = 2, \quad b_2(W_{19,7}(2,2,1,3)) = 1.$$

3. Milnor fibers of cyclic quotient surface singularities

A *Milnor fiber* of a germ of a cyclic quotient surface singularity $(X,0) = \frac{1}{n}(1,a)$ is roughly speaking a general fiber of its smooth deformation. Explicitly, a *smoothing* of $(X,0)$ is a proper flat map $\pi\colon \mathcal{X} \to \Delta$, where $\Delta = \{t \in \mathbb{C} : |t| < \epsilon\}$, such that $(\pi^{-1}(0), 0) \cong (X,0)$ and $\pi^{-1}(t)$ is smooth for every $t \neq 0$. Then the Milnor fiber M of a smoothing π of $(X,0)$ is a general fiber $\pi^{-1}(t)$ $(0 < t \ll \epsilon)$.

The link of $(X,0)$ is the lens space $L(n,a)$. So any Milnor fiber of $(X,0)$ is naturally a Stein (hence minimal symplectic) filling of $L(n,a)$. Conversely, Nemethi and Popescu-Pampu [6] (refer also [8]) prove that each minimal symplectic filling of $L(n, a)$ is diffeomorphic to a Milnor fiber of $(X, 0)$; hence, there is a one-to-one correspondence between the set of minimal symplectic fillings of $L(n, a)$ and the set of Milnor fibers of $(X, 0)$.

3.1. *P*-resolutions and *M*-resolutions

Kollar and Shepherd-Barron [4] show that every smoothing of $(X, 0)$ can be realized as a $\mathbb{Q}$-Gorenstein smoothing of a *P-resolution* of $(X, 0)$, which is a special partial resolution of $(X, 0)$.

Definition 3.1. A *singularity of class T* is a cyclic quotient surface singularity $\frac{1}{dp^2}(1, dpq-1)$ for some positive integers d, p, q with $d \geq 1$, $0 < q < p$, $(p, q) = 1$.

Definition 3.2. A *P-resolution* of $(X, 0)$ is a partial resolution $f\colon Y \to X$ such that Y has only singularities of class T, and K_Y is ample relative to f.

Furthermore Behnke–Christophersen [1] establish another one-to-one correspondence between minimal symplectic fillings and the so-called *M-resolutions* of $(X, 0)$.

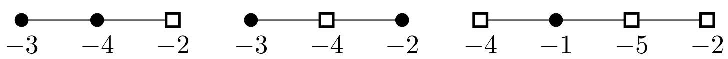
Definition 3.3. A *Wahl singularity* is a cyclic quotient surface singularity $\frac{1}{p^2}(1, pq-1)$ for some positive integers p, q satisfying $0 < q < p$ and $(p, q) = 1$.

We remark that a Wahl singularity admits a smoothing whose Milnor fiber M is a rational homology disk, i.e., $H^i(M, \mathbb{Q}) = 0$ for all $i \geq 1$.

Definition 3.4 (Behnke-Christophersen [1, p. 882])**.** An *M-resolution* of a quotient surface singularity $(X, 0)$ is a partial resolution $f\colon Y_M \to X$ such that

(1) Y_M has only Wahl singularities.
(2) K_{Y_M} is nef relative to f, i.e., $K_{Y_M} \cdot E \geq 0$ for all f-exceptional curves E.

Example 3.5 (Continued from Example 2.1)**.** There are three P-resolutions (which are also M-resolutions) of a cyclic quotient surface singularity $\frac{1}{19}(1, 7)$:



Here a linear chain of vertices decorated by a rectangle □ denotes curves on the minimal resolution of a P-resolution which are contracted to a singularity of class T on the P-resolution.

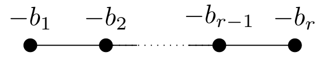
3.2. The rational blowdown surgery

The $\mathbb{Q}$-Gorenstein smoothing of a Wahl singularity may be regarded topologically as a *rational blowdown surgery* defined by Fintushel-Stern [2], and extended by J. Park [7].

