J. Appl. Math. & Informatics Vol. **41***(2023), No. 1, pp. 133 - 154*
https://doi.org/10.14317/jami.2023.133

ON TRIPOLAR FUZZY IDEALS IN ORDERED SEMIGROUPS

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ABSTRACT. In this paper, we introduce the concept of tripolar fuzzy subsemigroups, tripolar fuzzy ideals, tripolar fuzzy quasi-ideals, and tripolar fuzzy bi-ideals of an ordered semigroup and study some algebraic properties of them. Moreover, we prove that tripolar fuzzy bi-ideals and quasi-ideals coincide only in a particular class of ordered semigroups. Finally, we prove that every tripolar fuzzy quasi-ideal is the intersection of a tripolar fuzzy left and a tripolar fuzzy right ideal.

AMS Mathematics Subject Classification : 03E72, 20M12.
Key words and phrases : Ordered semigroup, tripolar fuzzy set, tripolar fuzzy left (right, two side, quasi-, bi-) ideal.

1. Introduction

The theory of fuzzy sets is the most appropriate theory for dealing with uncertainty was introduced by Zadeh [23] in 1965. After the introduction of the concept of fuzzy sets by Zadeh, several researchers researched the generalizations of the notions of fuzzy sets with huge applications in computer science, artificial intelligence, control engineering, robotics, automata theory, decision theory, finite state machine, graph theory, logic, operations research and many branches of pure and applied mathematics.

The fuzzification of the algebraic structure was introduced by Rosenfeld [19] in 1971. The notion of fuzzy subgroups was started. Followed by the concept of fuzzy semigroups, Kuroki [9] first used the concept of fuzzy sets to investigate semigroups. Apparently, the concept of ordered semigroups is a generalization of semigroups (see [2]). Kehayopulu and Tsinglis [5] firstly studied ordered semigroups in terms of fuzzy sets. Several researched ordered semigroups using fuzzy sets after inventing fuzzy ordered semigroups. Kehayopulu and Tsingelis [6] showed that, in ordered semigroups, any fuzzy right (resp. left) ideal is a

Received February 11, 2022. Revised September 10, 2022. Accepted September 17, 2022.
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fuzzy quasi-ideal, and any fuzzy quasi-ideal is a fuzzy bi-ideal. Moreover, they proved that in regular ordered semigroups, the notions of fuzzy quasi-ideals and fuzzy bi-ideals coincide. They finally showed that, in ordered semigroup, any fuzzy quasi-ideal is an intersection of a fuzzy right and a fuzzy left ideal. Jirojkul and Chinram [11] worked on the concept of quasi-ideals in ordered semigroups in 2009.

A bipolar fuzzy set is an extension of fuzzy sets whose membership degree range is $[-1, 1]$. In 1994, Zhang [24] initiated the concept of the bipolar fuzzy set as a generalization of fuzzy sets. Shabir and Iqbal proposed numerous concepts of bipolar fuzzy ideals in ordered semigroups. They explored some classes of ordered semigroups by using bipolar fuzzy left (resp. right, bi-) ideals. In 2017, Ibrar et al. [4] applied some types of bipolar fuzzy ideals to characterize many classes of ordered semigroups. Recently, Ibrar et al. [3] used an extended concept of bipolar fuzzy ideals to describe semi-simple, regular, and intra-regular ordered semigroups.

The concept of intuitionistic fuzzy sets is another generalization of fuzzy sets. This concept was introduced by Atanassov [1] in 1986. Similar to bipolar fuzzy sets, it can also be used to analyze the algebraic structure of ordered semigroups. In 2005, Jun [12] introduced the intuitionistic fuzzification of bi-ideals in ordered semigroups. Its essential characteristics of it were given. Intuitionistic fuzzy bi-ideals were used to describe completely regular ordered semigroups. Later, Park [18], in 2007, defined and investigated the topic of intuitionistic fuzzy interior ideals. The natural equivalence relation was investigated on the set of all intuitionistic fuzzy interior ideals. Ordered semigroups were characterized using intuitionistic fuzzy interior ideals by Shabir and Khan [21]. Some other types of intuitionistic ideals appeared in ordered semigroups (see [8]).

