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ON THE IMPROVED INSTABILITY REGION FOR THE CIRCULAR RAYLEIGH PROBLEM OF HYDRODYNAMIC STABILITY

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Abstract. We consider circular Rayleigh problem of hydrodynamic stability which deals with linear stability of axial flows of an incompressible iniviscid homogeneous fluid to axisymmetric disturbances. For this problem, we obtained two parabolic instability regions which intersect with Batchelor and Gill semi-circle under some condition. This has been illustrated with examples. Also, we derived upper bound for the amplification factor.

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1. Introduction

The study of circular Rayleigh problem is an important aspect of geophysical fluid dynamics and medical applications of (cf. [3, 4, 11]) . Circular Rayleigh problem is the inviscid case of Orr-Sommerfeld problem (cf. [11]). It deals with stability of axial flows of incompressible, inviscid homogeneous fluids with respect to axisymmetric disturbances in an annular region between concentric cylinders at the radial position $r = R_1, R_2$, and $0 < R_1 < R_2 < \infty$.

The following results are known for circular Rayleigh problem:

(i) A necessary condition for the instability is that $r\left(\frac{W'}{r}\right)'$ should changes its sign at least once in the flow domain (cf. [3]).

(ii) A necessary condition for instability is $r\left(\frac{W'}{r}\right)'(W - W_s) < 0$ (cf. [2]) .

(iii) A necessary condition for instability is $-\frac{r\left(\frac{W'}{r}\right)'}{(W-W_s)} > 0$ (cf. [3]).

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(iv) For an unstable mode, the phase velocity $c = c_r + ic_i$ with $(c_i > 0)$ must lie in the upper half of semi circle (cf. [2]).
(v) Howard's conjecture namely, growth rate approaches to zero as wave number approaches to infinity has been proved (cf. [6]).
(vi) Two parabolic instability regions are proved (cf. [8]).
(vii) A sufficient condition for stability has been proved (cf. [7]).
(viii) Short wave stability has been proved (cf. [7]).

[1, 5] obtained parabolic instability regions for standard Rayleigh problem of hydrodynamic stability which deals with stability of homogeneous shear flows. [9, 10] obtained parabolic instability regions for the extended Rayleigh problem of hydrodynamic stability which deals with stability of plane parallel shear flows under variable cross section. [8] followed the approach of [5] and derived parabolic instability regions which depend on conditions like $\phi(r) = r\left(\frac{W'}{r}\right)' + \frac{2R_1 \ (b-a)\pi^2}{R_2 (R_2 - R_1)^2} > 0$ or $\psi(r) = r\left(\frac{W'}{r}\right)' - \frac{2R_1 \ (b-a)\pi^2}{R_2 (R_2 - R_1)^2} < 0 \ \forall \ r \ \epsilon \ [R_1, R_2]$, where $a, \ b$ are the minimum and maximum basic velocities respectively.
It is necessary to improve their results by removing the conditions for circular Rayleigh problem. This has been done in this paper.

In this paper, we derived two different parabolic instability regions which intersect with Bachelor and Gill semi-circle [2] under some conditions. This has been illustrated with examples. Unlike parabolic instability region derived by [8] , new parabolic instability regions does not depend on any conditions like $\phi(r) > 0$ (or) $\psi(r) < 0$. In fact the instability regions derived in this paper depend on the parameters like wave number, minimum and maximum basic velocity profile, shear, radii. One of our parabolic instability regions depend on vorticity $\left(\frac{W'}{r}\right)'$ as $\left(\frac{W'}{r}\right)'$ plays an important role in stability (or) otherwise. On the other hand the Bachelor and Gill semi-circle depends only on minimum and maximum basic velocity profile. Also, we derived an upper bound for the amplification factor .

