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CONTINUOUS DATA ASSIMILATION FOR THE THREE-DIMENSIONAL LERAY-α MODEL WITH STOCHASTICALLY NOISY DATA

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Abstract. In this paper we study a nudging continuous data assimilation algorithm for the three-dimensional Leray-α model, where measurement errors are represented by stochastic noise. First, we show that the stochastic data assimilation equations are well-posed. Then we provide explicit conditions on the observation density (resolution) and the relaxation (nudging) parameter which guarantee explicit asymptotic bounds, as the time tends to infinity, on the error between the approximate solution and the actual solution which is corresponding to these measurements, in terms of the variance of the noise in the measurements.

1. Introduction

Data assimilation is a methodology to study and forecast the trend of natural phenomena, such as the weather, ocean models and environmental sciences. The idea of data assimilation is to combine observational data with dynamic principles related to the basic mathematical model. The classical method of data assimilation is to insert observational data directly into a model as the latter is being integrated in time, see e.g. [17, 23, 26] and references therein. However, this algorithm reveals some difficulties when measurements are gathered from a discrete set of nodal points, because it is impossible to accurately calculate the value of the spatial derivatives present in the model. In the pioneering work [7], the authors introduced a new approach for data assimilation problem, which is a feedback control algorithm [8] applied to data assimilation, and this method has overcome the disadvantages of the classical method. In this new algorithm, instead of directly inserting measurements into the model, a nudging parameter and the observational measurements are used to establish a new model whose approximation solution converges to the unknown solution of the original model. Such an approach has been developed later for

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data assimilation of many important equations in fluid mechanics, see, e.g. [1, 2, 5, 21, 22, 25, 27]. A similar data assimilation algorithm for stochastically noisy data was introduced in [9], where the problem for the 2D Navier-Stokes equations was investigated.

The three-dimensional (3D) Leray-α model was first introduced and studied in [13]. This model is one of a number of regularizations of the Navier-Stokes equation in which the amplitudes of high wave number components are suppressed (over and above the suppression already provided by the viscosity term). In recent years, the existence, regularity, convergence and long-time behavior of solutions to this model have attracted the attention of many mathematicians in both deterministic case [3, 4, 11, 12, 20, 28] and stochastic case [6, 10, 15, 18, 24]. The continuous data assimilation for the 3D Leray-α model was studied recently in [21], while the discrete data assimilation for this model was studied more recently in [5]. It is noticed that in these two works the observational data do not contain measurement errors.

In this paper we study the continuous data assimilation algorithm with stochastically noisy data for the following three-dimensional Leray-α model

$$\begin{cases} \dfrac{\partial v}{\partial t} - \nu\Delta v + (u\cdot\nabla)v + \nabla p = f, \\ \nabla\cdot u = \nabla\cdot v = 0, \\ v = u - \alpha^2\Delta u, \end{cases} \tag{1}$$

where $u = u(x,t)$ is the unknown velocity vector field, $p(x,t)$ is the scalar unknown pressure field, $\nu > 0$ is the kinematic viscosity, and $\alpha > 0$ is a scale parameter with dimension of length. We assume periodic boundary conditions with the fundamental domain $\mathcal{D} = [0, L]^3$ and take the initial condition $v(x, 0) = v_0(x)$ and the body forcing $f = f(x)$ to be an L-periodic function with zero spatial average.

In what follows, we will describe the data assimilation problem, which will be studied in the present paper. Assume that $v(t)$ is a solution lying on the global attractor of the 3D Leray-α model (1). Denote by $\mathcal{O}_h(v(t))$, for $t \geq 0$, the exact observational measurements without error of the exact solution v at time t. We assume $\mathcal{O}_h : V \to \mathbb{R}^D$ to be a linear operator, where V is the function space defined in Section 2 below, D is of the order $(L/h)^3$, L is a typical large length scale of the physical domain of interest, h is the observation density or resolution, and denote by $R_h(v(t))$ the interpolation of the observational data, i.e.,

$$R_h(v(t)) = \mathcal{L}_h \circ \mathcal{O}(v(t)),$$

where $\mathcal{L}_h : \mathbb{R}^D \to V$ is a bounded linear operator. Here we assume the interpolant operator R_h satisfies the approximating identity property

$$\|w - R_h(w)\|_{L^2}^2 \leq c_1 h^2 \|\nabla w\|_{L^2}^2 \quad \text{for all } w \in V. \tag{2}$$

Examples of such interpolant operators R_h are the orthogonal projections onto the low Fourier modes or finite volume elements (see [2, 7] for details).

In the absence of measurement errors, the data-assimilation algorithm proposed by Titi et al. [7] would construct the approximating solution z from the interpolant observables $R_h(v(t))$ dynamically as the solution to the following equations

$$\begin{cases} \dfrac{\partial z}{\partial t} - \nu\Delta z + (w\cdot\nabla)z + \nabla p = f - \mu(R_h(z) - R_h(v)), \\ \nabla\cdot z = 0, \end{cases} \tag{3}$$

with $z = w - \alpha^2\Delta w$ and arbitrary initial condition $w(0) = w_0$. Here $\mu > 0$ is a relaxation parameter that will be determined later, which forces the coarse spatial scales of z, i.e., $R_h(z)$, towards those of the observed data, i.e., $R_h(v)$.

Suppose now the exact measurements $\mathcal{O}_h(v(t))$ are subject to some random errors. Therefore, the only observations available for data assimilation are noisy observations $\tilde{O}_h(v(t))$ given by

$$\tilde{\mathcal{O}}_h(v(t)) = \mathcal{O}_h(v(t)) + \mathcal{E}(t), \tag{4}$$

where $\mathcal{E} : [0,\infty) \to \mathbb{R}^D$ represents the measurement error, for example, due to instrumental errors. This implies that the measurements of $v(t)$ contain random errors and are given by

$$\tilde{R}_h(v(t)) = \mathcal{L}_h(\tilde{\mathcal{O}}(v(t))) = \mathcal{L}_h(\mathcal{O}_h(v(t))) + \mathcal{L}_h(\mathcal{E}(t)) = R_h(v(t)) + \xi(t), \tag{5}$$

where the random vector $\xi(t)$ lies in the range of the interpolant operator R_h.

