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GEOMETRY OF LOCALLY PROJECTIVELY FLAT FINSLER SPACE WITH CERTAIN (α,β)-METRIC

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ABSTRACT. In view of solution to the Hilbert fourth problem, the present study engages to investigate the projectively flat special (α,β)-metric and the generalised first approximate Matsumoto (α,β)-metric, where α is a Riemannian metric and β is a differential one-form. Further, we concluded that α is locally Projectively flat and have β is parallel with respect to α for both the metrics. Also, we obtained necessary and sufficient conditions for the aforementioned metrics to be locally projectively flat.

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1. Introduction

The standard case of Hilbert's fourth problem is to characterise Finsler metrics on an open subset of $\mathbb{R}^n$ with positive geodesics that are straight lines. In $\mathbb{R}^n$, such Finsler metrics are known as locally projectively flat Finsler metrics. Beltrami addressed this problem in Riemannian geometry by stating that a Riemannian metric is locally projectively flat if and only if it has constant sectional curvature. This question becomes significantly more problematic when using Finsler metrics. The flag curvature in Finsler geometry is a natural generalisation of the sectional curvature in the Riemannian case, and locally projectively flat Finsler metrics must be of scalar flag curvature, that is, the flag curvature is a scalar function on the tangent bundle, which may or may not be a constant as in the Riemannian case [2]. Locally projectively flat Finsler metrics are an unique class of Finsler metrics. This class of metrics has yet to be classified. However, some improvement has been made in recent years ([5], [13]) under some limitations on Finsler metrics.

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If any geodesic in (M, F) stays to be a geodesic in (M, F) then a change $F \to \bar{F}$ of a Finsler metric on the same underlying manifold M is called projective. If a Finsler space is projective to a locally Minkowski space then it is said to be projectively flat. For a general (α, β)-metric $F = \alpha\phi\left(\frac{\beta}{\alpha}\right)$, if β is parallel with respect to α then F is locally projectively flat if and only if α is locally projectively flat. Suppose if β is not parallel with respect to α, Z. Shen [11] has given an equivalent definition for locally projectively flat (α, β)-metrics when β is not parallel with respect to α. Based on this results C. Yu completely characterized locally projectively flat (α, β)-metric by β-deformations [15]. S.K. Narasimhamurthy et al.[7] examined the condition for a Finsler space with (α, β)-metric to be projectively flat on the basis of Matsumoto's results. Yali Feng et al.[3] have discussed the equivalent characterization of locally projectively flat general (α, β)-metric on an $(n \geq 3)$-dim manifold. Ravindra Yadav et al.[14] discussed projective flatness of Finsler space with $L = \frac{\alpha^2}{\alpha - \beta} + \beta$ and $L = \alpha + \frac{\beta^{n+1}}{\alpha^n}$. Manoj Kumar [4] independently studied Matsumoto change of m-th root metric is locally projectively flat if and only if it is locally Minkowskian. Achal Singh et al.[12] proved necessary and sufficient condition of locally projectively flatness for generalized Kropina conformal change of m-th root metric. In the recent years several authors [1, 6, 8, 9, 10] have studied the projective relation and projective flat concept for different types of (α, β)-metrics.

In this article, we have focused locally projectively flat on special (α, β)-metric $F = \mu_1\alpha + \mu_2\beta + \mu_3\frac{\alpha^2}{\beta}$ and generalised (α, β)-metric $F = \mu_1\alpha + \mu_2\beta + \mu_3\frac{\beta^2}{\alpha}$, where μ_1, μ_2 and μ_3 are constants. Firstly, we have given a brief introduction to locally projectively flat in the section one. Section two covered the fundamental notations and conditions that must be met for a Finsler space F^n with an (α, β)-metric to be a projectively flat. Finally, we have proved locally projectively flat with special metric and generalised metric in sections three and four, respectively.