2. Circular Rayleigh Problem

The circular Rayleigh problem (cf. [3]) is given by

$$(W - c)\left(DD_* - k^2\right) u - rD\left(\frac{DW}{r}\right) u = 0, \tag{1}$$

with boundary conditions

$$u(R_1) = 0 = u(R_2). \tag{2}$$

Where $D_* = D + \frac{1}{r}$, $D = \frac{d}{dr}$, $k > 0$ is the wave number, $c = c_r + ic_i$ is the complex phase velocity, W is the basic velocity profile, u is the eigen function.

Using the transformation $u = (W-c)^{\frac{1}{2}} G$, we get

$$D\left[(W-c) D_* G\right] - \frac{1}{2} r D\left(\frac{DW}{r}\right) G - \frac{\frac{(W')^2}{4}}{W-c} G - k^2 (W-c) G = 0, \tag{3}$$

with boundary conditions

$$G(R_1) = 0 = G(R_2). \tag{4}$$

3. Parabolic Instability Region

Theorem 3.1. *For an unstable mode ($c_i > 0$), we have*

$$c_i^2 \leq \lambda\left[c_r + \frac{3\ W_{max}}{2} - \frac{W_{min}}{2}\right]$$

,

where $\lambda = \dfrac{(W')^2_{max}}{2|W_{max}+3W_{min}| \dfrac{R_1}{R_2}\left[\dfrac{\pi^2}{R_2(R_2-R_1)^2} + k^2\right]}$.

Proof. Multiplying (3) by rG^*, integrating by parts and applying (4), we get

$$\int_{R_1}^{R_2} (W-c)\left[|D_*G|^2 + k^2 |G|^2\right] r dr + \frac{1}{2}\int_{R_1}^{R_2} r^2 \left(\frac{W'}{r}\right)' |G|^2 dr + \int_{R_1}^{R_2} \frac{(W')^2}{4(W-c)} r |G|^2 dr = 0.$$

Equating the real and imaginary parts, we get

$$\int_{R_1}^{R_2} (W-c_r)\left[|D_*G|^2 + k^2 |G|^2\right] r dr + \frac{1}{2}\int_{R_1}^{R_2} r^2 \left(\frac{W'}{r}\right)' |G|^2 dr + \int_{R_1}^{R_2} \frac{(W')^2 (W-c_r)}{4|W-c|^2} r |G|^2 dr = 0, \tag{5}$$

and

$$-c_i \int_{R_1}^{R_2} \left[|D_*G|^2 + k^2 |G|^2\right] r dr + c_i \int_{R_1}^{R_2} \frac{(W')^2}{4|W-c|^2} r |G|^2 dr = 0. \tag{6}$$

Let an arbitrary value $W_s = \dfrac{W_{mim} + W_{max}}{2}$.

Multiplying (6) by $\left(\frac{c_r + W_s}{c_i}\right)$ and subtracting from (5), we get

$$\int_{R_1}^{R_2} (W+W_s)\left[|D_*G|^2 + k^2 |G|^2\right] r dr + \frac{1}{2}\int_{R_1}^{R_2} r^2 \left(\frac{W'}{r}\right)' |G|^2 dr$$

$$+\int_{R_1}^{R_2} \frac{(W')^2 (W-2c_r-W_s)}{4|W-c|^2} \, r|G|^2 \, dr = 0. \tag{7}$$

Multiplying (6) by $\left(\frac{W_{min}-W_{max}}{c_i}\right)$ and subtracting from (5), we get

$$\int_{R_1}^{R_2} (W-c_r+W_{min}-W_{max}) \left[|D_*G|^2 + k^2|G|^2\right] rdr$$

$$+\frac{1}{2}\int_{R_1}^{R_2} r^2 \left(\frac{W'}{r}\right)' |G|^2 \, dr + \int_{R_1}^{R_2} \frac{(W')^2 (W-c_r-W_{min}+W_{max})}{4|W-c|^2} \, r|G|^2 \, dr$$
$$= 0.$$

Since $W-c_r+W_{min}-W_{max}<0$, dropping the term from the above equation, we have

$$\frac{1}{2}\int_{R_1}^{R_2} r^2 \left(\frac{W'}{r}\right)' |G|^2 \, dr \geq \int_{R_1}^{R_2} \frac{(W')^2 (W_{min}-W_{max}+c_r-W)}{4|W-c|^2} \, r|G|^2 \, dr. \tag{8}$$

