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FIXED POINT THEOREMS FOR GENERALIZED G-METRIC SPACES[†]

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Abstract. A multi-dimensional metric, called a g-metric, as a generalization of the G-metric was introduced. We establish some well-known fixed point theorems in the frame work of g-metric spaces.

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1. Introduction

A metric is an important notion not only in mathematics but also in various other scientific fields. An ordinary metric is required to be generalized in order to improve the performance of clustering, classification, and information retrieval processes and also to be able to handle large, complex data sets effectively.

An ordinary metric assigns given two points in a space to a non-negative real number that represents how far they are. It is natural to generalize the ordinary metric in order to measure the distance between three or more points. Many authors, including Gahler [8] and Dhage [6], have been investigating generalizations of ordinary metrics in this context. Mustafa and Sims [11] have come up with a more general metric between three points, called a *G-metric*, that allows fundamental topological properties to be well embodied. See [9] for more details. The G-metric was generalized to a metric between n points, called a *g-metric* [4] in order to analyze complex high-dimensional data sets, such as grouped multivariate data.

Definition 1.1. [4] Let $\mathbb{R}_+$ be the set of all non-negative real numbers.

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(1) For a non-empty set Ω, a map $g : \Omega^n (= \prod_{i=1}^n \Omega) \longrightarrow \mathbb{R}_+$ is called a *g-metric with dimension n* ($n \geq 2$) on Ω if it satisfies the following conditions:
   ($g1$) (positive definiteness) $g(x_1, \dots, x_n) = 0$ if and only if $x_1 = \cdots = x_n$,
   ($g2$) (permutation invariancy) $g(x_1, \dots, x_n) = g(x_{\sigma(1)}, \dots, x_{\sigma(n)})$ for any permutation σ on the set $\{1, \dots, n\}$,
   ($g3$) (monotonicity) $g(x_1, \dots, x_n) \leq g(y_1, \dots, y_n)$ for all $(x_1, \dots, x_n)$, $(y_1, \dots, y_n) \in \Omega^n$ with $\{x_i \mid i = 1, \dots, n\} \subsetneq \{y_i \mid i = 1, \dots, n\}$,
   ($g4$) (triangle inequality) for all $x_1, \dots, x_s, y_1, \dots, y_t, w \in \Omega$ with $s + t = n$

$$g(x_1, \dots, x_s, y_1, \dots, y_t) \leq g(x_1, \dots, x_s, \underbrace{w, \dots, w}_{n-s \text{ times}}) + g(y_1, \dots, y_t, \underbrace{w, \dots, w}_{n-t \text{ times}}).$$

   We call the pair (Ω, g) a *g-metric space*.

(2) A g-metric on a non-empty set Ω is said to be *multiplicity-independent* if the following holds

$$g(x_1, \dots, x_n) = g(y_1, \dots, y_n)$$

   for all $(x_1, \dots, x_n), (y_1, \dots, y_n) \in \Omega^n$ with $\{x_i \mid i = 1, \dots, n\} = \{y_i \mid i = 1, \dots, n\}$.

The following are some basic examples of g-metrics.

Example 1.2. [4] Let Ω be a non-empty set, and let (Ω, δ) be a given ordinary metric space.

(1) The *discrete g-metric* is the map $d : \Omega^n \to \mathbb{R}_+$ defined by

$$d(x_1, \dots, x_n) = \begin{cases} 0 & \text{if } x_1 = \cdots = x_n, \\ 1 & \text{otherwise} \end{cases}$$

   for all $x_1, \dots, x_n \in \Omega$.

(2) The *diameter g-metric* is the map $d : \mathbb{R}_+^n \longrightarrow \mathbb{R}_+$ given by

$$d(x_1, \dots, x_n) = \max_{1 \leq i \leq n} x_i - \min_{1 \leq j \leq n} x_j$$

   for all $x_1, \dots, x_n \in \mathbb{R}_+$.