Fuzzy ideals and their generalizations have become more prevalent in the study of ordered semigroups by several researchers. The pioneer topic nowadays, for instance, is the concept of fuzzy soft ideals, bipolar fuzzy soft ideals, and almost ideals (see [7, 10, 22]). A further extension of the fuzzy set concept is the idea of tripolar fuzzy sets. Additionally, bipolar fuzzy sets and intuitionistic fuzzy sets can be regarded as tripolar fuzzy sets. Rao [14] introduced the notion of tripolar fuzzy sets as a generalization of fuzzy sets, bipolar fuzzy sets, and intuitionistic fuzzy sets. The author studied tripolar fuzzy interior ideals of Γ-semigroup. Rao and Venkateswarlu [15, 17] investigated tripolar fuzzy interior ideal, tripolar fuzzy soft ideal, and tripolar fuzzy soft interior ideal of Γ-semigroup and Γ-semiring. Rao [16] introduced the notion of tripolar fuzzy interior ideal, tripolar fuzzy soft ideal, tripolar fuzzy soft interior ideal over semigroup and studied some of their algebraic properties.

Lekkoksung and Lekkoksung [13] introduced the notion of interval-valued intuitionistic fuzzy hyperideals, bi-hyperideals, and quasi-hyperideals of an ordered semihypergroup. They characterized an interval-valued intuitionistic fuzzy hyperideal of an ordered semihypergroup in terms of its level subset. Moreover, they showed that interval-valued intuitionistic fuzzy bi-hyperideals and quasi-hypetideals coincide in a regular ordered semigroup. Finally, they also showed that every interval-valued intuitionistic fuzzy quasi-hyperideal is the intersection of an interval-valued fuzzy left hyperideal and interval-valued intuitionistic fuzzy right hyperideal.

In this present paper, we introduce the concept of tripolar fuzzy subsemigroups, tripoar fuzzy ideals, tripolar fuzzy quasi-ideals, and tripolar fuzzy bi-ideals of an ordered semigroup and study some algebraic properties of them. Moreover, we prove that tripolar fuzzy bi-ideals and quasi-ideals coincide only in a particular class of ordered semigroups. Finally, we prove that every tripolar fuzzy quasi-ideal is the intersection of a tripolar fuzzy left and a tripolar fuzzy right ideal.

2. Preliminaries

In this section, we will recall some of the fundamental concepts and definitions, which are necessary for this paper.

An ordered semigroup is a structure $(S;\cdot,\leq)$ such that

(1) $(S;\cdot)$ is a semigroup.
(2) $(S;\leq)$ is a partially ordered set, and
(3) $x \leq y$ implies $u \cdot x \leq v \cdot y$ and $x \cdot u \leq y \cdot v$ for all $u, v, x, y \in S$.

For simplicity, we will be written xy instead of $x \cdot y$ and an ordered semigroup $(S;\cdot,\leq)$, will be written in its universe set as a bold letter $\mathbf{S}$.

A fuzzy set on a nonempty set X is a mapping $f : X \to [0,1]$ from X to a unit closed interval, (see [6]).

A tripolar fuzzy set (tripolar fuzzy subset) A of S [14] is an object having the form

$$A := \{(x, \mu_A(x), \lambda_A(x), \delta_A(x)) \mid x \in S, 0 \leq \mu_A(x) + \lambda_A(x) \leq 1\},$$

where $\mu_A : S \to [0,1]$, $\lambda_A : S \to [0,1]$ and $\delta_A : S \to [-1,0]$.

A tripolar fuzzy set $A := \{(x, \mu_A(x), \lambda_A(x), \delta_A(x)) \mid x \in S, 0 \leq \mu_A(x) + \lambda_A(x) \leq 1\}$ of S, for the sake of simplicity, we shall use the symbol $A = (\mu_A, \lambda_A, \delta_A)$ for $A = \{(x, \mu_A(x), \lambda_A(x), \delta_A(x)) \mid x \in S, 0 \leq \mu_A(x) + \lambda_A(x) \leq 1\}$.