Substituting (8) in (7), we get

$$\int_{R_1}^{R_2} (W+W_s) \left[|D_*G|^2 + k^2|G|^2\right] rdr$$
$$\leq \int_{R_1}^{R_2} \frac{(W')^2 (c_r+W_s-W_{min}+W_{max})}{4|W-c|^2} \, r|G|^2 \, dr.$$

Using Rayleigh-Ritz inequality and $W_s = \frac{W_{min}+W_{max}}{2}$, we have

$$c_i^2 \leq \lambda \left[c_r + \frac{3\ W_{max}}{2} - \frac{W_{min}}{2}\right], \tag{9}$$

$$where,\ \lambda = \frac{(W')^2_{max}}{2|W_{max}+3W_{min}| \frac{R_1}{R_2}\left[\frac{\pi^2}{R_2(R_2-R_1)^2}+k^2\right]}. \tag{10}$$

□

Theorem 3.2. *If* $\lambda < \lambda_c$, *where* $\lambda_c = 4W_{max} - \sqrt{15\ W_{max}^2 - W_{min}^2 + 2\ W_{min}W_{max}}$ *then the parabola* $c_i^2 \leq \lambda \left[c_r + \frac{3\ W_{max}}{2} - \frac{W_{min}}{2}\right]$ *intersects Bachelor and Gill semi-circle.*

Proof. Batchelor and Gill semicircle (cf. [2]) is given by

$$\left[c_r - \frac{W_{max} + W_{min}}{2}\right]^2 + c_i^2 \leq \left[\frac{W_{max} - W_{min}}{2}\right]^2. \tag{11}$$

Substituting (9) in (11) , we get

$$c_r^2 + [\lambda - (W_{max} + W_{min})]\, c_r + \left[W_{min}W_{max} + \lambda\left(\frac{3W_{max}}{2} - \frac{W_{min}}{2}\right)\right] \leq 0.$$

The discriminant part of the above equation is greater than or equal to zero for real roots and hence solving for λ, we get

$\lambda = 4\ W_{max} \pm \sqrt{15\ W_{max}^2 - W_{min}^2 + 2\ W_{min}W_{max}}$.
In the above equation, if we keep positive sign then it leads to $c_r < W_{min}$ which is against the necessary condition $W_{min} < c_r < W_{max}$.
Hence, $\lambda < \lambda_c$, where $\lambda_c = 4\ W_{max} - \sqrt{15\ W_{max}^2 - W_{min}^2 + 2\ W_{min}W_{max}}$ then the parabola given by (9) intersects with Batchelor and Gill semicircle (11) . □

Example 1: Consider the following configuration,

$R_1 = 1.5,\ R_2 = 2, W(r) = sin(r^2)$

It easily seen that $W_{min} = -0.75680,\ W_{max} = 0.77807$.
From (10), we have $\lambda = 0.139512$ and $\lambda_c = 0.403935$, which implies that $\lambda < \lambda_c$, then the parabola intersects with Bachelor and Gill semi circle for all values of k from below figure 1 .

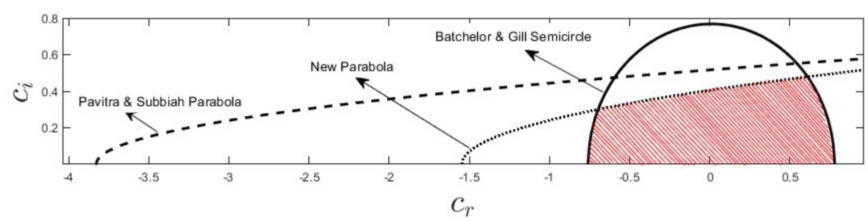

FIGURE 1. c_r *vs* c_i (The instability region is the shaded region)

Figure 2 illustrate the reduction of parabolic instability region for different values of k, as wave number k increases, the instability region reduces.