(3) The *max g-metric* is the map $d : \Omega^n \longrightarrow \mathbb{R}_+$ defined by

$$d(x_1, \dots, x_n) = \max_{1 \leq i, j \leq n} \delta(x_i, x_j)$$

   for all $x_1, \dots, x_n \in \Omega$.

(4) The *shortest path g-metric* is the map $d : \Omega^n \longrightarrow \mathbb{R}_+$ given by

$$d(x_1, \dots, x_n) = \min_{\pi \in S_n} \sum_{i=1}^{n-1} \delta(x_{\pi(i)}, x_{\pi(i+1)})$$

   for all $x_1, \dots, x_n \in \Omega$, where S_n is the symmetric group on the set $\{1, \dots, n\}$.

Note that $d(x_1, \dots, x_n)$ is the length of the shortest path connecting $x_1, \dots, x_n$, which is an important notion in computer science and operational research. See [12] for details.

In this paper, we generalize some well-known fixed point theorems such as the Banach contraction mapping principle, the weak contraction mapping principle, and the Ćirić fixed point theorem in a g-metric space.

2. Preliminaries

For a given g-metric space (Ω, g), the *open ball centered at $x \in \Omega$ with radius $r > 0$* is

$$B_g(x, r) = \{y \in \Omega \mid g(x, y, \dots, y) < r\}.$$

It was shown in [4] that the collection $\mathcal{B} = \{B_g(x, r) \mid x \in \Omega, r > 0\}$ forms a basis for a topology on Ω. The topology generated by the basis $\mathcal{B}$ is called the *g-metric topology* on Ω.

Lemma 2.1. [4] *Let g be a g-metric with dimension n on a non-empty set Ω. Then the following statements hold:*

(1) $g(\underbrace{x, \dots, x}_{s\ times}, y, \dots, y) \le g(\underbrace{x, \dots, x}_{s\ times}, w, \dots, w) + g(\underbrace{w, \dots, w}_{s\ times}, y, \dots, y)$,

(2) $g(x, y, \dots, y) \le g(x, w, \dots, w) + g(w, y, \dots, y)$,

(3) $g(\underbrace{x, \dots, x}_{s\ times}, w, \dots, w) \le sg(x, w, \dots, w)$ *and*

$g(\underbrace{x, \dots, x}_{s\ times}, w, \dots, w) \le (n - s)g(w, x, \dots, x)$,

(4) $g(x_1, x_2, \dots, x_n) \le \displaystyle\sum_{i=1}^{n} g(x_i, w, \dots, w)$,

(5) $\big|g(y, x_2, \dots, x_n) - g(w, x_2, \dots, x_n)\big| \le \max\{g(y, w, \dots, w), g(w, y, \dots, y)\}$,

(6) $\big|g(\underbrace{x, \dots, x}_{s\ times}, w, \dots, w) - g(\underbrace{x, \dots, x}_{\tilde{s}\ times}, w, \dots, w)\big| \le \big|s - \tilde{s}\big| g(x, w, \dots, w)$,

(7) $g(x, w, \dots, w) \le (1 + (s - 1)(n - s))g(\underbrace{x, \dots, x}_{s\ times}, w, \dots, w)$.

Definition 2.2. [4] Let (Ω, g) be a g-metric space. Let $x \in \Omega$ be a point and $\{x_k\} \subseteq \Omega$ be a sequence.

(1) $\{x_k\}$ *converges* to x, denoted by $\{x_k\} \xrightarrow{g} x$, if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$i_1, \dots, i_{n-1} \ge N \Longrightarrow g(x, x_{i_1}, \dots, x_{i_{n-1}}) < \varepsilon.$$

For such a case, $\{x_k\}$ is said to be *convergent* in Ω and x is called the *limit* of $\{x_k\}$.

(2) $\{x_k\}$ is said to be *Cauchy* if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$i_1, \dots, i_n \ge N \Longrightarrow g(x_{i_1}, \dots, x_{i_n}) < \varepsilon.$$

(3) (Ω, g) is *complete* if every Cauchy sequence in (Ω, g) is convergent in (Ω, g).