Now, we let $Tri(S)$ be the set of all tripolar fuzzy subsets of S and define the operation on $Tri(S)$ as follows: Let $A = (\mu_A, \lambda_A, \delta_A)$ and $B = (\mu_B, \lambda_B, \delta_B)$ be elements in $Tri(S)$. Then the product $A \circ B$ of A and B as the tripolar fuzzy subset $A \circ B = (\mu_A \circ \mu_B, \lambda_A \circ \lambda_B, \delta_A \circ \delta_B)$ of S and is defined by:

$$(\mu_A \circ \mu_B)(x) = \begin{cases} \bigvee\limits_{(a,b)\in \mathbf{S}_x} \{\min\{\mu_A(a), \mu_B(b)\}\} & \text{if } \mathbf{S}_x \neq \emptyset \\ 0 & \text{otherwise,} \end{cases}$$

$$(\lambda_A \circ \lambda_B)(x) = \begin{cases} \bigwedge\limits_{(a,b)\in \mathbf{S}_x} \{\max\{\lambda_A(a), \lambda_B(b)\}\} & \text{if } \mathbf{S}_x \neq \emptyset \\ 1 & \text{otherwise,} \end{cases}$$

$$(\delta_A \circ \delta_B)(x) = \begin{cases} \bigwedge\limits_{(a,b)\in \mathbf{S}_x} \{\max\{\delta_A(a), \delta_B(b)\}\} & \text{if } \mathbf{S}_x \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to verify that the structure $(Tri(S); \circ)$ is a semigroup. In the set of all tripolar fuzzy subsets of S we define the order relation on $Tri(S)$ as follows: $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$, $\lambda_A(x) \geq \lambda_B(x)$ and $\delta_A(x) \geq \delta_B(x)$ for all $x \in S$. Finally for tripolar fuzzy subsets A and B of S we define the operation $A \cap B$ as the tripolar fuzzy subsets of S defined by:

$$A \cap B := (\mu_A \cap \mu_B, \lambda_A \cup \lambda_B, \delta_A \cup \delta_B),$$

where $(\mu_A \cap \mu_B)(x) := \min\{\mu_A(x), \mu_B(x)\}$, $(\lambda_A \cup \lambda_B)(x) := \max\{\lambda_A(x), \lambda_B(x)\}$ and $(\delta_A \cup \delta_B)(x) := \max\{\delta_A(x), \delta_B(x)\}$.

The tripolar fuzzy subset $\mathcal{S} := (1_S^+, 0_S, 1_S^-)$ of S defined by $1_S^+(x) := 1$, $0_S(x) := 0$ and $1_S^-(x) = -1$ for all $x \in S$.

3. Main results

In this section, we introduce the notions of tripolar fuzzy subsemigroup, tripolar fuzzy ideal, tripolar fuzzy quasi-ideal, and tripolar fuzzy bi-ideal of the ordered semigroup. We study some of their algebraic properties and the relations between them.

Definition 3.1. Let $\mathbf{S}$ be an ordered semigroup. A tripolar fuzzy subset $A = (\mu_A, \lambda_A, \delta_A)$ of S is called a tripolar fuzzy subsemigroup of $\mathbf{S}$ if for any $x, y \in S$,

(1) $\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\}$,
(2) $\lambda_A(xy) \leq \max\{\lambda_A(x), \lambda_A(y)\}$,
(3) $\delta_A(xy) \leq \max\{\delta_A(x), \delta_A(y)\}$.