2. Preliminaries

A Finsler metric $F(x, y)$ is called a scalar field when satisfies the following conditions

i. It is differential for any point of $TM\backslash\{0\}$,

ii. It is positively homogeneous of degree in y^i, that is, $F(x, \lambda y) = \lambda F(x, y)$, for any positive number λ,

iii. It is regular, that is, $g_{ij}(x, y) = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j F^2$,

constitute the regular matrix g_{ij}, where $\dot{\partial}_i = \frac{\partial}{\partial y^i}$. The manifold M^n equipped with a fundamental function $F(x, y)$ is called Finsler space $F^n = (M^n, F)$.

There is a family of Finsler metrics termed (α, β)-metrics that are defined by a Riemannian metric and 1-form on a manifold and have significant curvature attributes. These metrics are computable. The Finsler space $F^n = (M^n, F)$ is said to be an (α, β)-metric if F is a positively homogeneous function of degree one in two variables $\alpha = \sqrt{a_{ij} y^i y^j}$ is a Riemannian metric and $\beta = b_i(x) y^i$ is differentiable one-form. The space $\mathbb{R}^n = (M^n, \alpha)$ is called the associated Riemannian space and the covariant vector field b_i is the associated vector field.
An (α, β)-metric is expressed in the following form

$$F = \alpha\phi(s), \quad s = \frac{\beta}{\alpha}$$

where $\phi = \phi(s)$ is a C^∞ positive function on an open interval $(-b_0, b_0)$. The norm $\|\beta_x\|_\alpha$ of β with respect to α is defined by

$$\|\beta_x\|_\alpha = \sup_{y \epsilon T_x M} \beta(x, y), \quad \alpha(x, y) = a_{ij}(x) b_i(x) b_j(x).$$

In order to define L, β must satisfy the condition $\|\beta_x\|_\alpha < b_0$ for all $x \epsilon M$.

Let G^i and G^i_α denotes the spray coefficients of F and α respectively and given by

$$G^i = \frac{g^{il}}{4}\left\{\left[F^2\right]_{x^k y^l} y^k - \left[F^2\right]_{x^k}\right\},$$
$$G^i_\alpha = \frac{a^{il}}{4}\left\{\left[\alpha^2\right]_{x^k y^l} y^k - \left[\alpha^2\right]_{x^l}\right\},$$

where $g_{ij} = \frac{1}{2}\left[F^2\right]_{y^i y^j}$ and $a^{ij} = a_{ij}$. We have the following

Lemma 2.1. [2] *The spray coefficients G^i are related to G^i_α by*

$$G^i = G^i_\alpha + \alpha Q s^i_0 + H\left\{-2Q\alpha s_0 + r_{00}\right\}\left\{b^i - s\frac{y^i}{\alpha}\right\} + J\left\{-2Q\alpha s_0 + r_{00}\right\}\frac{y^i}{\alpha}, \quad (1)$$

$$Q = \frac{\phi^{'}}{\phi - s\phi^{'}},$$
$$H = \frac{\phi^{''}}{2[(\phi - s\phi^{'}) + (b^2 - s^2)\phi^{''}]}, \quad (2)$$
$$J = \frac{\phi^{'}\left(\phi - s\phi^{'}\right)}{2\phi[(\phi - s\phi^{'}) + (b^2 - s^2)\phi^{''}]},$$

where $s = \frac{\beta}{\alpha}$, $b = \|\beta_x\|_\alpha$, $s_{ij} = \frac{1}{2}(b_{i;j} - b_{j;i})$, $s_{l0} = s_{li} y^i$, $s_0 = s_{l0} b^l$, $r_{ij} = \frac{1}{2}(b_{i;j} - b_{j;i})$ *and* $r_{00} = r_{ij} y^i y^j$.

Lemma 2.2. [2] *An (α,β)-metric $F=\alpha\phi(s)$, where $s=\dfrac{\beta}{\alpha}$, is projectively flat on an open subset $U\subset\mathbb{R}^n$ if and only if*

$$\left(a_{ml}\alpha^2-y_my_l\right)G^m_\alpha+\alpha^3Qs_{l0}+\alpha H\left(-2\alpha Qs_0+r_{00}\right)\left(b_l\alpha-sy_l\right)=0, \tag{3}$$

where $y_l=a_{lj}y^j$.