In this paper we will examine the data-assimilation method given by equation (3) when the noise-free interpolant observable $R_h(v(t))$ is replaced by $\tilde{R}_h(v(t))$. In this case, following the general lines of [9], the algorithm for constructing $z(t)$ from the observational measurements $\tilde{O}_h(v(t))$ is given by the following stochastic evolution equation

$$\begin{cases} dz + [-\nu\Delta z + (w\cdot\nabla)z + \nabla p]dt = f dt - \mu[R_h(z) - R_h(v)]dt + \mu\xi dt, \\ \nabla\cdot z = 0, \end{cases} \tag{6}$$

where $z = w - \alpha^2\Delta w$ and with arbitrary initial condition $z(0) = z_0$. Our aim here is to find explicit conditions on the relaxation parameter μ and the observation resolution h which guarantee explicit asymptotic bounds, as the time tends to infinity, on the error between the approximate solution z and the actual solution v which is corresponding to these measurements, in terms of the variance of the noise in the measurements.

The paper is organized as follows. In Section 2, for convenience of the reader, we recall the functional setting and some results for the deterministic 3D Leray-α model, and describe the noise term. The data-assimilation algorithm, including the well-posedness and the convergence results, is presented in Section 3 with observations of volume elements.

2. Preliminaries

2.1. The functional setting

In this subsection we introduce some notations which will be frequently used throughout this paper.

Denote by $\mathcal{V}$ the space of all divergence-free $\mathbb{R}^3$-valued L-periodic trigonometric polynomials with zero spatial averages. Let

$$H = \text{the closuse of } \mathcal{V} \text{ in } [L^2(\mathcal{D})]^3,$$
$$V = \text{the closuse of } \mathcal{V} \text{ in } [H^1(\mathcal{D})]^3,$$

with the inner products given by

$$(u,v) := \int_{\mathcal{D}} \sum_{i=1}^{3} u_i v_i \, dx,$$
$$((u,v)) := \int_{\mathcal{D}} \sum_{i=1}^{3} \nabla u_i \cdot \nabla v_i \, dx, \text{ respectively}$$

and the associated norms $|u|^2 := (u,u)$ and $\|u\|^2 := ((u,u))$.

For $\varphi \in L^1$ we define the average

$$\langle \varphi \rangle = \frac{1}{L^3} \int_{\mathcal{D}} \varphi(x) \, dx,$$

and for every subset $Z \subset L^1$, we denote $\dot{Z} = \{\varphi \in Z : \langle \varphi \rangle = 0\}$.

Let $\Pi : [\dot{L}^2(\mathcal{D})]^3 \to H$ be the Leray-Helmholtz orthogonal projector, and the Stokes operator A subject to the periodic boundary conditions with domain $D(A) = [\dot{H}^2(\mathcal{D})]^3 \cap V$ is defined by $Au = -\Pi\Delta u = -\Delta u$. The norm in $D(A)$ is $\|u\|_{D(A)} = |Au|$, $\forall u \in D(A)$. Moreover, the Stokes operator A is a positive self-adjoint operator with compact inverse, thus there exists a complete orthonormal set of eigenfunctions $\{\psi_j\}_{j=1}^{\infty} \subset H$ such that $A\psi_j = \lambda_j \psi_j$ and

$$\frac{4\pi^2}{L^2} = \lambda_1 \le \lambda_2 \le \cdots, \ \lambda_j \to +\infty \text{ as } j \to \infty.$$

We have the following Poincaré inequalities

$$\begin{aligned} &\|u\|_{V'}^2 \le \lambda_1^{-1} |u|^2, \quad \forall u \in H, \\ &|u|^2 \le \lambda_1^{-1} \|u\|^2, \quad \forall u \in V, \end{aligned} \tag{7}$$

where V' denotes the dual space of V (see e.g. [14, 29]).

For all $v = u + \alpha^2 Au, v \in H$, we have

$$|v|^2 = |u|^2 + 2\alpha^2 \|u\|^2 + \alpha^4 |Au|^2.$$

Thus, from (7) we get

$$|u| \le |v|, \quad \|u\| \le \frac{1}{2^{1/2}\alpha} |v|, \quad |Au| \le \frac{1}{\alpha^2} |v|. \tag{8}$$

Let $b(\cdot,\cdot,\cdot): V\times V\times V\to\mathbb{R}$ be the continuous trilinear form defined by

$$b(u,v,w)=\sum_{i,j=1}^{3}\int_{\mathcal{D}} u_i\frac{\partial v_j}{\partial x_i}w_j dx,\quad \forall u,v,w\in V.$$

It is well-known that there exists a continuous bilinear operator $B(\cdot,\cdot): V\times V\to V'$ such that

$$\langle B(u,v),w\rangle_{V',V}=b(u,v,w),\quad \forall w\in V.$$

Lemma 2.1 ([14, 29]). *We have*

$$\langle B(u,v),z\rangle=-\langle B(u,z),v\rangle \text{ and } \langle B(u,v),v\rangle=0,\quad \forall u,v,w\in V.$$

Furthermore,

$$|\langle B(u,v),z\rangle|\le C_L|u|^{1/4}\|u\|^{3/4}\|v\||z|^{1/4}\|z\|^{3/4},\quad \forall u,v,w\in V,$$

and

$$|\langle B(u,v),z\rangle|\le C_L\|u\|\|v\|^{1/2}|Av|^{1/2}|z|,\quad \forall u\in D(A), v\in H, w\in V. \tag{9}$$

Applying the Leray-Helmholtz orthogonal projector Π to the Leray-α model (1) to obtain the functional evolution equation

$$\begin{cases}\dfrac{dv}{dt}+\nu Av+B(u,v)=f,\\ v(0)=v_0,\end{cases}\tag{10}$$

where $v=u+\alpha^2Au$, $f\in H$, and $v_0\in H$.

Similarly, the stochastic data assimilation equation (6) becomes

$$dz+[\nu Az+B(w,z)]dt=[f-\mu\Pi R_h(z-v)]dt+\mu dW,\tag{11}$$

where $z=w+\alpha^2Aw$ and $dW(t)=\Pi\xi(t)dt$ is the noise term.

2.2. The deterministic Leray-α model

Let $f\in H$. We denote the Grashof number in three dimensions by

$$Gr=\frac{|f|}{\nu^2\lambda_1^{3/4}}.$$

The following result was proved in [13].

Theorem 2.2. *Let $f\in H$ and $v_0\in H$. Then for any $T>0$, problem* (10) *has a unique weak solution v that satisfies*

$$v\in C([0,T];H)\cap L^2(0,T;V)\quad \textit{and}\quad \frac{dv}{dt}\in L^2(0,T;V').$$

Additionally, the associated semigroup $S(t): H\to H$ has a global attractor $\mathcal{A}$ in H. And for any $v\in\mathcal{A}$, we have

$$|v|^2\le M_0^2:=\frac{2\nu^2Gr^2}{\lambda_1^{1/2}}.\tag{12}$$

2.3. The noise term

We now describe the error term $\mathcal{E} : [0,\infty) \to \mathbb{R}^D$ that gives rise to the noisy observations $\tilde{\mathcal{O}}_h$ in (4) in terms of a Brownian motion and by using $\tilde{R}_h = \mathcal{L}_h \circ \tilde{\mathcal{O}}$ to obtain the noise term dW in (11).