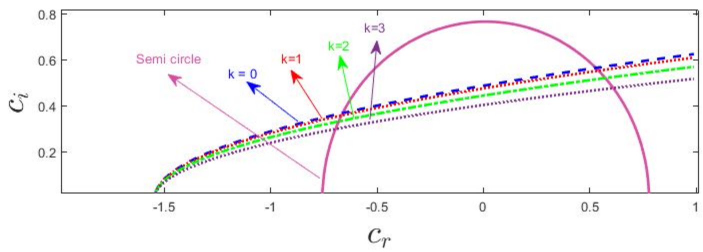

FIGURE 2. c_r *vs* c_i (Parabolic instability regions)

Theorem 3.3. *For an unstable mode* ($c_i > 0$), *we have* $c_i^2 \leq \lambda^* \left[c_r + \frac{W_{max}}{4} - \frac{W_{min}}{4}\right]$, *where* $\lambda^* = \dfrac{(W')^2_{max}}{|3\ W_{min} + W_{max}| \left[\frac{R_1\pi^2}{R_2(R_2-R_1)^2} + k^2\right] + R_1\left(\frac{W'}{r}\right)'_{min}}$.

Proof. Multiplying (6) by $\left(\frac{c_r + W_s}{-c_i}\right)$ and adding with (5), we get

$$\int_{R_1}^{R_2} (W + W_s)\left[|D_*G|^2 + k^2|G|^2\right] r dr + \frac{1}{2}\int_{R_1}^{R_2} r^2\left(\frac{W'}{r}\right)'|G|^2 dr + \int_{R_1}^{R_2} \frac{(W')^2(W - 2c_r - W_s)}{4|W-c|^2}\ r|G|^2 dr = 0.$$

Using Rayleigh-Ritz inequality, we have

$$\int_{R_1}^{R_2}\left[(W + W_s)\left[\frac{R_1\pi^2}{R_2(R_2-R_1)^2} + k^2\right] + \frac{1}{2}r\left(\frac{W'}{r}\right)'\right] r|G|^2 dr \leq \int_{R_1}^{R_2} \frac{(W')^2(2c_r - W + W_s)}{4|W-c|^2}\ r|G|^2 dr.$$

$$i.e.,\ c_i^2 \leq \lambda^*\left[c_r + \frac{W_{max}}{4} - \frac{W_{min}}{4}\right], \tag{12}$$

where

$$\lambda^* = \frac{(W')^2_{max}}{|3\ W_{min} + W_{max}|\left[\frac{R_1\pi^2}{R_2(R_2-R_1)^2} + k^2\right] + R_1\left(\frac{W'}{r}\right)'_{min}}. \tag{13}$$

□

Theorem 3.4. *If* $\lambda^* < \lambda_c^*$,
where $\lambda_c^* = \dfrac{(W_{min} + 3\ W_{max}) - \sqrt{5\ W_{max}^2 - 3W_{min}^2 + 14\ W_{min}W_{max}}}{2}$, *then the parabola* $c_i^2 \leq \lambda^* \left[c_r + \dfrac{W_{max}}{4} - \dfrac{W_{min}}{4}\right]$ *intersects Bachelor and Gill semi-circle.*

Proof. Proceeding in the same way as in theorem 3.2 , we get

$$\lambda_c^* = \frac{(W_{min} + 3\ W_{max}) - \sqrt{5\ W_{max}^2 - 3W_{min}^2 + 14\ W_{min}W_{max}}}{2}.$$

Then the parabola given by (12) intersects with Batchelor and Gill semicircle (11) . ☐

Example 2: Consider the following configuration,

$R_1 = 0.5,\ R_2 = 1, W(r) = r - \frac{1}{2}$

It easily seen that $W_{min} = 0,\ W_{max} = 0.5$.
From (13), we have $\lambda^* = 0.03297$ and $\lambda_c^* = 0.19098$, which implies that $\lambda^* < \lambda_c^*$, then the parabola intersects with Bachelor and Gill semi circle for all values of k from below figure 3 .