3. Locally Projectively flat (α,β)-metric $F=\mu_1\alpha+\mu_2\beta+\mu_3\dfrac{\alpha^2}{\beta}$

Let us consider the special (α,β)-metric

$$F=\mu_1\alpha+\mu_2\beta+\mu_3\frac{\alpha^2}{\beta},\quad \mu_1,\mu_2>0. \tag{4}$$

This can be written as

$$F=\alpha\phi(s),\quad \text{where}\ \ \phi(s)=\mu_1+\mu_2 s+\mu_3\frac{1}{s}, \tag{5}$$

where $s<1$ so that ϕ must be positive function. Let b_0 be the largest number such that $\phi(s)-s\phi^{'}(s)+(b^2-s^2)\phi^{''}>0,\quad |s|\le b<b_0$ that is,

$$\mu_1s^3+2b^2\mu_3>0,\quad |s|\le b<b_0.$$

Lemma 3.1. *The special (α,β)-metric $F=\mu_1\alpha+\mu_2\beta+\mu_3\dfrac{\alpha^2}{\beta}$, μ_1, $\mu_3>0$ is a Finsler metric if and only if $\|\beta_x\|_\alpha<\dfrac{2\mu_3}{\mu_1}$.*

Proof. If F is a Finsler metric then $\mu_1s^3+2b^2\mu_3>0$.
Let $s=b$, we get $b<\dfrac{2\mu_3}{\mu_1}$, $\mu_1>0$, $\forall b<b_0$.
Let $b\to b_0$, then $b_0<\dfrac{2\mu_3}{\mu_1}$. So, $\|\beta_x\|_\alpha<\dfrac{2\mu_3}{\mu_1}$.
Now, if $|s|\le b<\dfrac{2\mu_3}{\mu_1}$ then

$$\frac{\mu_1s^3+2b^2\mu_3}{s^3}=\frac{\left(\mu_1s+2\mu_3\right)s^2+2\mu_3\left(b^2-s^2\right)}{s^3}>0.$$

Therefore, $F=\mu_1\alpha+\mu_2\beta+\mu_3\dfrac{\alpha^2}{\beta}$ is a Finsler metric. □

By lemma (2.1) the spray co-efficients G^i of F are given by (2)

$$\begin{aligned} Q &= \frac{\left(\mu_2 s^2-\mu_3\right)}{s\left(\mu_1 s+2\mu_3\right)} = \frac{\left(\mu_2\beta^2-\mu_3\alpha^2\right)}{\beta\left(\mu_1\beta+2\mu_3\alpha\right)}, \\ H &= \frac{\mu_3}{\mu_1 s^3+2b^2\mu_3} = \frac{\mu_3\alpha^3}{\mu_1\beta^3+2b^2\mu_3\alpha^3}, \\ J &= \frac{s\left(\mu_2 s^2-\mu_3\right)\left(\mu_1 s+2\mu_3\right)}{2\left(\mu_1 s+\mu_2 s^2+\mu_3\right)\left(\mu_1 s^3+2b^2\mu_3\right)} \\ &= \frac{\alpha\beta\left(\mu_2\beta^2-\mu_3\alpha^2\right)\left(\mu_1\beta+2\mu_3\alpha\right)}{2\left(\mu_1\alpha\beta+\mu_2\beta^2+\mu_3\alpha^2\right)\left(\mu_1\beta^3+2b^2\mu_3\alpha^3\right)}\ . \end{aligned} \tag{6}$$

Substituting (6) in (3), we obtain

$$\begin{aligned} &\left(a_{ml}\alpha^2-y_m y_l\right)G^m_\alpha+\alpha^3\left[\frac{\mu_2\beta^2-\mu_3\alpha^2}{\beta\left(\mu_1\beta+2\mu_3\alpha\right)}\right]s_{l0}+\frac{\mu_3\alpha^3}{\mu_1\beta^3+2b^2\mu_3\alpha^3} \\ &\left[-2\alpha\left(\frac{\mu_2\beta^2-\mu_3\alpha^2}{\beta\left(\mu_1\beta+2\mu_3\alpha\right)}\right)s_0+r_{00}\right]\left(b_l\alpha^2-y_l\beta\right)=0. \end{aligned} \tag{7}$$