Lemma 2.3. [4] *Let (Ω, g) be a g-metric space. Let $\{x_k\} \subseteq \Omega$ be a sequence and $x \in \Omega$. Then the following statements are equivalent:*

(1) $\{x_k\} \xrightarrow{g} x$.
(2) *For a given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $x_k \in B_g(x, \varepsilon)$ for all $k \geq N$.*
(3) $\lim_{k_1,\ldots,k_s \to \infty} g(\underbrace{x_{k_1}, \ldots, x_{k_s}}_{s\ times}, x, \ldots, x) = 0$ *for a fixed $1 \leq s \leq n-1$. That is, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $k_1, \ldots, k_s \geq N \Longrightarrow g(x_{k_1}, \ldots, x_{k_s}, x, \ldots, x) < \varepsilon$.*

Lemma 2.4. [4] *Let (Ω, g) be a g-metric space. Let $\{x_k\} \subseteq \Omega$ be a sequence. Then the following statements are equivalent:*

(1) *$\{x_k\}$ is Cauchy.*
(2) $g(x_k, x_{k+1}, x_{k+1}, \ldots, x_{k+1}) \longrightarrow 0$ *as $k \longrightarrow \infty$.*
(3) $\lim_{k,\ell \to \infty} g(\underbrace{x_k, \ldots, x_k}_{s\ times}, x_\ell, \ldots, x_\ell) = 0$ *for a fixed $1 \leq s \leq n-1$.*

A g-metric space (Ω, g) is said to *have the fixed point property* if every continuous map $T : \Omega \longrightarrow \Omega$ has a fixed point.

Proposition 2.5. *The fixed point property is a topological invariant.*

Proof. Let (Ω_1, g_1) and (Ω_2, g_2) be g-metric spaces, and let $h : \Omega_1 \longrightarrow \Omega_2$ be a homeomorphism. Suppose that Ω_1 has the fixed point property.
Let $\widetilde{T} : \Omega_2 \longrightarrow \Omega_2$ be a continuous map. We consider the map $T : \Omega_1 \longrightarrow \Omega_1$ given by $T(x) = (h^{-1} \circ \widetilde{T} \circ h)(x)$. Since Ω_1 has the fixed point property and T is continuous, there exists a fixed point $x \in \Omega_1$ under T, i.e., $T(x) = x$. Denote $h(x)$ by y. Then we have

$$\widetilde{T}(y) = \widetilde{T}(h(x)) = (h \circ h^{-1} \circ \widetilde{T} \circ h)(x) = h(T(x)) = h(x) = y,$$

implying that y is a fixed point under $\widetilde{T}$. Therefore, Ω_2 has the fixed point property. □

Lemma 2.6. *If (Ω, g) is a g-metric space, then the map g is jointly continuous in all n variables, i.e., if for each $i = 1, \ldots, n$, $\{x_i^{(k)}\}_{k \in \mathbb{N}}$ is a sequence in Ω such that $\{x_i^{(k)}\} \xrightarrow{g} x_i$, then $\{g(x_1^{(k)}, \ldots, x_n^{(k)})\} \longrightarrow \{g(x_1, \ldots, x_n)\}$ as $k \longrightarrow \infty$.*

Proof. Assume that $\{x_i^{(k)}\} \xrightarrow{g} x_i$ as $k \longrightarrow \infty$ for each $i = 1, \ldots, n$. For a given $\varepsilon > 0$, there exists $N_i \in \mathbb{N}$ such that $g(x_i^{(k)}, x_i, \ldots, x_i) < \dfrac{\varepsilon}{n}$ if $k \geq N_i$ by Lemma 2.3 (3). We let $N = \max\{N_1, \ldots, N_n\}$. Then by the conditions $(g2), (g4)$, if $k \geq N$, then

$$g(x_1^{(k)}, \ldots, x_n^{(k)}) \leq \sum_{i=1}^{n} g(x_i^{(k)}, x_i, \ldots, x_i) + g(x_1, x_2, \ldots, x_n) < \varepsilon + g(x_1, x_2, \ldots, x_n).$$

In a similar way, we have $g(x_1, x_2, \ldots, x_n) < \varepsilon + g(x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)})$.