We briefly review the rational blowdown surgery. Let $(Y,0) = \frac{1}{p^2}(1, pq-1)$ be a Wahl singularity. Suppose that

$$\frac{p^2}{pq-1} = [b_1, \dots, b_r].$$

Let $C_{p,q}$ be a regular neighborhood of the linear chain of smooth 2-spheres u_i in a smooth 4-manifold Z whose dual graph is given by:



Let $B_{p,q}$ be the Milnor fiber of $(Y,0)$ associated to the $\mathbb{Q}$-Gorenstein smoothing of $(Y,0)$. Then $B_{p,q}$ is a smooth 4-manifold with the lens space $L(p^2, pq-1)$ as its boundary such that $H_*(B_{p,q};\mathbb{Q}) \cong H_*(B^4;\mathbb{Q})$.

One may cut $C_{p,q}$ from Z and paste $B_{p,q}$ along the boundary $L(p^2, pq-1)$ so that one obtains a new smooth 4-manifold $Z_{p,q} = (Z - C_{p,q}) \cup_{L(p^2,pq-1)} B_{p,q}$, which is called a *rational blow-down surgery* along $C_{p,q}$. The surgery can be performed compatibly with a symplectic structure; Symington [9]. That is, if Z is a symplectic 4-manifold and if each 2-spheres u_i's in $C_{p,q}$ are symplectic 2-spheres intersecting positively with each other, then the rational blowdown $Z_{p,q}$ is also a symplectic 4-manifold.

3.3. Milnor fibers via the rational blowdown surgery

Let $(X,0)$ be a cyclic quotient surface singularity and let M be its Milnor fiber. Then M is a general fiber of the $\mathbb{Q}$-Gorenstein smoothing of the corresponding M-resolution Y of X. Since the $\mathbb{Q}$-Gorenstein smoothing of Y is induced from the $\mathbb{Q}$-Gorenstein smoothings of each Wahl singularities of Y, the Milnor fiber M is diffeomorphic to the symplectic 4-manifold obtained by applying rational blowdown surgeries to each Wahl singularities of Y.

Lemma 3.6. *Let $(X,0)$ be a cyclic quotient surface singularity and let M be its Milnor fiber. Let Y be the M-resolution of X corresponding to M. Then $b_2(M)$ is equal to the number of irreducible curves in Y.*

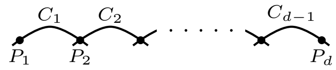
Proof. The assertion follows easily from the fact that $B_{p,q}$ is a rational homology ball with the boundary $L(p^2, pq-1)$, which is a rational homology sphere. □

Corollary 3.7. *If W is a minimal symplectic filling of a cyclic quotient surface singularity $(X,0)$ with $b_2(W) = 1$, then its corresponding P-resolution Y has only one irreducible curve.*

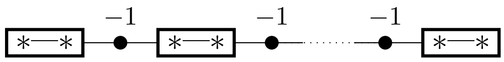
The following corollary is a well-known fact; cf. [1] for instance.

Corollary 3.8. *Let $(Z,0)$ be a cyclic quotient surface singularity $\frac{1}{dp^2}(1, dpq-1)$. Then it has a Milnor fiber M with $b_2(M) = d-1$.*

Proof. One may construct an M-resolution Y of $(Z, 0)$ with $d-1$ irreducible curves $C_i \cong \mathbb{CP}^1$ $(i = 1, \dots, d-1)$ and d singular points P_i of type $\frac{1}{p^2}(1, pq-1)$ as described in the following figure:



The proper transforms of C_i's in the minimal resolution $\widetilde{Y}$ of Y are (-1)-curves. So the minimal resolution $\widetilde{Y}$ is given by



where $\boxed{*-*}$ is the minimal resolution of the singularity $\frac{1}{p^2}(1, pq-1)$. One can check that the above linear chain contracts to the singularity $Z = \frac{1}{dp^2}(1, dpq - 1)$. □

3.4. Extremal P-resolutions

According to Lemma 3.6, the P-resolution corresponding to a minimal symplectic filling with $b_2 = 1$ has a special property. So one can define:

Definition 3.9. An *extremal P-resolution* of a cyclic quotient surface singularity $(X, 0)$ is a P-resolution Y of $(X, 0)$ such that it has only Wahl singularities and it has only one exceptional curve C^+.

Therefore there is a one-to-one correspondence between minimal symplectic fillings with $b_2 = 1$ and extremal P-resolutions.