Example 3.2. Let $S = \{a, b, c\}$. Define the binary operation $\circ$ on S by the following table:

$\circ$	a	b	c
a	a	a	a
b	a	a	a
c	a	c	c

and define an order on S as follows:

$$\leq := \{(a, b), (a, c)\} \cup \Delta_S,$$

where Δ_S is an equality relation on S. That is, $\Delta_S := \{(x, x) \in S \times S \mid x \in S\}$. Then, $\mathbf{S} := (S; \circ, \leq)$ is an ordered semigroup. We define a tripolar fuzzy subset $A = (\mu_A, \lambda_A, \delta_A)$ of S by:

S	μ_A	λ_A	δ_A
a	0.8	0.1	-0.8
b	0.7	0.2	-0.7
c	0.7	0.2	-0.7

Then A is a tripolar fuzzy subsemigroup of $\mathbf{S}$.

Definition 3.3. Let $\mathbf{S}$ be an ordered semigroup. A tripolar fuzzy subset $A = (\mu_A, \lambda_A, \delta_A)$ of S is called a tripolar fuzzy right ideal (resp. tripolar fuzzy left ideal) of $\mathbf{S}$ if for every $x, y \in S$,

(1) $\mu_A(xy) \geq \mu_A(x)$ (resp. $\mu_A(xy) \geq \mu_A(y)$),
(2) $\lambda_A(xy) \leq \lambda_A(x)$ (resp. $\lambda_A(xy) \leq \lambda_A(y)$),
(3) $\delta_A(xy) \leq \delta_A(x)$ (resp. $\delta_A(xy) \leq \delta_A(y)$),
(4) if $x \leq y$, then $\mu_A(x) \geq \mu_A(y)$, $\lambda(x) \leq \lambda(y)$, $\delta(x) \leq \delta(y)$.

A tripolar fuzzy subset A of S is called a *tripolar fuzzy two-side ideal* (*tripolar fuzzy ideal*) of $\mathbf{S}$ if A is both a tripolar fuzzy right and a tripolar fuzzy left ideal of $\mathbf{S}$.

Example 3.4. Let $S = \{a, b, c\}$. Define the binary operation $\circ$ on S by the following table:

$\circ$	a	b	c
a	a	a	a
b	a	a	a
c	a	b	c

and define an order on S as follows:

$$\leq := \{(a, b)\} \cup \Delta_S,$$

where Δ_S is an equality relation on S. That is, $\Delta_S := \{(x, x) \in S \times S \mid x \in S\}$. Then, $\mathbf{S} := (S; \circ, \leq)$ is an ordered semigroup. We define a tripolar fuzzy subset $A = (\mu_A, \lambda_A, \delta_A)$ of S by:

S	μ_A	λ_A	δ_A
a	0.7	0.1	-0.6
b	0.5	0.2	-0.5
c	0.7	0.1	-0.6

Then A is a tripolar fuzzy left ideal of $\mathbf{S}$ but, A is not a tripolar fuzzy right ideal of $\mathbf{S}$ since $\mu_A(c \circ b) = \mu_A(b) = 0.5 < 0.7 = \mu_A(c)$.

Example 3.5. Let $A = (\mu_A, \lambda_A, \delta_A)$ be as Example 3.2. Then it is not difficult to show that A is a tripolar fuzzy ideal of $\mathbf{S}$.

Definition 3.6. Let $\mathbf{S}$ be an ordered semigroup. A tripolar fuzzy subset $A = (\mu_A, \lambda_A, \delta_A)$ of S is called a tripolar fuzzy quasi-ideal of $\mathbf{S}$ if satisfies the following conditions:

(1) $(A \circ \mathcal{S}) \cap (\mathcal{S} \circ A) \subseteq A$, and
(2) if $x \leq y$, then $\mu_A(x) \geq \mu_A(y)$, $\lambda(x) \leq \lambda(y)$, $\delta(x) \leq \delta(y)$ for all $x, y \in S$.