$\square$

3. Fixed point theorems in a g-metric space

Fixed point theorems on a G-metric space have extensively been studied (see [2] and references therein). The interested reader can also refer to [1, 3, 7, 10]. In this section we generalize several fixed point theorems on a g-metric space under the g-metric topology.

The following is a generalization of the Banach contractive mapping principle in a g-metric space.

Theorem 3.1 (Banach contractive mapping principle in a g-metric space)**.** *Let (Ω, g) be a complete g-metric space and let $T : \Omega \longrightarrow \Omega$ be a map such that there exists $\lambda \in [0, 1)$ satisfying*

$$g(T(x_1), T(x_2), \ldots, T(x_n)) \leq \lambda g(x_1, x_2, \ldots, x_n) \quad \textit{for all } x_1, \ldots, x_n \in \Omega.$$

Then T has a unique fixed point in Ω.

Proof. Let y_0 be an arbitrary point in Ω. Set $y_{k+1} = T(y_k)$ for all $k \in \mathbb{N}$.

(Existence of a fixed point) If $y_{m+1} = y_m$ for some $m \in \mathbb{N}$, then y_m is a fixed point of T. We assume that $y_{k+1} \neq y_k$ for all $k \in \mathbb{N}$. Then, by the condition (3.1) it follows that

$$g(y_{k+1}, y_{k+2}, y_{k+2}, \ldots, y_{k+2}) \leq \lambda g(y_k, y_{k+1}, y_{k+1}, \ldots, y_{k+1}) \quad \text{for all } k \in \mathbb{N}.$$

So, by induction we have $g(y_k, y_{k+1}, y_{k+1}, \ldots, y_{k+1}) \leq \lambda^k g(y_0, y_1, y_1, \ldots, y_1)$, implying

$$g(y_k, y_{k+1}, y_{k+1}, \ldots, y_{k+1}) \longrightarrow 0 \quad \text{as } k \longrightarrow \infty.$$

Thus, $\{y_k\}$ is a Cauchy sequence in (Ω, g) by Lemma 2.4. Since (Ω, g) is complete, there exists $y \in \Omega$ such that $\{y_k\} \xrightarrow{g} y$. It follows that

$$g(y_{k+1}, T(y), T(y), \ldots, T(y)) \leq \lambda g(y_k, y, y, \ldots, y).$$

As $k \longrightarrow \infty$, by Lemma 2.6

$$g(y, T(y), T(y), \ldots, T(y)) \leq \lambda g(y, y, y, \ldots, y) = 0.$$

Therefore, $T(y) = y$ by the positive definiteness for the g-metric.

(Uniquness of a fixed point) Suppose that $y, \tilde{y}$ are distinct fixed points. Then

$$\begin{aligned} g(\tilde{y}, y, y, \ldots, y) &= g(T(\tilde{y}), T(y), T(y), \ldots, T(y)) \\ &\leq \lambda g(\tilde{y}, y, y, \ldots, y) < g(\tilde{y}, y, y, \ldots, y), \end{aligned}$$

which is a contradiction. Thus, $y = \tilde{y}$. $\square$

Definition 3.2. Let (Ω, g) be a g-metric space. A map $T : \Omega \to \Omega$ is said to be *weakly contractive* if

$$g(T(x_1), \ldots, T(x_n)) < g(x_1, \ldots, x_n)$$

for which any two of $x_1, \ldots, x_n \in \Omega$ are distinct.