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a probability space on which is defined a sequence of independent one-dimensional Brownian motions $\beta_d(t), d = 1, 2, \dots, D$, relative to the filtration $(\mathcal{F}_t)$, which is assumed to be complete and right continuous, such that

$$\mathbb{E}(\beta_d(t)) = 0 \text{ and } \mathbb{E}(\beta_d^2(t)) = \frac{t\sigma^2}{3} \text{ for } t \geq 0.$$

The measurement errors may now be described by

$$\mathcal{E}(t)dt = (d\beta_1(t), d\beta_2(t), \dots, d\beta_D(t)). \tag{13}$$

Note that σ is a dimensional constant whose units of measurement must be chosen so that the units of measurement for $\mathcal{O}_h(v(t))$ are the same as $\mathcal{E}$.

Writing the linear operator $\mathcal{L}_h : \mathbb{R}^D \to [\dot{H}^1(\mathcal{D})]^3$ as

$$\mathcal{L}_h(\zeta)(\cdot) = \sum_{d=1}^{D} \zeta_d \ell_d(\cdot), \quad \zeta \in \mathbb{R}^D \text{ and } \ell_d \in [\dot{H}^1(\mathcal{D})]^3, \tag{14}$$

it follows that the noise term in (11) is the Wiener process

$$W(t) = \sum_{d=1}^{D} \beta_d(t)\gamma_d, \quad \gamma_d = \Pi \ell_d. \tag{15}$$

We do not assume γ_d are orthogonal or even linearly independent.

We can see that W is an $[\dot{L}^2(\mathcal{D})]^3$-valued Q-Brownian motion with $\mathbb{E}(W(t)) = 0$. Following [16] (see also [19]), we have

$$tQ = \operatorname{Cov}(W(t)) = \mathbb{E}\left(\sum_{d,p=1}^{D} \beta_d(t)\gamma_d \otimes \beta_p(t)\gamma_p\right).$$

Note that Q is a nonnegative and symmetric linear operator with finite trace. By some computations as in [9], we get that

$$\operatorname{Tr}(Q) = \frac{\sigma^2}{3} \sum_{d=1}^{D} |\gamma_d|^2 < \infty.$$

We now give an example for interpolant observable based on volume elements. Suppose the observations of volume elements $\mathcal{O}_h : [\dot{H}^1(\mathcal{D})]^3 \to \mathbb{R}^{3N}$ are given by

$$\mathcal{O}_h(\Phi) = (\bar{\varphi}_1, \bar{\varphi}_2, \dots, \bar{\varphi}_{3N}), \tag{16}$$

where $\begin{bmatrix} \bar{\varphi}_{3n-2} \\ \bar{\varphi}_{3n-1} \\ \bar{\varphi}_{3n} \end{bmatrix} = \frac{1}{|Q_n|}\int_{Q_n} \Phi(x)dx = \frac{N}{L^3}\int_{Q_n} \Phi(x)dx$ for $n = 1, \dots, N$, where the domain $\mathcal{D} = [0, L]^3$ has been divided into $N = K^3$ disjoint equal cubes with Q_n with edges $\frac{L}{\sqrt[3]{N}}$ and so $|Q_n| = \frac{L^3}{N}$. Define $R_h = \mathcal{L}_h \circ \mathcal{O}_h$, where $\mathcal{L}_h : \mathbb{R}^{3N} \to [\dot{L}^2(\mathcal{D})]^3$ with $\mathcal{L}_h(\zeta)$ is the L-periodic function on $\mathcal{D}$ given by

$$\mathcal{L}_h(\zeta)(x) = \sum_{n=1}^{N} \begin{bmatrix} \zeta_{3n-2} \\ \zeta_{3n-1} \\ \zeta_{3n} \end{bmatrix} \left(\chi_{Q_n}(x) - \frac{h^3}{L^3} \right).$$

Let

$$\begin{aligned} &\ell_{3n-2}(x) = \begin{bmatrix} \chi_{Q_n}(x) - h^3/L^3 \\ 0 \\ 0 \end{bmatrix}, \quad \ell_{3n-1}(x) = \begin{bmatrix} 0 \\ \chi_{\Omega_n}(x) - h^3/L^3 \\ 0 \end{bmatrix}, \\ &\text{and } \ell_{3n}(x) = \begin{bmatrix} 0 \\ 0 \\ \chi_{\Omega_n}(x) - h^3/L^3 \end{bmatrix} \end{aligned} \tag{17}$$

for $n = 1, 2, \dots, N$. This implies $D = 3N$ functions are needed in (14). We have the following proposition.

Proposition 2.3. *Let $W(t)$ be the Wiener process in* (15)*, where ℓ_d is as in* (17) *for $d = 1, \dots, 3N$. Then W is an $[\dot{L}^2(\mathcal{D})]^3$-valued Q-Brownian motion with covariance operator Q that satisfies*

$$\operatorname{Tr}(Q) \le \sigma^2 L^3.$$

Proof. We have

$$\begin{aligned} \operatorname{Tr}(Q) &= \frac{\sigma^2}{3} \sum_{d=1}^{3N} |\gamma_d|^2 \\ &\le \frac{\sigma^2}{3} \sum_{d=1}^{3N} |\ell_d|_{L^2}^2 \\ &= \sigma^2 \sum_{n=1}^{N} \int_\Omega \left| \chi_{\Omega_n}(x) - h^3/L^3 \right|^2 dx \\ &= \sigma^2 \sum_{n=1}^{N} \int_\Omega \left[\left(1 - 2\frac{h^3}{L^3}\right) \chi_{\Omega_n(x)} + \frac{h^6}{L^6} \right] dx \\ &\le \sigma^2 (L^3 - h^3) \le \sigma^2 L^3. \end{aligned}$$

□

3. Continuous data assimilation algorithm

Let v be the weak solution of the 3D Leray-α model (10) given by Theorem 2.2, and let R_h be an interpolation operator satisfying (2). Suppose the only knowledge we have about v is from the noisy observational measurements $R_h(v(t)) + \xi(t)$ that have been continuously recorded for times $t \in [0, T]$.

Our first goal is to show that the data-assimilation algorithm given by equation (11) for computing the approximating solution z is global well-posed. The second goal is to prove that z, the solution of the data assimilation equation (11), approximates the unknown reference solution v of (10), when $t \to \infty$, within some tolerance depending on the error in the observations.