Following [3, §4], the extremal P-resolution Y has at most two Wahl singularities $\frac{1}{m_i^2}(1, m_i a_i - 1)$ for $i = 1, 2$ on the curve C^+. Here if we have smooth points, then we set $m_i = a_i = 1$. Let

$$\frac{m_1^2}{m_1 a_1 - 1} = [f_1, \dots, f_s] \text{ and } \frac{m_2^2}{m_2 a_2 - 1} = [g_1, \dots, g_t]$$

and $c = -(C^+ \cdot C^+)$ on the minimal resolution of Y. Then we have

$$\frac{n}{a} = [f_s, \dots, f_1, c, g_1, \dots, g_t].$$

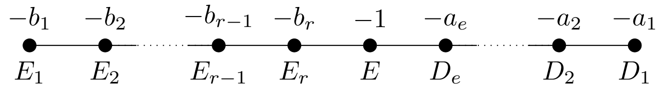
4. From symplectic fillings to M-resolutions

Let $(X, 0)$ be a cyclic quotient surface singularity $\frac{1}{n}(1, a)$. In [8], the authors with J. Park and G. Urzua created an algorithm for constructing the P-resolution Y of $(X, 0)$ for a given $\underline{k} \in K(n/n-a)$ whose Milnor fiber is diffeomorphic to $W_{n,a}(\underline{k})$. The algorithm is based on the semi-stable minimal model program for complex 3-folds. We briefly introduce the algorithm. For details, refer [8, §10].

Let $n/a = [b_1, \dots, b_r]$ and $n/(n-a) = [a_1, \dots, a_e]$. Let

$$d_i := a_i - k_i.$$

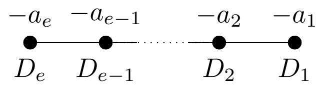
Since $[b_1, \dots, b_r, 1, a_e, \dots, a_1] = 0$, we have a chain of $\mathbb{CP}^1$'s contracting to a smooth point whose dual graph is given by:



Notice that $\cup_{i=1}^r E_i$ is the minimal resolution $\widetilde{X}$ of $(X, 0)$.

Each P-resolution Y of $(X, 0)$ is dominated by the maximal resolution of $(X, 0)$ so that the minimal resolution $\widetilde{Y}$ is also a linear chain of $\mathbb{CP}^1$'s. Hence we can think of the singularities and the $\mathbb{CP}^1$'s in the P-resolution Y are *near or far* from the (-1)-curve E. So we can explain the P-resolution Y associated to $\underline{k} = (k_1, \dots, k_e)$ by constructing the singularities of class T and $\mathbb{CP}^1$'s in Y in the order in which they are closest to the (-1)-curve E.

We consider the dual part:



Let us attach d_i disjoint (-1)-curves to D_i, each transversally at one point. The (recursive) algorithm for constructing the corresponding P-resolution from $\underline{k}$ is as follows:

Step I. (a) If $d_e \neq 0$, then we have an $A_{d_e - 1}$ singularity in the first $\mathbb{CP}^1$.

(b) If $d_e = 0$, we find the smallest nonnegative integer r such that $d_{e-(r+1)} \neq 0$. Then we have a T-singularity

$$\frac{1}{d_{e-(r+1)} n'^2}(1, d_{e-(r+1)} n' a' - 1)$$

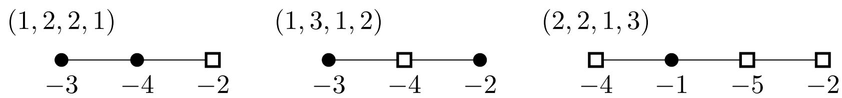
with

$$\frac{n'}{a'} = [a_e, \dots, a_{e-r}].$$

Step II. Now we contract all (-1)-curves attached to D_e (if Step I(a)) or to $D_{e-(r+1)}$ (if Step I(b)), and all (-1)-curves after that coming from $D_e, D_{e-1}, \dots, D_1$, until there are none.

After this, we obtain the new cyclic quotient surface singularity whose dual exceptional divisor is what is left in Step II from $D_e, D_{e-1}, \dots, D_1$. We then repeat the above procedure.

Example 4.1 (Continued from Example 2.1)**.** The P-resolutions of the cyclic quotient surface singularity $\frac{1}{19}(1, 7)$ corresponding to $\underline{k} \in K(19/19 - 7)$ are as follows:



In detail, let $\underline{k} = (2, 2, 1, 3)$. Since $19/19 - 7 = [2, 3, 2, 3]$, we have $d_1 = 0, d_2 = 1, d_3 = 1, d_4 = 0$. Since $d_3 = 1 \neq 0$, we have $n'/a' = [3]$. So we have a T-singularity $1/3^2(1, 2) = [5, 2]$ on the rightmost part of the corresponding P-resolution to $\underline{k}$. Hence we have a partial resolution $4 - 1 - [5, 2]$ of the P-resolution. Applying Step II, the remained dual part is $2 - 2 - 2$ with $d_1 = 0, d_2 = 1, d_3 = 0$. Hence we have a T-singularity [4]. Therefore the P-resolution corresponding to $\underline{k} = (2, 2, 1, 3)$ is $[4] - 1 - [5, 2]$.