Example 3.7. Let $S = \{a, b, c, d, f\}$. Define the binary operation $*$ on S by the following table:

$*$	a	b	c	d	f
a	a	a	a	a	a
b	a	b	a	d	a
c	a	f	c	c	f
d	a	b	d	d	b
f	a	f	a	c	a

and define an order on S as follows:

$$\leq := \{(a, b), (a, c), (a, d), (a, f)\} \cup \Delta_S,$$

where Δ_S is an equality relation on S. That is, $\Delta_S := \{(x, x) \in S \times S \mid x \in S\}$. Then, $\mathbf{S} := (S; *, \leq)$ is an ordered semigroup. We define a tripolar fuzzy subset $A = (\mu_A, \lambda_A, \delta_A)$ of S by:

S	μ_A	λ_A	δ_A
a	0.8	0.1	-0.7
b	0.8	0.1	-0.7
c	0.5	0.4	-0.4
d	0.3	0.3	-0.2
f	0.8	0.1	-0.7

Then A is a tripolar fuzzy quasi-ideal of $\mathbf{S}$. By simple calculation, we see that A is not a tripolar fuzzy right ideal of $\mathbf{S}$. Indeed, there are $b, d \in S$ such that $\mu_A(b * d) = \mu_A(d) = 0.3 < 0.8 = \mu_A(b)$. Moreover, A is not a tripolar fuzzy left ideal of $\mathbf{S}$ since there are $c, d \in S$, such that $\mu_A(d * c) = \mu_A(d) = 0.3 < 0.5 = \mu_A(c)$.

Definition 3.8. Let $\mathbf{S}$ be an ordered semigroup. A tripolar fuzzy subsemigroup $A = (\mu_A, \lambda_A, \delta_A)$ of S is called a tripolar fuzzy bi-ideal of $\mathbf{S}$ if for every $x, y, z \in S$,

(1) $\mu_A(xyz) \geq \min\{\mu_A(x), \mu_A(z)\}$,
(2) $\lambda_A(xyz) \leq \max\{\lambda_A(x), \lambda_A(z)\}$,
(3) $\delta_A(xyz) \leq \max\{\delta_A(x), \delta_A(z)\}$,
(4) if $x \leq y$, then $\mu_A(x) \geq \mu_A(y)$, $\lambda(x) \leq \lambda(y)$, $\delta(x) \leq \delta(y)$.

Example 3.9. Let $S = \{a, b, c, d\}$. Define the binary operation $\circ$ on S by the following table:

$\circ$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

and define an order on S as follows:

$$\leq := \{(a, b)\} \cup \Delta_S,$$

where Δ_S is an equality relation on S. That is, $\Delta_S := \{(x,x) \in S \times S \mid x \in S\}$. Then, $\mathbf{S} := (S; \circ, \leq)$ is an ordered semigroup. We define a tripolar fuzzy subset $A = (\mu_A, \lambda_A, \delta_A)$ of S by:

S	μ_A	λ_A	δ_A
a	0.8	0.1	-0.7
b	0.3	0.1	-0.2
c	0.5	0.4	-0.4
d	0.8	0.1	-0.7

Then A is a tripolar fuzzy bi-ideal of $\mathbf{S}$ but not a tripolar fuzzy quasi-ideal of $\mathbf{S}$.

Lemma 3.10. *Let $\mathbf{S}$ be an ordered semigroup and $A = (\mu_A, \lambda_A, \delta_A)$ a tripolar fuzzy subset of S. Then the following conditions are equivalent.*

(1) *A is a tripolar fuzzy right ideal of $\mathbf{S}$.*

(2) *A satisfies*

 (2.1) *$A \circ \mathcal{S} \subseteq A$,*

 (2.2) *if $x \leq y$, then $\mu_A(x) \geq \mu_A(y)$,$\lambda_A(x) \leq \lambda_A(y)$ and $\delta_A(x) \leq \delta_A(y)$ for all $x, y \in S$.*