3.1. Well-posedness

In this subsection we will prove the existence and uniqueness of a global solution to the stochastic data assimilation equation (11) with arbitrary initial condition $z(0) = z_0 \in H$ and $z = w + \alpha^2 Aw$, where $dW(t) = \Pi\xi(t)dt$ is the noise term.

We first give the definition of weak solutions to problem (11).

Definition. A stochastic process $(z(t))_{t\in[0,T]}$ is called a weak solution on $(0,T)$ of the stochastic problem (11) if the following conditions holds:

(i) z is progressively measurable;
(ii) $z = w + \alpha^2 Aw$ belongs to $C([0,T];H) \cap L^2(0,T;V)$ a.s.;
(iii) for all $t \in [0,T]$ and $\mathbb{P}$-a.s.,

$$\begin{aligned}&(z(t),\varphi) + \nu\int_0^t (Az(s),\varphi)ds - \int_0^t \langle B(w(s),z(s)),\varphi\rangle ds\\ &= (z_0,\varphi) + \int_0^t (f,\varphi)ds - \mu\int_0^t (R_h(z(s)-v(s)),\varphi)ds + \mu\int_0^t (dW(s),\varphi)\end{aligned}$$

for all test functions $\varphi \in V$.

The following theorem gives the well-posedness of problem (11).

Theorem 3.1. *Suppose that the interpolant operator* $R_h : (\dot{H}^1(\mathcal{D}))^3 \to (\dot{L}^2(\mathcal{D}))^3$ *satisfies* (2) *and that* $2\mu c_1 h^2 \le \nu$ *with* $\mu \ge \frac{2^{3/2}C_L^2 M_0}{\nu\alpha^3}$. *Then for any* $z_0 \in H$ *and* $T > 0$ *given, there exists a unique stochastic process solution* $z \in C([0,T];V)$ *of problem* (11). *Moreover,*

$$\mathbb{E}\left(\sup_{0\le t\le T}(|z(t)|^2)\right) < \infty, \tag{18}$$

and

$$\mathbb{E}\left(\int_0^T \|z(t)\|^2 dt\right) < \infty. \tag{19}$$

Proof. The proof is based on a pathwise argument. Consider the auxiliary process y which is a solution of the following problem

$$\begin{cases} dy + \nu Ay dt = \mu dW,\\ y(0) = 0.\end{cases} \tag{20}$$

It is known (see [16]) that

$$y(t) = \mu\int_0^t e^{-\nu A(t-\tau)}dW(\tau)$$

is a stationary ergodic solution to (20) with continuous trajectories taking values in H. In particular, we have

$$\mathbb{E}(|y(t)|^2) \leq \frac{\mu^2}{2\nu}\mathrm{Tr}(Q).$$

Indeed, we write

$$y = \sum_{j=1}^{\infty} y_j e_j \quad \text{and} \quad W = \sum_{j=1}^{\infty} W_j e_j = \sum_{j=1}^{\infty} \left(\sum_{d=1}^{D} \gamma_{d,j}\beta_d \right) e_j,$$

where $y_j(t) = \langle y(t), e_j \rangle$ and $\gamma_{d,j} = \langle \gamma_d, e_j \rangle$. Then

$$y_j(t) = \mu \int_0^t e^{-\nu\lambda_j(t-\tau)} dW_j(\tau) = \mu \sum_{d=1}^{D} \gamma_{d,j} \int_0^t e^{-\nu\lambda_j(t-\tau)} d\beta_d(\tau).$$

By the independence of β_d and the Itô isometry, we get that

$$\begin{aligned}
\mathbb{E}|y_j(t)|^2 &= \mu^2 \sum_{d=1}^{D} \gamma_{d,j}^2 \mathbb{E}\left| \int_0^t e^{-\nu\lambda_j(t-\tau)} d\beta_d(\tau) \right|^2 \\
&= \frac{\mu^2\sigma^2}{3} \sum_{d=1}^{D} \gamma_{d,j}^2 \int_0^t e^{-2\nu\lambda_j(t-\tau)} d\tau \\
&= \frac{\mu^2\sigma^2}{6\nu\lambda_j} \sum_{d=1}^{D} \gamma_{d,j}^2 \int_0^t e^{-2\nu\lambda_j(t-\tau)} d(-2\nu\lambda_j(t-\tau)) \\
&\leq \frac{\mu^2\sigma^2}{6\nu\lambda_j} \sum_{d=1}^{D} \gamma_{d,j}^2 = \frac{\mu^2}{2\nu\lambda_j} \left(\frac{\sigma^2}{3} \sum_{d=1}^{D} \gamma_{d,j}^2 \right).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\mathbb{E}|y(t)|^2 &= \sum_{j=1}^{\infty} \lambda_j \mathbb{E}|y_j|^2 \\
&\leq \sum_{j=1}^{\infty} \lambda_j \left(\frac{\mu^2\sigma^2}{6\nu\lambda_j} \sum_{d=1}^{D} \gamma_{d,j}^2 \right) = \frac{\mu^2\sigma^2}{6\nu} \sum_{d=1}^{D} \left(\sum_{j=1}^{\infty} \gamma_{d,j}^2 \right) \\
&\leq \frac{\mu^2}{2\nu} \frac{\sigma^2}{3} \sum_{d=1}^{D} \left(\sum_{j=1}^{\infty} \gamma_{d,j}^2 \right) = \frac{\mu^2}{2\nu} \mathrm{Tr}(Q).
\end{aligned}$$

Now, using the change of variable $\tilde{z} = z - y$, we find that $\tilde{z}$ is a solution of the random differential equation

$$\frac{d}{dt}\tilde{z} + \nu A\tilde{z} + B(w, \tilde{z} + y) + \mu\Pi R_h(\tilde{z} + y) = \tilde{f} \tag{21}$$

with $\tilde{z}(0) = \tilde{z}_0 = z_0$, where $\tilde{f} = f + \mu\Pi R_h(v)$. Since $v \in C([0,T];H)$ and since R_h is a bounded linear operator, we have

$$|\Pi R_h(v)| \leq C|v|,$$

which implies that $\Pi R_h(v) \in C([0,T];H)$. Therefore, $\tilde{f} \in C([0,T];H)$.

For every $\omega \in \Omega$, there exists a unique weak solution $\tilde{u}$ of equation (21) and it depends continuously in $C([0,T];H) \cap L^2(0,T;V)$, for any given $T > 0$, on the initial condition $\tilde{z}(0) = z_0$ in H. The rigorous proof is based on the Galerkin approximation procedure and then passing to the limit using the appropriate compactness lemmas. Because the proof is quite standard, in what follows we only give the necessary *a priori* estimates.