Proposition 3.3. *Let (Ω, g) be a g-metric space. Suppose that $T : \Omega \longrightarrow \Omega$ is weakly contractive. Then the map $f : \Omega \longrightarrow \mathbb{R}_+$ given by $f(x) = g(x, T(x), \ldots, T(x))$ is continuous.*

Proof. Let $x \in \Omega$. We need to show that for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\left|g(y, T(y), \ldots, T(y)) - g(x, T(x), \ldots, T(x))\right| < \varepsilon$ if $y \in B_g(x, \delta)$. We let $\delta = \dfrac{\varepsilon}{n}$. For $y \in B_g(x, \delta)$, we first assume that $g(y, T(y), \ldots, T(y)) \leq g(x, T(x), \ldots, T(x))$. Then

$$
\begin{aligned}
&|g(x, T(x), \ldots, T(x)) - g(y, T(y), \ldots, T(y))| \\
&= g(x, T(x), \ldots, T(x)) - g(y, T(y), \ldots, T(y)) \\
&\leq g(x, y, \ldots, y) + g(y, T(x), \ldots, T(x)) - g(y, T(y), \ldots, T(y)) \text{ (by Lemma 2.1 (2))} \\
&\leq g(x, y, \ldots, y) + g(T(y), T(x), \ldots, T(x)) \quad \text{(by Lemma 2.1 (2))} \\
&\leq g(x, y, \ldots, y) + g(y, x, \ldots, x) \quad \text{(by the weak contractivity of } T\text{)} \\
&\leq g(x, y, \ldots, y) + (n-1)g(x, y, \ldots, y) \quad \text{(by Lemma 2.1 (3))} \\
&< n\delta = \varepsilon.
\end{aligned}
$$

In a similar way, it can be proved that $|g(x, T(x), \ldots, T(x)) - g(y, T(y), \ldots, T(y))| < \varepsilon$ holds when $g(y, T(y), \ldots, T(y)) \geq g(x, T(x), \ldots, T(x))$. Hence, f is continuous. □

Theorem 3.4 (Weak contraction mapping principle in a g-metric space)**.** *Let T be a weakly contractive map on a compact g-metric space (Ω, g). Then T has a unique fixed point.*

Proof. The map $f : \Omega \longrightarrow \mathbb{R}_+$ defined by $f(x) = g(x, T(x), \ldots, T(x))$ is continuous by Proposition 3.3. Since Ω is compact, the continuous map f attains its minimum at some $\bar{x} \in \Omega$. If $\bar{x} \neq T(\bar{x})$, then

$$
\begin{aligned}
g(\bar{x}, T(\bar{x}), \ldots, T(\bar{x})) &= \min_{x \in \Omega} g(x, T(x), \ldots, T(x)) \\
&\leq g(T(\bar{x}), T(T(\bar{x})), \ldots, T(T(\bar{x}))) \\
&< g(\bar{x}, T(\bar{x}), \ldots, T(\bar{x})),
\end{aligned}
$$

which is a contradiction. So, $\bar{x}$ is a fixed point of T. The uniqueness argument follows exactly same as in the proof of Theorem 3.1. □

We next generalize the Ćirić fixed point theorem [5] in a g-metric space. Let (Ω, g) be a g-metric space and $T : \Omega \longrightarrow \Omega$ a map. For each $x \in \Omega$, we denote $O(x, N) = \{x,\ T(x),\ T^2(x), \ldots,\ T^N(x)\}$ and $O(x, \infty) = \{x,\ T(x),\ T^2(x), \ldots\}$, where $T^{k+1} = T \circ T^k$ for all $k \in \mathbb{N}$ and T^0 is the identity map on Ω.

Definition 3.5. (1) A g-metric space Ω is said to be *T-orbitally complete* if every Cauchy sequence contained in $O(x, \infty)$ for some $x \in \Omega$ is convergent in Ω.

(2) A map $T:\Omega\longrightarrow\Omega$ is called a *quasi-contraction* if there exists $\lambda\in[0,1)$ such that for all $x_1,\ldots,x_n\in\Omega$,

$$g(T(x_1),\ldots,T(x_n))\leq\frac{\lambda}{n}\max\Big[\{g(x_1,\ldots,x_n)\}\\ \cup\{g(x_i,T(x_j),\ldots,T(x_j))\mid i,j=1,\ldots,n\}\Big].$$

For $A\subseteq\Omega$, we denote $\sup\{g(a_1,\ldots,a_n)\mid a_1,\ldots,a_n\in A\}$ by $\mu(A)$.