Proof. (1)$\Rightarrow$(2). Let $A = (\mu_A, \lambda_A, \delta_A)$ be a tripolar fuzzy right ideal of $\mathbf{S}$ and $x \in S$. If $S_x = \emptyset$, we obtain that $(\mu_A \circ 1_S^+)(x) = 0 \leq \mu_A(x)$, $(\lambda_A \circ 0_S)(x) = 1 \geq \lambda_A(x)$ and $(\delta_A \circ 1_S^-)(x) = 0 \geq \delta_A(x)$. If $S_x \neq \emptyset$, we have that

$$
\begin{aligned}
(\mu_A \circ 1_S^+)(x) &= \bigvee_{(a,b)\in \mathbf{S}_x} \{\min\{\mu_A(a), 1_S^+(b)\}\} = \bigvee_{(a,b)\in \mathbf{S}_x} \mu_A(a) \\
&\leq \bigvee_{(a,b)\in \mathbf{S}_x} \mu_A(ab) \leq \bigvee_{(a,b)\in \mathbf{S}_x} \mu_A(x) = \mu_A(x),
\end{aligned}
$$

$$
\begin{aligned}
(\lambda_A \circ 0_S)(x) &= \bigwedge_{(a,b)\in \mathbf{S}_x} \{\max\{\lambda_A(a), 0_S(b)\}\} = \bigwedge_{(a,b)\in \mathbf{S}_x} \lambda_A(a) \\
&\geq \bigwedge_{(a,b)\in \mathbf{S}_x} \lambda_A(ab) \geq \bigwedge_{(a,b)\in \mathbf{S}_x} \lambda_A(x) = \lambda_A(x),
\end{aligned}
$$

and

$$
\begin{aligned}
(\delta_A \circ 1_S^-)(x) &= \bigwedge_{(a,b)\in \mathbf{S}_x} \{\max\{\delta_A(a), 1_S^-(b)\}\} = \bigwedge_{(a,b)\in \mathbf{S}_x} \delta_A(a) \\
&\geq \bigwedge_{(a,b)\in \mathbf{S}_x} \delta_A(ab) \geq \bigwedge_{(a,b)\in \mathbf{S}_x} \delta_A(x) = \delta_A(x).
\end{aligned}
$$

For any two cases, we obtain $A \circ \mathcal{S} \subseteq A$ and (2.2) is clear by hypothesis.

(2)$\Rightarrow$(1). Let $x, y \in S$. By assumption, we have that

$$
\begin{aligned}
\mu_A(xy) &\geq (\mu_A \circ 1_S^+)(xy) = \bigvee_{(a,b)\in \mathbf{S}_{xy}} \{\min\{\mu_A(a), 1_S^+(b)\}\} \\
&\geq \min\{\mu_A(x), 1_S^+(y)\} = \mu_A(x),
\end{aligned}
$$

$$\begin{aligned}\lambda_A(xy) &\leq (\lambda_A \circ 0_S)(xy) = \bigwedge_{(a,b)\in \mathbf{S}_{xy}} \{\max\{\lambda_A(a), 0_S(b)\}\} \\ &\leq \max\{\lambda_A(x), 0_S(y)\} = \lambda_A(x),\end{aligned}$$

and

$$\begin{aligned}\delta_A(xy) &\leq (\delta_A \circ 1_S^-)(xy) = \bigwedge_{(a,b)\in \mathbf{S}_{xy}} \{\max\{\delta_A(a), 1_S^-(b)\}\} \\ &\leq \max\{\delta_A(x), 1_S^-(y)\} = \delta_A(x).\end{aligned}$$

Let $x, y \in S$ be such that $x \leq y$. Then, by assumption, we have that $\mu_A(x) \geq \mu_A(y)$,$\lambda_A(x) \leq \lambda_A(y)$ and $\delta_A(x) \leq \delta_A(y)$. Hence A is a tripolar fuzzy right ideal of $\mathbf{S}$. □

By a similar method to Lemma 3.10, we have the following lemma.

Lemma 3.11. *Let* $\mathbf{S}$ *be an ordered semigroup and* $A = (\mu_A, \lambda_A, \delta_A)$ *a tripolar fuzzy subset of* S. *Then the following conditions are equivalent.*

(1) *A is a tripolar fuzzy left ideal of S.*
(2) *A satisfies*
  (2.1) $\mathcal{S} \circ A \subseteq A$,
  (2.2) *if $x \leq y$, then $\mu_A(x) \geq \mu_A(y)$,$\lambda_A(x) \leq \lambda_A(y)$ and $\delta_A(x) \leq \delta_A(y)$ for all $x, y \in S$.*

Combining Lemma 3.10 and 3.11, we obtain the following result.