Lemma 3.2. *If $\left(a_{ml}\alpha^2-y_m y_l\right)G^m_\alpha=0$ then α is projectively flat.*

Proof. If $\left(a_{ml}\alpha^2-y_m y_l\right)G^m_\alpha=0$ then $a_{ml}\alpha^2 G^m_\alpha=y_m y_l G^m_\alpha$.
Contracting with a^{il}, we have $\alpha^2 G^i_\alpha=y_m y^i G^m_\alpha$.

Let $\lambda(x,y)=y_m\dfrac{G^m_\alpha}{\alpha^2}$, then $G^i_\alpha=\lambda y^i$.
Therefore, α is projectively flat. □

Theorem 3.3. *A Special (α,β)-metric $F=\mu_1\alpha+\mu_2\beta+\mu_3\dfrac{\alpha^2}{\beta}$, $\mu_1,\ \mu_3\neq 0$ is locally Projectively flat if and only if*

i. *β is parallel with respect to α,*
ii. *α is locally projectively flat, that is., α is constant curvature.*

Proof. Assume that F is locally projectively flat. By rewriting (7) as a polynomial in y^i and α, we get

$$\begin{aligned} &\beta\left(\mu_1\beta+2\mu_3\alpha\right)\left(\mu_1\beta^3+2b^2\mu_3\alpha^3\right)\left(a_{ml}\alpha^2-y_m y_l\right)G^m_\alpha+\alpha^3\left(\mu_2\beta^2-\mu_3\alpha^2\right) \\ &\left(\mu_1\beta^3+2b^2\mu_3\alpha^3\right)s_{l0}+\mu_3\alpha^3\left[-2\alpha\left(\mu_2\beta^2-\mu_3\alpha^2\right)s_0+\beta\left(\mu_1\beta+2\mu_3\alpha\right)r_{00}\right] \\ &\left(b_l\alpha^2-y_l\beta\right)=0. \end{aligned} \tag{8}$$

Envisaging the coefficients of α necessarily be zero, then from (8), we have

$$2\mu_3\left(b^2\alpha^2+\beta^2\right)\left(a_{ml}\alpha^2-y_my_l\right)G^m_\alpha = -\alpha^2\{\beta\left(\mu_2\beta^2-\mu_3\alpha^2\right)s_{l0}+\mu_3\left(b_l\alpha^2-y_l\beta\right)r_{00}\} \tag{9}$$

and

$$\left(\mu_1^2\beta^5+4\mu_3^2b^2\alpha^4\beta\right)\left(a_{ml}\alpha^2-y_my_l\right)G^m_\alpha = -\{2b^2\mu_2\mu_3\alpha^6\beta^2-2b^2\mu_3^2\alpha^8\}s_{l0} + 2\mu_3\alpha^4\left(\mu_2\beta^2-\mu_3\alpha^2\right)\left(b_l\alpha^2-y_l\beta\right)s_0-2\mu_3^2\alpha^4\beta\left(b_l\alpha^2-y_l\beta\right)r_{00}. \tag{10}$$

Contracting (9) and (10) with b^l yields

$$2\mu_3\left(b^2\alpha^2+\beta^2\right)\left(\alpha^2b_m-\beta y_m\right)G^m_\alpha = -\alpha^2\{\beta\left(\mu_2\beta^2-\mu_3\alpha^2\right)s_0+\mu_3\left(b^2\alpha^2-\beta^2\right)r_{00}\} \tag{11}$$

and

$$\left(\mu_1^2\beta^4+4\mu_3^2b^2\alpha^4\right)\left(\alpha^2b_m-\beta y_m\right)G^m_\alpha = -2\alpha^4\left[\beta\left(\mu_2\mu_3\beta^2-\mu_3^2\alpha^2\right)s_0+\mu_3^2\left(b^2\alpha^2-\beta^2\right)r_{00}\right]. \tag{12}$$

Subtracting (12) from the (11) $\times\ 2\mu_3\alpha^2$, we get

$$\left(4\mu_3^2\alpha^2-\mu_1^2\beta^2\right)\left(\alpha^2b_m-\beta y_m\right)G^m_\alpha=0.$$

Since, $\left(4\mu_3^2\alpha^2-\mu_1^2\beta^2\right)\neq 0$, we get $\left(\alpha^2b_m-\beta y_m\right)G^m_\alpha=0$.