Taking the inner product of equation (21) by $\tilde{z}$, we get

$$\frac{1}{2}\frac{d}{dt}|\tilde{z}|^2 + \nu\|\tilde{z}\|^2 = -\langle B(w, \tilde{z}+y), \tilde{z}\rangle - \mu\langle \Pi R_h(\tilde{z}+y), \tilde{z}\rangle + (\tilde{f}, \tilde{z}).$$

We now estimate each term on the right-hand side. First, by using Lemma 2.1 and (8),

$$\begin{aligned}
|\langle B(w, \tilde{z}+y), \tilde{z}\rangle| &= |\langle B(w, y), \tilde{z}\rangle| \\
&\leq C_L|w|^{1/2}|Aw|^{1/2}|y|\|\tilde{z}\| \\
&\leq C_L 2^{-1/4}\alpha^{-3/2}|y||z|\|\tilde{z}\| \\
&\leq \frac{\nu}{4}\|\tilde{z}\|^2 + \frac{C_L^2}{2^{1/2}\nu\alpha^3}|y|^2|z|^2 \\
&\leq \frac{\nu}{4}\|\tilde{z}\|^2 + \frac{2^{1/2}C_L^2}{\nu\alpha^3}|y|^2(|\tilde{z}|^2 + |y|^2).
\end{aligned}$$

Next, by (2) and the Cauchy inequality, we have

$$\begin{aligned}
-\mu\langle \Pi R_h(\tilde{z}+y), \tilde{z}\rangle &= -\mu\langle R_h(y), \tilde{z}\rangle - \mu\langle R_h(\tilde{z}), \tilde{z}\rangle \\
&\leq \mu|y - R_h(y)|_H|\tilde{z}| + \mu|y||\tilde{z}| + \mu\langle \tilde{z} - R_h(\tilde{z}), \tilde{z}\rangle - \mu|\tilde{z}|^2 \\
&\leq \frac{\mu}{2}|\tilde{z}|^2 + \mu(|y|^2 + |y - R_h(y)|^2) \\
&\quad + \frac{\mu}{2}|\tilde{z} - R_h(\tilde{z})|^2 + \frac{\mu}{2}|\tilde{z}|^2 - \mu|\tilde{z}|^2 \\
&\leq \mu(\lambda_1^{-1} + c_1h^2)\|y\|^2 + \frac{c_1h^2\mu}{2}\|\tilde{z}\|^2,
\end{aligned}$$

and

$$(\tilde{f}, \tilde{z}) \leq |\tilde{f}||\tilde{z}| \leq \lambda_1^{-1/2}|\tilde{f}|\|\tilde{z}\|^2 \leq \frac{1}{\lambda_1\nu}|\tilde{f}|^2 + \frac{\nu}{4}\|\tilde{z}\|^2.$$

Therefore, if we choose h and μ such that $\nu \geq 2c_1h^2\mu$, then we get

$$\frac{d}{dt}|\tilde{z}|^2 + \frac{\nu}{4}\|\tilde{z}\|^2 \leq \frac{2^{1/2}C_L^2}{\nu\alpha^3}|y|^2(|\tilde{z}|^2 + |y|^2) + \mu(\lambda_1^{-1} + c_1h^2)\|y\|^2 + \frac{1}{\lambda_1\nu}|\tilde{f}|^2.$$

Since $\tilde{f} \in C([0,T];H)$ and $y \in C([0,T];V)$, by the Gronwall inequality, we obtain

$$\sup_{t\in[0,T]} |\tilde{z}(t)|^2 \leq C.$$

And thus, we also obtain

$$\int_0^T \|\tilde{z}(s)\|^2 ds \leq C.$$

Since $z = \tilde{z} + y$, we have $\mathbb{P}$-a.s., $z \in C([0,T];H) \cap L^2(0,T;V)$.

We now prove estimate (18). Using the Itô formula on $|z|^2$, we have

$$\begin{aligned} &d|z|^2 + 2[(\nu Az + B(w,z) - f, z)]dt \\ &= 2(dW(t),z) + \mu^2 \mathrm{Tr}(Q) + 2\mu(R_h(z-v),z)dt. \end{aligned}$$

Integrating over $(0,T)$, we obtain

$$\begin{aligned} &\sup_{0\leq t\leq T} |z(t)|^2 + 2\nu \int_0^T \|z(\tau)\|^2 d\tau \\ &= |z_0|^2 + 2\int_0^T (f, z(\tau))d\tau - 2\mu \int_0^T (R_h(z(\tau) - v(\tau)), z)d\tau \\ &\quad + 2\mu \sup_{0\leq t\leq T} \int_0^t (dW(\tau), z(\tau))d\tau + \mu^2 \mathrm{Tr}(Q)T. \end{aligned}$$

By the Burkhölder-Gundy-Davis inequality (see [19]), we have

$$\begin{aligned} 2\mu\mathbb{E}\left(\sup_{0\leq t\leq T} \int_0^t (dW(\tau), z(\tau))d\tau\right) &\leq 2\mu\sqrt{\mathrm{Tr}(Q)}\mathbb{E}\left(\int_0^t |z(\tau)|^2 d\tau\right)^{\frac{1}{2}} \\ &\leq 2\mu\mathbb{E}\sup_{0\leq t\leq T} |z(t)|\sqrt{T\mathrm{Tr}(Q)} \\ &\leq \frac{1}{2}\mathbb{E}\sup_{0\leq t\leq T} |z(t)|^2 + 2\mu^2 T\mathrm{Tr}(Q). \end{aligned}$$

On the other hand, using (2) and the Poincaré inequality, we get

$$\begin{aligned} -2\mu(R_h(z-v),z) &\leq 2\mu\|z - R_h(z)\|_{L^2}|z| - 2\mu|z|^2 + 2\mu\|R_h(v)\|_{L^2}|z| \\ &\leq 2\mu c_1 h^2\|z\|^2 - \mu|z|^2 + 2\mu\big(\|v - R_h(v)\|_{L^2}^2 + |v|^2\big) \\ &\leq 2\mu c_1 h^2\|z\|^2 - \mu|z|^2 + 2\mu(c_1 h^2 + \lambda_1^{-1})\|v\|^2, \end{aligned}$$

and by the Cauchy inequality,

$$2(f,z) \leq \mu|z|^2 + \frac{1}{\mu}|f|^2.$$

Since $2\mu c_1 h^2 \leq \nu$, combining the previous estimates with the Gronwall inequality, we obtain

$$\mathbb{E}\left(\sup_{0\leq t\leq T} (|z(t)|^2)\right) \leq C,$$

therefore, we also obtain

$$\mathbb{E}\left(\int_0^T \|z(t)\|^2 dt\right) < C$$

as claimed in (19).