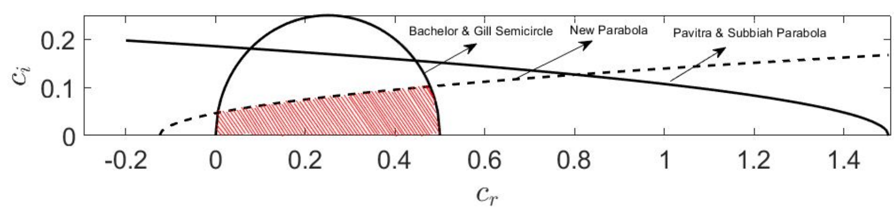

FIGURE 3. c_r *vs* c_i (The instability region is the shaded region)

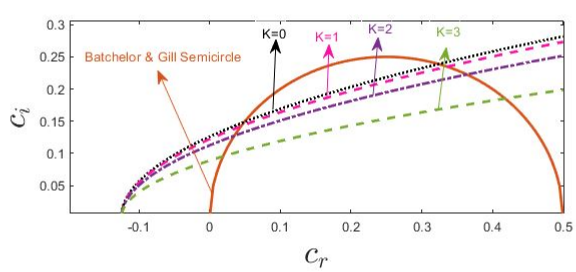

FIGURE 4. c_r *vs* c_i (Parabolic instability region)

Figure 4 illustrate the reduction of parabolic instability region for different values of k, as wave number k increases, the instability region reduces.

For an unstable mode $c_i > 0$ the time spend by the wave is proportional to phase speed or eigen values. Hence, it is necessary to know the bound or location of eigen values. The two parabolic instability regions given in Theorem 3.1 and 3.3 gives the location of the eigen values. The regions are unbounded hence we showed that it intersect with Batchelor and Gill semi circle ([2]). The region of location of eigen values is further reduced. The results are valid for constant velocity profile and exchange flows also.

4. Bounds for c_i

Theorem 4.1. *The upper bound for the amplification factor is given by*

$$c_i^2 \leq \left[\frac{\left[k^2 \ r \ \left(\frac{W'}{r}\right)' (W - W_s) + \left[r \ \left(\frac{W'}{r}\right)' \right] \right]^2}{\frac{R_1 \ \pi^2}{R_2 \left(R_2 - R_1\right)^2} \left[\frac{R_1 \pi^2}{R_2 \left(R_2 - R_1\right)^2} + k^2 \right]} \right]_{max} .$$

Proof. Multiplying (1) by $\left[D\left(\frac{D\left(ru^*\right)}{r}\right)\right] r$, integrating over (R_1, R_2) using by parts and applying (2), we get

$$\int_{R_1}^{R_2} \left| \left[\left(\frac{(ru)'}{r} \right)' \right] \right|^2 rdr - \int_{R_1}^{R_2} \left[k^2 + \frac{r \left(\frac{W'}{r}\right)'}{W - c} \right] u \ \left[D \left(\frac{D\left(ru^*\right)}{r} \right) \right] rdr = 0. \tag{14}$$

From (1), taking conjugate , we get

$$\left[\left(\frac{(ru^*)'}{r} \right)' \right] = \left[k^2 + \frac{r \left(\frac{W'}{r}\right)'}{W - c^*} \right] u^*. \tag{15}$$

Substituting (14) in (15), we get

$$\int_{R_1}^{R_2} \left| \left[\left(\frac{(ru)'}{r} \right)' \right] \right|^2 rdr + k^2 \int_{R_1}^{R_2} \frac{\left|(ru)'\right|^2}{r} dr - k^2 \int_{R_1}^{R_2} \frac{r \left(\frac{W'}{r}\right)'}{W - c} \left|u\right|^2 rdr$$

$$-\int_{R_1}^{R_2} \frac{\left[\left(\frac{W'}{r}\right)'\right]^2}{|W-c|^2} |u|^2 r^3 dr = 0.$$