Lemma 3.6. *Suppose that $T:\Omega\longrightarrow\Omega$ is a quasi-contraction on a g-metric space (Ω,g). Then for each $x\in\Omega$ the following inequalities hold:*

(1) $g(T^{k_1}(x),\ldots,T^{k_n}(x))\leq\dfrac{\lambda}{n}\mu(O(x,N))$ *for all* $k_1,\ldots,k_n\in\{1,\ldots,N\}$.

(2) $\mu(O(x,\infty))\leq\dfrac{n}{1-\lambda}g(x,T(x),\ldots,T(x))$.

Proof. (1) Let $x\in\Omega$. Since $\{T^{k_1}(x),T^{k_1-1}(x),\ldots,T^{k_n}(x),T^{k_n-1}(x)\}$ is a subset of $O(x,N)$ and the map T is a quasi-contraction, there exists $\lambda\in[0,1)$ such that

$$\begin{aligned}g(T^{k_1}(x),\ldots,T^{k_n}(x))&=g(TT^{k_1-1}(x),\ldots,TT^{k_n-1}(x))\\&\leq\frac{\lambda}{n}\max\Big[\{g(T^{k_1-1}(x),\ldots,T^{k_n-1}(x))\}\\&\qquad\cup\{g(T^{k_i-1}(x),T^{k_j}(x),\ldots,T^{k_j}(x))\mid i,j=1,\ldots,n\}\Big]\\&\leq\frac{\lambda}{n}\mu(O(x,N)).\end{aligned}$$

(2) Let $x\in\Omega$. Since the sequence $\{\mu(O(x,N))\}_{N\in\mathbb{N}}$ is monotonically increasing, $\mu(O(x,\infty))=\sup\{\mu(O(x,N))\mid N\in\mathbb{N}\}$. For a fixed positive integer N_0, the statement (1) implies that there exist $k_1,k_2,\ldots,k_{n-1}\in\{0,1,\ldots,N_0\}$ such that $g(x,T^{k_1}(x),\ldots,T^{k_{n-1}}(x))=\mu(O(x,N_0))$. Without loss of generality, we can assume that $k_1\leq k_2\leq\cdots\leq k_{n-1}$. If $k_{n-1}=0$ (i.e. $k_i=0$ for all i), then $\mu(O(x,N_0))=g(x,x,\ldots,x)=0$. Suppose that there exists $1\leq j\leq n-1$ such that $k_j\neq0$ and $k_{j-1}=0$. Then by Lemma 2.1 (4) and the statement (1) we have

$$\begin{aligned}g(x,T^{k_1}(x),\ldots,T^{k_{n-1}}(x))&\leq g(x,T(x),\ldots,T(x))+\sum_{i=1}^{n-1}g(T^{k_i}(x),T(x),\ldots,T(x))\\&=jg(x,T(x),\ldots,T(x))+\sum_{i=j}^{n-1}g(T^{k_i}(x),T(x),\ldots,T(x))\\&\leq jg(x,T(x),\ldots,T(x))+(n-j)\frac{\lambda}{n}\mu(O(x,N_0))\\&=jg(x,T(x),\ldots,T(x))+(n-j)\frac{\lambda}{n}g(x,T^{k_1}(x),\ldots,T^{k_{n-1}}(x)).\end{aligned}$$

Thus, it follows that

$$\begin{aligned}\mu(O(x,N_0)) &= g(x,T^{k_1}(x),\dots,T^{k_{n-1}}(x))\\ &\le \frac{j}{1-\frac{n-j}{n}\lambda}g(x,T(x),\dots,T(x))\\ &\le \frac{n}{1-\lambda}g(x,T(x),\dots,T(x)).\end{aligned}$$