Which implies $\left(a_{ml}\alpha^2-y_my_l\right)G^m_\alpha=0$.

By lemma(3.2), α is projectively flat, that is., $G^i_\alpha=0$. Now from equation(9), we can have

$$\beta\left(\mu_2\beta^2-\mu_3\alpha^2\right)s_{l0}+\mu_3\left(b_l\alpha^2-y_l\beta\right)r_{00}=0.$$

Which gives $\beta\left(\mu_2\beta^2-\mu_3\alpha^2\right)s_{l0}=0$.

Since $\beta\neq 0$ and $\left(\mu_2\beta^2-\mu_3\alpha^2\right)\neq 0$, then $s_{l0}=0$. Hence, β is closed.

On the other hand, if β is closed and α is projectively flat then by lemma(3.2) we can see that F is locally projectively flat. □

4. Locally Projectively flat (α,β)-metric $F=\mu_1\alpha+\mu_2\beta+\mu_3\dfrac{\beta^2}{\alpha}$

In this section, we find the necessary and sufficient conditions for generalized (α,β)-metric $F=\mu_1\alpha+\mu_2\beta+\mu_3\dfrac{\beta^2}{\alpha}$ to be locally projectively flat. Consider the generalized metric

$$F=\mu_1\alpha+\mu_2\beta+\mu_3\frac{\beta^2}{\alpha} \tag{13}$$

The above equation can be expressed as

$$F=\alpha\left(\mu_1+\mu_2\frac{\beta}{\alpha}+\mu_3\frac{\beta^2}{\alpha^2}\right),$$

$$F=\alpha\phi(s),\quad s=\frac{\beta}{\alpha},$$

which implies $\phi(s)=\mu_1+\mu_2 s+\mu_3 s^2$.

Where $s<1$ so that ϕ must be a positive function. Let b_0 be the largest number such that

$$\phi(s)-s\phi^{'}(s)+(b^2-s^2)\phi^{''}>0,\quad |s|\leq b<b_0,$$

that is,

$$\mu_1+2b^2\mu_3-3\mu_3 s^2>0,\quad |s|\leq b<b_0,$$

Lemma 4.1. *The generalized (α,β)-metric $F=\mu_1\alpha+\mu_2\beta+\mu_3\dfrac{\beta^2}{\alpha}$, $\mu_1,\ \mu_3>0$ is a Finsler metric if and only if $\|\beta_x\|_\alpha<\sqrt{\dfrac{\mu_1}{\mu_3}}$.*

Proof. If F is a Finsler metric, then $\mu_1+\mu_3(2b^2-3s^2)>0$.
Let $s=b$, we get $b<\sqrt{\dfrac{\mu_1}{\mu_3}}$, $\mu_3>0$, $\forall b<b_0$.
Let $b\to b_0$, then $b_0<\sqrt{\dfrac{\mu_1}{\mu_3}}$. So, $\|\beta_x\|_\alpha<\sqrt{\dfrac{\mu_1}{\mu_3}}$.
Now, if $|s|\leq b<\sqrt{\dfrac{\mu_1}{\mu_3}}$, $\mu_3\neq 0$ then

$$\mu_1+\mu_3(2b^2-3s^2)>0.$$

Therefore, $F=\mu_1\alpha+\mu_2\beta+\mu_3\dfrac{\beta^2}{\alpha}$ is a Finsler metric. □