Lemma 3.12. *Let* $\mathbf{S}$ *be an ordered semigroup and* $A = (\mu_A, \lambda_A, \delta_A)$ *a tripolar fuzzy subset of* S. *Then the following conditions are equivalent.*

(1) *A is a tripolar fuzzy ideal of S.*
(2) *A satisfies*
  (2.1) *$\mathcal{S} \circ A \subseteq A$ and $A \circ \mathcal{S} \subseteq A$,*
  (2.2) *if $x \leq y$, then $\mu_A(x) \geq \mu_A(y)$,$\lambda_A(x) \leq \lambda_A(y)$ and $\delta_A(x) \leq \delta_A(y)$ for all $x, y \in S$.*

Lemma 3.13. *Let* $\mathbf{S}$ *be an ordered semigroup and* $A = (\mu_A, \lambda_A, \delta_A)$ *a tripolar fuzzy subset of* S. *Then the following conditions are equivalent.*

(1) *A is a tripolar fuzzy bi-ideal of $\mathbf{S}$.*
(2) *A satisfies*
  (2.1) $A \circ A \subseteq A$.
  (2.2) $A \circ \mathcal{S} \circ A \subseteq A$,
  (2.3) *if $x \leq y$, then $\mu_A(x) \geq \mu_A(y)$,$\lambda_A(x) \leq \lambda_A(y)$ and $\delta_A(x) \leq \delta_A(y)$ for all $x, y \in S$.*

Proof. (1)⇒(2). Let A be a tripolar fuzzy bi-ideal of $\mathbf{S}$ and $x \in S$. (2.1). If $\mathbf{S}_x = \emptyset$, we obtain $(\mu_A \circ \mu_A)(x) = 0 \leq \mu_A(x), (\lambda_A \circ \lambda_A)(x) = 1 \geq \lambda_A(x)$ and $(\delta_A \circ \delta)(x) = 0 \geq \delta_A(x)$. If $\mathbf{S}_x \neq \emptyset$, we obtain

$$(\mu_A \circ \mu_A)(x) = \bigvee_{(a,b)\in \mathbf{S}_x} \{\min\{\mu_A(a), \mu_A(b)\}\}$$

$$
\begin{aligned}
&\leq \bigvee_{(a,b)\in \mathbf{S}_x} \{\mu_A(ab)\} \\
&\leq \bigvee_{(a,b)\in \mathbf{S}_x} \{\mu_A(x)\} \\
&= \mu_A(x),
\end{aligned}
$$

$$
\begin{aligned}
(\lambda_A \circ \lambda_A)(x) &= \bigwedge_{(a,b)\in \mathbf{S}_x} \{\max\{\lambda_A(a), \lambda_A(b)\}\} \\
&\geq \bigwedge_{(a,b)\in \mathbf{S}_x} \{\lambda_A(ab)\} \\
&\geq \bigwedge_{(a,b)\in \mathbf{S}_x} \{\lambda_A(x)\} \\
&= \lambda_A(x),
\end{aligned}
$$

and

$$
\begin{aligned}
(\delta_A \circ \delta_A)(x) &= \bigwedge_{(a,b)\in \mathbf{S}_x} \{\max\{\delta_A(a), \delta_A(b)\}\} \\
&\geq \bigwedge_{(a,b)\in \mathbf{S}_x} \{\delta_A(ab)\} \\
&\geq \bigwedge_{(a,b)\in \mathbf{S}_x} \{\delta_A(x)\} \\
&= \delta_A(x).
\end{aligned}
$$

For any two cases we have $A \circ A \subseteq A$. The proof of (2.1) is completed.