This completes the proof. $\square$

3.2. The convergence theorem

Let $\sigma = w - u, \delta = z - v$ with $z = w + \alpha^2 Aw, v = u + \alpha^2 Au$. Then $\delta = \sigma + \alpha^2 A\sigma$. Thus, from (10) and (11) we have

$$d\delta + [\nu A\delta + B(\sigma,\delta) + B(\sigma, v) + B(u,\delta)]dt = -\mu\Pi R_h(\delta)dt + \mu dW(t), \tag{22}$$

with $\delta_0 \in V$ is chosen arbitrarily.

Theorem 3.2. *Assume that v is a solution of* (10) *and $R_h : (\dot{H}^1(\mathcal{D}))^3 \to (\dot{L}^2(\mathcal{D}))^3$ is a linear interpolant operator satisfying assumption* (2). *Assume that μ is large enough and h is small enough such that*

$$\frac{1}{h^2} \geq \frac{2c_1\mu}{\nu} \geq \frac{2^{3/2}c_1 C_L^2 M_0^2}{\nu^2\alpha^3}, \tag{23}$$

where c_1, C_L are given in (2) *and* (9)*, respectively. Then the solution z of the data assimilation equation* (11) *given by Theorem 3.1 satisfies*

$$\limsup_{t\to\infty} \mathbb{E}\left(|z(t) - v(t)|^2\right) \leq \mu \mathrm{Tr}(Q), \tag{24}$$

and

$$\limsup_{t\to\infty} \frac{\nu}{T}\int_t^{t+T} \mathbb{E}(\|z(s) - v(s)\|^2)ds \leq \frac{\mu}{T}\mathrm{Tr}(Q) + \mu^2\mathrm{Tr}(Q). \tag{25}$$

Proof. Applying the Itô formula on $|\delta|^2$, and from (22) we get

$$d|\delta(t)|^2 = 2\langle\delta(t), d\delta(t)\rangle + \mu^2\mathrm{Tr}(Q)dt.$$

This is equivalent to

$$\begin{aligned} &d|\delta|^2 + 2\nu\|\delta\|^2 \\ &= -2\langle B(\sigma, v), \delta\rangle dt - 2\mu(R_h(\delta), \delta)dt + 2\mu(dW(t), \delta) + \mu^2\mathrm{Tr}(Q)dt. \end{aligned} \tag{26}$$

Using Lemma 2.1, (8), (12) and Young's inequality, we get

$$\begin{aligned} -2b(\sigma, v, \delta) &\leq 2C_L\|\sigma\|^{1/2}|A\sigma|^{1/2}|v|\|\delta\| \\ &\leq 2C_L M_0\|\sigma\|^{1/2}|A\sigma|^{1/2}\|\delta\| \\ &\leq 2.2^{-1/4}C_L\alpha^{-3/2}M_0|\delta|\|\delta\| \\ &= \left(2^{1/4}C_L\alpha^{-3/2}\nu^{-1/2}M_0|\delta|\right)\left(2^{1/2}\nu^{1/2}\|\delta\|\right) \\ &\leq \frac{1}{2}\left(\frac{2^{1/2}C_L^2M_0^2}{\alpha^3\nu}|\delta|^2 + 2\nu\|\delta\|^2\right) \end{aligned} \tag{27}$$

$$= \frac{C_L^2 M_0^2}{2^{1/2}\nu\alpha^3}|\delta|^2 + \nu\|\delta\|^2.$$

By (2) and the Cauchy inequality, we obtain

$$\begin{aligned}
-2\mu(\delta, R_h(\delta)) &= -2\mu|\delta|^2 + 2\mu(\delta - R_h(\delta), \delta) \\
&\le -2\mu|\delta|^2 + 2\mu|\delta - R_h(\delta)||\delta| \\
&\le -2\mu|\delta|^2 + 2\mu\left(|\delta - R_h(\delta)|^2 + \frac{1}{4}|\delta|^2\right) \\
&= -2\mu|\delta|^2 + 2\mu|\delta - R_h(\delta)|^2 + \frac{\mu}{2}|\delta|^2 \\
&\le -\frac{3\mu}{2}|\delta|^2 + 2\mu c_1 h^2\|\delta\|^2 \\
&\le -\frac{3\mu}{2}|\delta|^2 + \nu\|\delta\|^2,
\end{aligned} \tag{28}$$

where we have used the fact that $2\mu c_1 h^2 \le \nu$ due to (23).

Therefore, from (26), (27) and (28) we deduce that

$$d|\delta|^2 + \left(\frac{3\mu}{2} - \frac{C_L^2 M_0^2}{2^{1/2}\nu\alpha^3}\right)|\delta|^2 dt \le 2\mu(dW(t), \delta) + \mu^2 \mathrm{Tr}(Q)dt.$$

Noting that $\frac{C_L^2 M_0^2}{2^{1/2}\nu\alpha^3} - \frac{\mu}{2} \le 0$ by (23) and integrating from t_0 to t, we obtain

$$|\delta(t)|^2 + \mu\int_{t_0}^t |\delta(s)|^2 ds \le |\delta(t_0)|^2 + 2\mu\int_{t_0}^t (\delta(s), dW(s)) + \int_{t_0}^t \mu^2 \mathrm{Tr}(Q)ds.$$

Taking the expected value, we obtain

$$\mathbb{E}(|\delta(t)|^2) + \mu\mathbb{E}\int_{t_0}^t |\delta(s)|^2 ds \le \mathbb{E}|\delta(t_0)|^2 + \mathbb{E}\mu^2\int_{t_0}^t \mathrm{Tr}(Q)ds,$$

and using the Gronwall inequality we get

$$\begin{aligned}
\mathbb{E}(|\delta(t)|^2) &\le \mathbb{E}(|\delta(0)|^2)e^{-\mu(t-t_0)} + \mu^2\mathrm{Tr}(Q)\int_{t_0}^t e^{-\mu(t-s)}ds \\
&= \mathbb{E}(|\delta(0)|^2)e^{-\mu(t-t_0)} + \mu\mathrm{Tr}(Q)\int_{t_0}^t e^{-\mu(t-s)}d(\mu(t-s)) \\
&\le \mathbb{E}(|\delta(0)|^2)e^{-\mu(t-t_0)} + \mu\mathrm{Tr}(Q).
\end{aligned}$$

Hence

$$\limsup_{t\to\infty} \mathbb{E}(|\delta(t)|^2) \le \mu\mathrm{Tr}(Q),$$

or

$$\limsup_{t\to\infty} \mathbb{E}(|z(t) - v(t)|^2) \le \mu\mathrm{Tr}(Q).$$

This proves (24).