By lemma (2.1), the spray coefficients G^i of F are given by (2)

$$
\begin{aligned}
Q &= \frac{(\mu_2 + 2\mu_3 s)}{(\mu_1 - \mu_3 s^2)} = \frac{\alpha\,(\mu_2\alpha + 2\mu_3\beta)}{(\mu_1\alpha^2 - \mu_3\beta^2)},\\
H &= \frac{\mu_3}{2\,(\mu_1 + 2b^2\mu_3 - 3\mu_3 s^2)} = \frac{\mu_3\alpha^2}{\mu_1\alpha^2 + 2b^2\mu_3\alpha^2 - 3\mu_3\beta^2},\\
J &= \frac{(\mu_2 + 2\mu_3 s)\left(\mu_1 - \mu_3 s^2\right)}{2\,(\mu_1 + \mu_2 s + \mu_3 s^2)\,(\mu_1 + 2b^2\mu_3 - 3\mu_3 s^2)}\\
&= \frac{\alpha\,(\mu_2\alpha + 2\mu_3\beta)\left(\mu_1\alpha^2 - \mu_3\beta^2\right)}{2\,(\mu_1\alpha^2 + \mu_2\alpha\beta + \mu_3\beta^2)\,(\mu_1\alpha^2 + 2b^2\mu_3\alpha^2 - 2\mu_3\beta^2)}\ .
\end{aligned}
\tag{14}
$$

Substituting (14) in (3), we have

$$
\begin{aligned}
&\left(a_{ml}\alpha^2 - y_m y_l\right) G^m_\alpha + \alpha^3 \left[\frac{\alpha\,(\mu_2\alpha + 2\mu_3\beta)}{(\mu_1\alpha^2 - \mu_3\beta^2)}\right] s_{l0} + \frac{\mu_3\alpha^3}{\mu_1\alpha^2 + 2b^2\mu_3\alpha^2 - 3\mu_3\beta^2}\\
&\left[-2\alpha^2\left(\frac{\mu_2\alpha + 2\mu_3\beta}{\mu_1\alpha^2 - \mu_3\beta^2}\right) s_0 + r_{00}\right]\left(b_l\alpha^2 - y_l\beta\right) = 0.
\end{aligned}
\tag{15}
$$

By Lemma (3.2), α is projectively flat. Now, we prove the following

Theorem 4.2. *The generalized (α, β)-metric $F = \mu_1\alpha + \mu_2\beta + \mu_3\dfrac{\beta^2}{\alpha}$, μ_1, $\mu_3 \neq 0$ is locally Projectively flat if and only if*

i. $b_{i;j} = \tau\left[\left(\mu_1 + 2\mu_3 b^2\right) a_{ij} - 3\mu_3 b_i b_j\right]$,
ii. $G^i_\alpha = \theta y^i - \mu_3\tau\alpha^2 b^i$.

Proof. Suppose F is projectively flat. By rewriting (15) as a polynomial in y^i and α, we have

$$
\begin{aligned}
&\left(\mu_1\alpha^2 - \mu_3\beta^2\right)\left(\mu_1\alpha^2 + 2b^2\mu_3\alpha^2 - 3\mu_3\beta^2\right)\left(a_{ml}\alpha^2 - y_m y_l\right) G^m_\alpha + \alpha^4\,(\mu_2\alpha + 2\mu_3\beta)\\
&\left(\mu_1\alpha^2 + 2b^2\mu_3\alpha^2 - 3\mu_3\beta^2\right) s_{l0} + \mu_3\alpha^2\left(-2\alpha^2\,(\mu_2\alpha + 2\mu_3\beta)\, s_0 + \left(\mu_1\alpha^2 - \mu_3\beta^2\right) r_{00}\right)\\
&\left(b_l\alpha^2 - y_l\beta\right) = 0.
\end{aligned}
\tag{16}
$$