5. Symplectic fillings with $b_2 = 1$

We classify lens spaces $L(n, a)$ that admit minimal symplectic fillings with $b_2 = 1$ using the algorithm in the previous Section 4.

According to Corollary 2.3, a lens space $L(n, a)$ has a minimal symplectic filling W with $b_2(W) = 1$ if and only if

Case I. There are two different indices α, β $(1 \leq \alpha < \beta \leq e)$ such that $a_\alpha - k_\alpha = a_\beta - k_\beta = 1$ and $a_i - k_i = 0$ for all $i \neq \alpha, \beta$; or

Case II. There is one index γ such that $a_\gamma - k_\gamma = 2$ and $a_i - k_i = 0$ for all $i \neq \gamma$.

5.1. Case I

Assume Case I. Let

$$\frac{m_1}{a_1} = [a_1, \dots, a_{\alpha-1}] \text{ and } \frac{m_2}{a_2} = [a_e, \dots, a_{\beta+1}].$$

Here if $\alpha = 1$ or $\beta = e$, then we set $m_1 = a_1 = 1$ or $m_2 = a_2 = 1$, respectively. Let

$$\frac{\delta}{\epsilon} = [a_{\alpha+1}, \dots, a_{\beta-1}]$$

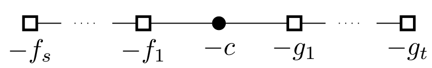
if $\alpha + 1 < \beta$; or we set $\delta = 1$ if $\alpha + 1 = \beta$. Finally, let

$$c = \frac{\delta + m_1 a_2 + m_2 a_1}{m_1 m_2}.$$

Proposition 5.1. *Assume Case I. Then one of the following holds*:

(a) $n/a = [f_s, \dots, f_1, c, g_1, \dots, g_t]$ *for* $\alpha \neq 1$ *and* $\beta \neq e$; *or*
(b) $n/a = [c, g_1, \dots, g_t]$ *for* $\alpha = 1$ *and* $\beta \neq e$; *or*
(c) $n/a = [f_s, \dots, f_1, c]$ *for* $\alpha \neq 1$ *and* $\beta = e$; *or*
(d) $n/a = [c]$ *for* $\alpha = 1$ *and* $\beta = e$.

Proof. According to the algorithm in the previous section and [3, Proposition 4.1], the corresponding extremal P-resolution of $(X, 0)$ to the sequence $\underline{k}$ for Case I is given by:

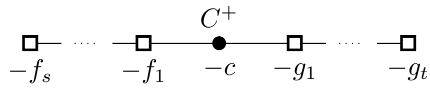


Here we have smooth points if $m_i = a_i = 1$. Hence the assertion follows. □

Conversely,

Proposition 5.2. *Suppose that the Hirzebruch-Jung continued fraction of n/a for a lens space $L(n, a)$ is equal to one of the Hirzebruch-Jung fractions given in Proposition 5.1. If $\delta \geq 1$, then $L(n, a)$ admits a minimal symplectic filling with $b_2 = 1$.*

Proof. Let Y be a partial resolution of the cyclic quotient surface singularity $(X, 0) = \frac{1}{n}(1, a)$ given by:



According to [3, §4], we have $K_Y \cdot C^+ = \frac{\delta}{m_1 m_2} > 0$. Hence Y is an extremal P-resolution of $(X, 0)$. So $L(n, a)$ admits a minimal symplectic filling with $b_2 = 1$ corresponding to the extremal P-resolution Y. □

Case I is also treated in [3] (see also Urzua-Vilches [10]) in a different context. Notice that if we attach (-1)-curves $(a_i - k_i)$-times to each vertices a_i, then after contracting all the (-1)-curves that are attached we get a sequence $\{k_1, \dots, k_e\}$, which represents a zero Hirzebruch-Jung continued fraction.