In order to obtain the estimate (25), we now estimate

$$-2\langle B(\sigma, v), \delta\rangle \le 2C_L\|\sigma\|^{1/2}|A\sigma|^{1/2}|v|\|\delta\|$$

$$
\begin{aligned}
&\le C_L M_0 \|\sigma\|^{1/2} |A\sigma|^{1/2} \|\delta\| \\
&\le 2.2^{-1/4} C_L \alpha^{-3/2} M_0 |\delta| \|\delta\| \\
&= \left(2^{3/4} C_L \alpha^{-3/2} \nu^{-1/2} M_0 |\delta|\right) \left(\nu^{1/2} \|\delta\|\right) \\
&\le \frac{1}{2} \left[\frac{2^{3/2} C_L^2 M_0^2}{\alpha^3 \nu} |\delta|^2 + \nu \|\delta\|^2\right] \\
&= \frac{2^{1/2} C_L^2 M_0^2}{\nu \alpha^3} |\delta|^2 + \frac{\nu}{2} \|\delta\|^2,
\end{aligned}
$$

and

$$
\begin{aligned}
-2\mu(\delta, R_h(\delta)) &= -2\mu|\delta|^2 + 2\mu(\delta - R_h(\delta), \delta) \\
&\le -2\mu|\delta|^2 + 2\mu|\delta - R_h(\delta)||\delta| \\
&\le -2\mu|\delta|^2 + \mu\left(|\delta - R_h(\delta)|^2 + |\delta|^2\right) \\
&= -\mu|\delta|^2 + \mu|\delta - R_h(\delta)|^2 \\
&\le -\mu|\delta|^2 + \mu c_1 h^2 \|\delta\|^2 \\
&\le -\mu|\delta|^2 + \frac{\nu}{2} \|\delta\|^2.
\end{aligned}
$$

Thus, from (26) and by two estimates above, we get

$$
\begin{aligned}
&d|\delta|^2 + \nu\|\delta\|^2 dt \\
&\le \left(\frac{2^{1/2} C_L^2 M_0^2}{\nu \alpha^3} - \mu\right) |\delta|^2 dt + 2\mu(dW(t), \delta) + \mu^2 \mathrm{Tr}(Q) dt \\
&\le 2\mu(dW(t), \delta) + \mu^2 \mathrm{Tr}(Q) dt.
\end{aligned} \tag{29}
$$

By taking expected values and integrating from t to $t+T$ (29), we have

$$
\mathbb{E}(|\delta(t+T)|^2) + \nu \int_t^{t+T} \mathbb{E}(\|\delta(s)\|^2) ds \le \mathbb{E}(|\delta(t)|^2) + \mu^2 T \mathrm{Tr}(Q).
$$

Therefore,

$$
\limsup_{t\to\infty} \nu \int_t^{t+T} \mathbb{E}(\|\delta(s)\|^2) ds \le \mu \mathrm{Tr}(Q) + \mu^2 T \mathrm{Tr}(Q),
$$

and this is equivalent to

$$
\limsup_{t\to\infty} \nu \int_t^{t+T} \mathbb{E}(\|z(s) - v(s)\|^2) ds \le \mu \mathrm{Tr}(Q) + \mu^2 T \mathrm{Tr}(Q).
$$

This completes the proof. □

Corollary 3.3. *Suppose that the observational measurements are given by finite volume elements in* (16) *plus a noise term as in* (13), *where each* β_d *is* *an independent one-dimensional Brownian motion with variance* $\sigma^2/3$. *Interpolate these noisy observations using* (14) *where* ℓ_d *are given by* (17). *Let* $\mu = \frac{2^{1/2}C_L^2 M_0^2}{\nu\alpha^3}$, *and choose* $N = K^3$ *large enough such that*

$$h = \frac{L}{K} \leq \sqrt{\frac{\nu}{2c_1\mu}}.$$

Then the solution z *to the data assimilation equation* (11) *satisfies*

$$\limsup_{t\to\infty} \mathbb{E}\left(|z(t) - v(t)|^2\right) \leq \frac{k_1 \nu Gr^2 \sigma^2 L^4}{\alpha^3},$$

and

$$\limsup_{t\to\infty} \frac{\nu}{T}\int_t^{t+T} \mathbb{E}(\|z(s) - v(s)\|^2) ds \leq \left(\frac{k_1\nu Gr^2}{\alpha^3 T} + \frac{k_1^2\nu^2 Gr^4 L}{\alpha^6}\right)\sigma^2 L^4,$$

where $k_1 = \frac{2^{1/2}C_L^2}{\pi}$.

Proof. By Proposition 2.3 and the above choice of μ and h, we have

$$\mu \mathrm{Tr}(Q) \leq \mu\sigma^2 L^3 = \frac{2^{1/2}C_L^2 M_0^2}{\nu\alpha^3}\sigma^2 L^3 = \frac{2^{3/2}C_L^2 \nu Gr^2}{\alpha^3 \lambda_1^{1/2}}\sigma^2 L^3 = \frac{k_1\nu Gr^2\sigma^2 L^4}{\alpha^3},$$

and similarly, we also get that

$$\begin{aligned}
\left(\frac{\mu}{T} + \mu^2\right)\mathrm{Tr}(Q) &\leq \left(\frac{\mu}{T} + \mu^2\right)\sigma^2 L^3 \\
&= \left(\frac{2^{1/2}C_L^2 M_0^2}{\nu\alpha^3 T} + \frac{2C_L^4 M_0^4}{\nu^2\alpha^6}\right)\sigma^2 L^3 \\
&= \left(\frac{2^{1/2}C_L^2 \frac{2\nu^2 Gr^2}{\lambda_1^{1/2}}}{\nu\alpha^3 T} + \frac{2C_L^4 \frac{2^2\nu^4 Gr^4}{\lambda_1}}{\nu^2\alpha^6}\right)\sigma^2 L^3 \\
&= \left(\frac{2^{1/2}C_L^2 \nu Gr^2 L}{\alpha^3\pi T} + \frac{2C_L^4\nu^2 Gr^4 L^2}{\alpha^6\pi^2}\right)\sigma^2 L^3 \\
&= \left(\frac{k_1\nu Gr^2}{\alpha^3 T} + \frac{k_1^2\nu^2 Gr^4 L}{\alpha^6}\right)\sigma^2 L^4,
\end{aligned}$$

where we have used the fact that $\lambda_1 = 4\pi^2/L^2$ and $k_1 = \frac{2^{1/2}C_L^2}{\pi}$. $\square$

Remark 3.4. It is observed that the upper bound on the error given by Corollary 3.3 is independent of h. In particular, if we increase the observation density, there is no improvement in the quality of the approximation. This is not surprising since increasing the resolution of the observations did not lead to any decrease in the size of the measurement errors present in the interpolant observables $\tilde{R}_h$ given by (5). We remedy this defect in the following corollary.