Envisaging the coefficients of α and α^3 necessarily be zero then we obtain

$$
\left(\mu_1\alpha^2 + 2b^2\mu_3\alpha^2 - 3\mu_3\beta^2\right) s_{l0} - 2\mu_3\left(b_l\alpha^2 - y_l\beta\right) s_0 = 0 \tag{17}
$$

and

$$\left(\mu_1\alpha^4 + 2b^2\mu_1\mu_3\alpha^4 - 4\mu_1\mu_3\alpha^2\beta^2 - 2\mu_3^2b^2\alpha^2\beta^2 + 3\mu_3^2\beta^4\right)\left(a_{ml}\alpha^2 - y_my_l\right)G_\alpha^m$$
$$+ 2\mu_3\alpha^4\beta\left(\mu_1\alpha^2 + 2b^2\mu_3\alpha^2 - 3\mu_3\beta^2\right)s_{l0} + \left(-4\mu_3^2\alpha^4\beta s_0 + \left(\mu_1\alpha^2 - \mu_3\beta^2\right)r_{00}\right)$$
$$\left(b_l\alpha^2 - y_l\beta\right) = 0. \tag{18}$$

Contracting (17) with b^l which gives

$$\left(\mu_1\alpha^2 - \mu_3\beta^2\right)s_0 = 0. \tag{19}$$

Since $\left(\mu_1\alpha^2 - \mu_3\beta^2\right) \neq 0$, we have $s_0 = 0$. From equation (17), we have $s_{l0} = 0$. Thus, β is closed.

Now equation (16) can be re-written as

$$\left((\mu_1 + 2b^2\mu_3)\alpha^2 - 3\mu_3\beta^2\right)\left(a_{ml}\alpha^2 - y_my_l\right)G_\alpha^m + \mu_3\alpha^2\left(b_l\alpha^2 - y_l\beta\right)r_{00} = 0. \tag{20}$$

Contracting (20) with b^l, we get

$$-\left((\mu_1 + 2b^2\mu_3)\alpha^2 - 3\mu_3\beta^2\right)\left(b_m\alpha^2 - y_m\beta\right)G_\alpha^m = \mu_3\alpha^2\left(b^2\alpha^2 - \beta^2\right)r_{00} = 0.$$

Noting that the polynomial $-\left((\mu_1 + 2b^2\mu_3)\alpha^2 - 3\mu_3\beta^2\right)$ is not divisible by α^2, μ_3 and $\left(b^2\alpha^2 - \beta^2\right)$. Hence $\left(b_m\alpha^2 - y_m\beta\right)G_\alpha^m$ is divisible by $\mu_3\alpha^2\left(b^2\alpha^2 - \beta^2\right)$. Therefore, their exist a scalar function $\tau = \tau(x)$ such that

$$r_{00} = \tau\{\left(\mu_1 + 2\mu_3b^2\right)\alpha^2 - 3\mu_3\beta^2\}, \tag{21}$$

which gives $b_{i;j} = \tau\{\left(\mu_1 + 2\mu_3b^2\right)a_{ij} - 3\mu_3b_ib_j\}$.

By (19) and (21) , equation (1) can be simplified as

$$G^i = G_\alpha^i + \mu_3\tau\alpha^2b^i + \tau\Psi\alpha y^i, \tag{22}$$

where

$$\Psi = \frac{(\mu_2\alpha + 2\mu_3\beta)\left(\mu_1\alpha^2 - \mu_3\beta^2\right) - 2\mu_3\beta\left(\mu_1\alpha^2 + \mu_2\alpha\beta + \mu_3\beta^2\right)}{2\alpha\left(\mu_1\alpha^2 + \mu_2\alpha\beta + \mu_3\beta^2\right)}.$$

Note that F is projectively flat if and only if $G^i = Py^i$.
By (22)

$$G^i_\alpha = -\mu_3\tau\alpha^2 b^i + (P - \tau\alpha\Psi)\, y^i$$
$$= \theta y^i - \mu_3\tau\alpha^2 b^i,$$

where $\theta = P - \tau\Psi\alpha$.

On the other hand, if β is closed and α is locally projectively flat, then by lemma(2.2) one can see that F is locally projectively flat. Hence proved. □

5. Conclusion

Every locally projectively flat Finsler metric has a scalar flag curvature, which means that the flag curvature is a scalar function on the tangent bundle $TM\backslash 0$. As a result, locally projectively flat construct a rich class of Finsler metrics. In this paper, we obtained α is locally projectively flat and further shown β is parallel with respect to α. Also, proved locally projectively flat for both the aforementioned metrics.

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