Since N_0 was arbitrarily chosen, $\mu(O(x,\infty)) \le \dfrac{n}{1-\lambda}g(x,T(x),\dots,T(x))$. □

Theorem 3.7 (Ćirić fixed point theorem in a g-metric space)**.** *Let Ω be a g-metric space. Suppose that Ω is T-orbitally complete and $T : \Omega \longrightarrow \Omega$ is a quasi-contraction. Then the following are true:*

(1) $\{T^N(x)\} \xrightarrow{g} y$ *as* $N \longrightarrow \infty$.
(2) T *has a unique fixed point* y *in* Ω.
(3) $g(T^N(x),y,\dots,y) \le \dfrac{\lambda^N}{n^{N-1}(1-\lambda)}g(x,T(x),\dots,T(x))$.

Proof. (1) Let $x \in \Omega$. Since T is a quasi-contraction, by Lemma 3.6 (1) it follows that

$$\begin{aligned}g(T^{k_1}(x),\dots,T^{k_n}(x)) &= g(TT^{k_1-1}(x),T^{k_2-k_1+1}T^{k_1-1}(x),\dots,T^{k_n-k_1+1}T^{k_1-1}(x))\\ &\le \frac{\lambda}{n}\mu(O(T^{k_1-1}(x),k_n-k_1+1))\end{aligned}$$

for positive integers $k_1,k_2,\dots,k_n$ with $k_1 < k_2 < \cdots < k_n$. By Lemma 3.6 (1), there exist $\ell_1,\dots,\ell_{n-1} \in \{0,\dots,k_n-k_1+1\}$ (without loss of generality, we assume that $\ell_1 \le \cdots \le \ell_{n-1}$) such that

$$\mu(O(T^{k_1-1}(x),k_n-k_1+1)) = g(T^{k_1-1}(x),T^{l_1}T^{k_1-1}(x),\dots,T^{l_{n-1}}T^{k_1-1}(x)).$$

Then by Lemma 3.6 (1), we have

$$\begin{aligned}&g(T^{k_1-1}(x),T^{\ell_1}T^{k_1-1}(x),\dots,T^{\ell_{n-1}}T^{k_1-1}(x))\\ &\quad= g(TT^{k_1-2}(x),T^{\ell_1+1}T^{k_1-2}(x),\dots,T^{\ell_{n-1}+1}T^{k_1-2}(x))\\ &\quad\le \frac{\lambda}{n}\mu(O(T^{k_1-2}(x),\ell_{n-1}+1)) \le \frac{\lambda}{n}\mu(O(T^{k_1-2}(x),k_n-k_1+2)).\end{aligned}$$

By repeating process, we eventually obtain the following inequalities:

$$\begin{aligned}g(T^{k_1}(x),\dots,T^{k_n}(x)) &\le \frac{\lambda}{n}\mu(O(T^{k_1-1}(x),k_n-k_1+1))\\ &\le \left(\frac{\lambda}{n}\right)^2\mu(O(T^{k_1-2}(x),k_n-k_1+2))\\ &\ \ \vdots\\ &\le \left(\frac{\lambda}{n}\right)^{k_1}\mu(O(x,k_n)).\end{aligned}$$

Then it follows from Lemma 3.6 (2) that

$$g(T^{k_1}(x),\ldots,T^{k_n}(x)) \le \left(\frac{\lambda}{n}\right)^{k_1}\frac{n}{1-\lambda}g(x,T(x),\ldots,T(x)). \qquad (\star)$$

The sequence of iterates $\{T^N(x)\}$ is Cauchy because $\left(\frac{\lambda}{n}\right)^{k_1}$ tends to 0 as $k_1 \longrightarrow \infty$. Therefore, since Ω is T-orbitally complete, $\{T^N(x)\}$ has the limit y in Ω.