Proposition 5.3 ([3, p. 325])**.** *For any sequence of integers $\{a_1, \dots, a_e\}$ with $a_i \geq 2$ $(i = 1, \dots, e)$, there exist* at most *two pairs (α, β) with $\alpha < \beta$ such that*

$$[a_1, \dots, a_\alpha - 1, \dots, a_\beta - 1, \dots, a_e] = 0.$$

5.2. Case II

Assume that we are in Case II.

Proposition 5.4. *Case II occurs if and only if either $(n, a) = (2, 1)$ or $(n, a) = (2m^2, 2ma - 1)$ for some integers m, a with $0 < a < m$ and $(m, a) = 1$.*

Proof. At first, suppose $\delta = e$. According to the algorithm in the previous section, we have an A_1-singularity (that is, the $\frac{1}{2}(1, 1)$-singularity) on the corresponding P-resolution Y of the cyclic quotient surface singularity $(X, 0) = \frac{1}{n}(1, a)$. But the Milnor fiber of the A_1-singularity has already $b_2 = 1$. Hence there are no other exceptional curves and singularities on Y. Hence $(X, 0)$ is the A_1-singularity.

Suppose now that $\delta < e$. Let $m/a = [a_e, \dots, a_{\delta-1}]$. Then we have the T-singularity $\frac{1}{2m^2}(1, 2ma - 1)$ on the P-resolution Y according to the algorithm. By Corollary 3.8, the T-singularity $\frac{1}{2m^2}(1, 2ma - 1)$ has $b_2 = 1$. Therefore Y cannot have any other exceptional curves and singularities on it. Hence $(X, 0)$ is the T-singularity $\frac{1}{2m^2}(1, 2ma - 1)$. □

5.3. Classification

We now classify lens spaces admitting minimal symplectic fillings with $b_2 = 1$.

Theorem 5.5. *A lens space $L(n, a)$ admits a minimal symplectic filling with $b_2 = 1$ if and only if either* (1) $(n, a) = (2, 1)$; *or* (2) $(n, a) = (2m^2, 2ma - 1)$ *for some integers m, a with $0 < a < m$ and $(m, a) = 1$; or* (3) *the Hirzebruch-Jung continued fraction of n/a is one of the Hirzebruch-Jung fractions in Proposition 5.1.*

Proof. This is an easy consequence of Proposition 5.1, Proposition 5.2, and Proposition 5.4. □

Theorem 5.6. *A lens space $L(n, a)$ may have at most two different minimal symplectic fillings with $b_2 = 1$ up to deformations and symplectomorphisms.*

Proof. The assertion follows from Proposition 5.3 and Proposition 5.4. □

References

[1] K. Behnke and J. A. Christophersen, *M-resolutions and deformations of quotient surface singularities*, Amer. J. Math. **116** (1994), no. 4, 881–903.

[2] R. Fintushel and R. Stern, *Rational blowdowns of smooth 4-manifolds*, J. Differential Geom. **46** (1997), no. 2, 181–235.

[3] R. Hacking, J. Tevelev, and G. Urzua, *Flipping surfaces*, J. Algebraic Geom. **26** (2017), no. 2, 279–345.

[4] J. Kollar and N. I. Shepherd-Barron, *Threefolds and deformations of surface singularities*, Invent. Math. **91** (1988), no. 2, 299–338.

[5] P. Lisca, *On symplectic fillings of lens spaces*, Trans. Amer. Math. Soc. **360** (2008), no. 2, 765–799. https://doi.org/10.1090/S0002-9947-07-04228-6

[6] A. Nemethi and P. Popescu-Pampu, *On the Milnor fibres of cyclic quotient singularities*, Proc. Lond. Math. Soc. **101** (2010), no. 2, 554–588.

[7] J. Park, *Seiberg-Witten invariants of generalised rational blow-downs*, Bull. Austral. Math. Soc. **56** (1997), no. 3, 363–384. https://doi.org/10.1017/S0004972700031154

[8] H. Park, J. Park, D. Shin, and G. Urzua, *Milnor fibers and symplectic fillings of quotient surface singularities*, Adv. Math. **329** (2018), 1156–1230.

[9] M. Symington, *Generalized symplectic rational blowdowns*, Algebr. Geom. Topol. **1** (2001), 503–518. https://doi.org/10.2140/agt.2001.1.503

[10] G. Urzua and N. Vilches, *On wormholes in the moduli space of surfaces*, arXiv:2102.02177.

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