Taking real part, we get

$$\int_{R_1}^{R_2} \left|\left[\left(\frac{(ru)'}{r}\right)'\right]\right|^2 r dr + k^2 \int_{R_1}^{R_2} \frac{|(ru)'|^2}{r} dr - k^2 \int_{R_1}^{R_2} \frac{r^2 \left(\frac{W'}{r}\right)' (W-c_r)}{|W-c|^2} |u|^2 dr$$

$$-\int_{R_1}^{R_2} \frac{\left[\left(\frac{W'}{r}\right)'\right]^2}{|W-c|^2} |u|^2 r^3 dr = 0. \tag{16}$$

Multiplying (1) by ru^*, integrating using by parts and using (2), we get

$$\int_{R_1}^{R_2} \frac{|(ru)'|^2}{r} dr + k^2 \int_{R_1}^{R_2} |u|^2 r dr + \int_{R_1}^{R_2} \frac{\left(\frac{W'}{r}\right)'}{W-c} |u|^2 r^2 dr = 0.$$

Equating the imaginary parts, we get

$$c_i \int_{R_1}^{R_2} \frac{\left(\frac{W'}{r}\right)'}{|W-c|^2} |u|^2 r^2 dr = 0. \tag{17}$$

Multiplying (17) by $\left(\frac{c_r - W_s}{c_i}\right)$ and subtracting from (16) , we get

$$\int_{R_1}^{R_2} \left|\left[\left(\frac{(ru)'}{r}\right)'\right]\right|^2 r dr + k^2 \int_{R_1}^{R_2} \frac{|(ru)'|^2}{r} dr$$

$$-\int_{R_1}^{R_2} \frac{k^2 \left(\frac{W'}{r}\right)' (W-W_s) + r\left[\left(\frac{W'}{r}\right)'\right]^2}{|W-c|^2} r^2 |u|^2 dr = 0.$$

Since $\frac{1}{|W-c|^2} \leq \frac{1}{c_i^2}$ and using Rayleigh-Ritz inequality, we get

$$\frac{R_1^2\,\pi^4}{R_2^2 (R_2-R_1)^4} + k^2 \frac{R_1 \pi^2}{R_2 (R_2-R_1)^2} \leq \frac{\left[k^2\ r\left(\frac{W'}{r}\right)'(W-W_s) + \left[r\left(\frac{W'}{r}\right)'\right]^2\right]_{max}}{c_i^2},$$

$$i.e.,\ c_i^2 \leq \left[\frac{k^2\ r\ \left(\frac{W'}{r}\right)' (W - W_s) + \left[r\ \left(\frac{W'}{r}\right)' \right]^2}{\frac{R_1\ \pi^2}{R_2\,(R_2 - R_1)^2} \left[\frac{R_1 \pi^2}{R_2\,(R_2 - R_1)^2} + k^2 \right]} \right]_{max} .$$

□

$c_i > 0$ refers to unstable mode, the bound for amplification factor will give the upper bound for c_i

5. Concluding remarks

In this paper, we derived two parabolic instability regions for the circular Rayleigh problem of hydrodynamic stability. Unlike the previously known parabolic instability regions, new parabolic instability regions does not depend on any conditions like $\phi(r) > 0$ (or) $\psi(r) < 0$. In fact the instability regions derived in this paper depend on the parameters like wave number, minimum and maximum basic velocity profile, shear, radii. One of our instability depends on $\left(\frac{W'}{r}\right)'$ which is an important parameter in deciding whether flow is stable or unstable. The two parabolic instability regions intersect with Batchelor and Gill semi-circle under some condition. This has been illustrated with examples. The example shows that the instability region is further reduced. Also, we derived an upper bound for the amplification factor. The results can be extended to heterogeneous flows with heterogeneous factor taken in to consideration and these will be submitted later.

Conflicts of interest : The authors declare no conflict of interest.

Data availability : Not applicable

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