(2) (**Existence of a fixed point**) We shall show that the limit y is a fixed point of T. Let us consider the following inequalities:

$$\begin{aligned}
g(y,T(y),\ldots,T(y)) &\le g(y,T^{N+1}(y),\ldots,T^{N+1}(y)) + g(TT^N(y),T(y),\ldots,T(y))\\
&\le g(y,T^{N+1}(y),\ldots,T^{N+1}(y)) + \frac{\lambda}{n}\max\Big\{g(T^N(y),y,\ldots,y),\\
&\qquad g(T^N(y),T^{N+1}(y),\ldots,T^{N+1}(y)),\ g(y,T(y),\ldots,T(y)),\\
&\qquad g(T^N(y),T(y),\ldots,T(y)),\ g(y,T^{N+1}(y),\ldots,T^{N+1}(y))\Big\}\\
&\le g(y,T^{N+1}(y),\ldots,T^{N+1}(y)) + \frac{\lambda}{n}\Big(g(T^N(y),y,\ldots,y)\\
&\quad + g(T^N(y),T^{N+1}(y),\ldots,T^{N+1}(y)) \ + g(y,T(y),\ldots,T(y))\\
&\quad + g(y,T^{N+1}(y),\ldots,T^{N+1}(y))\Big) \quad \text{(by Theorem 2.1 (2))}.
\end{aligned}$$

Then for every positive integer N, we have

$$\begin{aligned}
&g(y,T(y),\ldots,T(y))\\
&\le \frac{\lambda}{n-\lambda}\Big[g(T^N(y),y,\ldots,y) + g(T^N(y),T^{N+1}(y),\ldots,T^{N+1}(y))\\
&\qquad + \Big(\frac{n}{\lambda}+1\Big)g(y,T^{N+1}(y),\ldots,T^{N+1}(y))\Big].
\end{aligned}$$

Note that for any $x\in\Omega$, $\{T^N(x)\} \xrightarrow{g} y$. Thus, $g(y,T(y),\ldots,T(y))=0$, i.e., $T(y)=y$. Therefore, y is a fixed point of T.

(**Uniqueness of a fixed point**) Suppose that y and $\widetilde{y}$ are fixed points under T, i.e., $T(y)=y$ and $T(\widetilde{y})=\widetilde{y}$. The quasi-contractivity of T gives rise to the following:

$$\begin{aligned}
g(\widetilde{y},y,\ldots,y) &= g(T(\widetilde{y}),T(y),\ldots,T(y))\\
&\le \frac{\lambda}{n}\max\Big\{g(\widetilde{y},y,\ldots,y), g(\widetilde{y},T(\widetilde{y}),\ldots,T(\widetilde{y})), g(y,T(y),\ldots,T(y)),\\
&\qquad\qquad g(\widetilde{y},T(y),\ldots,T(y)), g(y,T(\widetilde{y}),\ldots,T(\widetilde{y}))\Big\}\\
&\le \frac{\lambda}{n}\max\Big\{g(\widetilde{y},y,\ldots,y), g(y,\widetilde{y},\ldots,\widetilde{y})\Big\}\\
&\le \frac{\lambda}{n}\max\Big\{g(\widetilde{y},y,\ldots,y), (n-1)g(\widetilde{y},y,\ldots,y)\Big\} \quad \text{(by Theorem 2.1 (3))}
\end{aligned}$$

$$= \frac{\lambda}{n}(n-1)g(\widetilde{y}, y, \ldots, y) \leq \lambda g(\widetilde{y}, y, \ldots, y).$$

Since $0 \leq \lambda < 1$, it holds that $g(\widetilde{y}, y, \ldots, y) = 0$. Therefore, $y = \widetilde{y}$ as desired.

(3) Taking the limit as $k_2 \longrightarrow \infty$ on the both side of $(\star)$, one can obtain the inequality

$$g(T^{k_1}(x), y, \ldots, y) \leq \left(\frac{\lambda}{n}\right)^{k_1} \left(\frac{n}{1-\lambda}\right) g(x, T(x), \ldots, T(x)),$$

as desired.

□

Conflicts of interest : The authors declare no conflict of interest.

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