Corollary 3.5. *Suppose that the observational measurements are given by finite volume elements in* (16) *plus a noise term as in* (13)*, where each β_d is an independent one-dimensional Brownian motion with variance $\sigma^2/3$. Let μ be as in Corollary 3.3 and $\epsilon \in (0,1)$. Then, there exists an interpolant observable based on volume elements with observation density h such that*

$$\frac{c' Gr^3}{L^3} \leq \frac{\epsilon}{h^3} \leq \frac{\max\{\epsilon, 64c' Gr^3\}}{L^3},$$

where $c' = \left(\frac{2^{5/2} c_1 C_L^2 L^3}{2\pi^{1/2}\alpha^3}\right)^{3/2}$, *and*

$$\limsup_{t\to\infty} \mathbb{E}(|z(t) - v(t)|^2) \leq \mu\sigma^2 L^3 \epsilon. \tag{30}$$

Proof. If $\sqrt{\nu/(2c_1\mu)} \geq L$, then we may take $h = L$ in Theorem 3.2. In this case v is a steady state and consequently no observational data is needed to accurately recover v. Otherwise, let $M = K_1^3$, where $K_1 \geq 3$ is the unique integer such that

$$h' = \frac{L}{K_1} \leq \sqrt{\frac{\nu}{2c_1\mu}} < \frac{L}{K_1 - 1}.$$

Let Q'_m be cubes with edges h', where $m = 1, \dots, M$. Choose $h = h'/q$, where q is the integer satisfying

$$q^3 \geq \frac{1}{\epsilon} > (q-1)^3. \tag{31}$$

With these choices of K_1 and q, we have

$$\sqrt{\frac{2c_1\mu}{\nu}} \leq \frac{1}{h'} = \frac{K_1}{L} \leq \frac{2(K_1-1)}{L} < 2\sqrt{\frac{2c_1\mu}{\nu}},$$

and

$$\epsilon^{-1/3} \leq q = (q-1) + 1 \leq \epsilon^{-1/3} + 1 \leq 2\epsilon^{-1/3},$$

or

$$\frac{\epsilon^{1/3}}{2} \leq \frac{1}{q} \leq \epsilon^{1/3}.$$

Thus,

$$\sqrt{\frac{\nu}{32c_1\mu}}\epsilon^{1/3} \leq h = \frac{h'}{q} \leq \sqrt{\frac{\nu}{2c_1\mu}}\epsilon^{1/3},$$

which is equivalent to

$$\left(\frac{2c_1\mu}{\nu}\right)^{3/2} \leq \frac{\epsilon}{h^3} \leq \left(\frac{32c_1\mu}{\nu}\right)^{3/2}.$$

By $\mu = \frac{2^{1/2} C_L^2 M_0^2}{\nu\alpha^3} = \frac{2^{3/2} C_L^2 \nu Gr^2}{\lambda_1^{1/2}\alpha^3}$, we arrive at

$$\left(\frac{2^{5/2} c_1 C_L^2 \nu Gr^2}{\nu \lambda_1^{1/2}\alpha^3}\right)^{3/2} \leq \frac{\epsilon}{h^3} \leq \left(\frac{16.2^{5/2} c_1 C_L^2 \nu Gr^2}{\nu \lambda_1^{1/2}\alpha^3}\right)^{3/2}.$$

Therefore, we obtain

$$\frac{c' Gr^3}{L^3} \le \frac{\epsilon}{h^3} \le \frac{64c' Gr^3}{L^3},$$

where $c' = \left(\frac{2^{5/2} c_1 C_L^2 L^3}{2\pi^{1/2}\alpha^3}\right)^{3/2}$.

Let Q_m be the cubes with edge h, where $m = 1, \dots, N$ and $N = L^3/h^3 = q^3 M$. Define the averaging operator $\mathcal{A} : \mathbb{R}^{3N} \to \mathbb{R}^{3M}$ by

$$\begin{bmatrix} \mathcal{A}(\varphi)_{3m-2} \\ \mathcal{A}(\varphi)_{3m-1} \\ \mathcal{A}(\varphi)_{3m} \end{bmatrix} = \frac{1}{q^3} \sum_{Q_n \subseteq Q'_m} \begin{bmatrix} \varphi_{3n-2} \\ \varphi_{3n-1} \\ \varphi_{3n} \end{bmatrix} \quad \text{for } m = 1, 2, \dots, M.$$

We note that $\mathcal{O}'_h = \mathcal{A} \circ \mathcal{O}_h$, where $\mathcal{O}_h$ are the noise-free observations of volume elements given in (16), for the Q_n and $\mathcal{O}_{h'}$ are analogous observations for Q'_m.

Let $\tilde{\mathcal{O}}_h(v(t))$ be the noisy observations defined by (4), where $\mathcal{E}(t)$ is given by (13). It follows that $\mathcal{A} \circ \tilde{\mathcal{O}}_h(v(t)) = \mathcal{O}_{h'}(v(t)) + \mathcal{F}(t)$, where

$$\mathcal{F}(t)dt = (d\beta_1(t), d\beta_2(t), \dots, d\beta_{3M}(t))$$

and β_j's are one-dimensional independent Brownian motions such that

$$E(\beta_j(t)) = 0 \quad \text{and} \quad \mathcal{E}(\beta_j^2(t)) = t\frac{\sigma^2}{3q^3}, \quad j = 1, 2, \dots, 3M.$$

Therefore, by taking averages of volume elements we have reduced the variance in the noise term of the measurements. In particular, from (31), the noise term is equivalent to an $[\dot{L}^2]^3$-valued Q'-Brownian motion with

$$\mathrm{Tr}(Q') \le \sigma^2 L^3/q^3 \le \sigma^2 L^3 \epsilon.$$

We now define the interpolant observable

$$R_{h'} = \mathcal{L}_{h'} \circ \mathcal{A} \circ \mathcal{O}_h.$$

Since $R_{h'}$ satisfies (2) with the same constants as before, applying Theorem 3.2 we get (30) as desired. □

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