(2.2) If $\mathbf{S}_x = \emptyset$, we obtain $(\mu_A \circ 1_S^+ \circ \mu_A)(x) = 0 \leq \mu_A(x), (\lambda_A \circ 0_S \circ \lambda_A)(x) = 1 \geq \lambda_A(x)$, and $(\delta_A \circ 1_S^- \circ \delta_A)(x) = 0 \geq \delta_A(x)$. If $\mathbf{S}_x \neq \emptyset$, we obtain

$$
\begin{aligned}
(\mu_A \circ 1_S^+ \circ \mu_A)(x) &= \bigvee_{(a,b)\in \mathbf{S}_x} \{\min\{\mu_A(a), (1_S^+ \circ \mu_A)(b)\}\} \\
&= \bigvee_{(a,b)\in \mathbf{S}_x} \{\min\{\mu_A(a), \bigvee_{(u,v)\in \mathbf{S}_b} \{\min\{1_S^+(u), \mu_A(v)\}\}\}\} \\
&= \bigvee_{(a,b)\in \mathbf{S}_x} \bigvee_{(u,v)\in \mathbf{S}_b} \{\min\{\mu_A(a), 1_S^+(u), \mu_A(v)\}\} \\
&= \bigvee_{(a,uv)\in \mathbf{S}_x} \{\min\{\mu_A(a), 1_S^+(u), \mu_A(v)\}\} \\
&= \bigvee_{(a,uv)\in \mathbf{S}_x} \{\min\{\mu_A(a), \mu_A(v)\}\} \\
&\leq \bigvee_{(a,uv)\in \mathbf{S}_x} \{\mu_A(auv)\}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \bigvee_{(a,uv)\in \mathbf{S}_x} \{\mu_A(x)\} \\
&= \mu_A(x),
\end{aligned}
$$

$$
\begin{aligned}
(\lambda_A \circ 0_S \circ \lambda_A)(x) &= \bigwedge_{(a,b)\in \mathbf{S}_x} \{\max\{\lambda_A(a), (0_S \circ \lambda_A)(b)\}\} \\
&= \bigwedge_{(a,b)\in \mathbf{S}_x} \{\max\{\mu_A(a), \bigwedge_{(u,v)\in \mathbf{S}_b} \{\max\{0_S(u), \lambda_A(v)\}\}\}\} \\
&= \bigwedge_{(a,b)\in \mathbf{S}_x} \bigwedge_{(u,v)\in \mathbf{S}_b} \{\max\{\lambda_A(a), 0_S(u), \lambda_A(v)\}\} \\
&= \bigwedge_{(a,uv)\in \mathbf{S}_x} \{\max\{\lambda_A(a), 0_S(u), \lambda_A(v)\}\} \\
&= \bigwedge_{(a,uv)\in \mathbf{S}_x} \{\max\{\lambda_A(a), \lambda_A(v)\}\} \\
&\geq \bigwedge_{(a,uv)\in \mathbf{S}_x} \{\lambda_A(auv)\} \\
&\geq \bigwedge_{(a,uv)\in \mathbf{S}_x} \{\lambda_A(x)\} \\
&= \lambda_A(x),
\end{aligned}
$$

and

$$
\begin{aligned}
(\delta_A \circ 1_S^- \circ \delta_A)(x) &= \bigwedge_{(a,b)\in \mathbf{S}_x} \{\max\{\delta_A(a), (1_S^- \circ \delta_A)(b)\}\} \\
&= \bigwedge_{(a,b)\in \mathbf{S}_x} \{\max\{\delta_A(a), \bigwedge_{(u,v)\in \mathbf{S}_b} \{\max\{1_S^-(u), \delta_A(v)\}\}\}\} \\
&= \bigwedge_{(a,b)\in \mathbf{S}_x} \bigwedge_{(u,v)\in \mathbf{S}_b} \{\max\{\delta_A(a), 1_S^-(u), \delta_A(v)\}\} \\
&= \bigwedge_{(a,uv)\in \mathbf{S}_x} \{\max\{\delta_A(a), 1_S^-(u), \delta_A(v)\}\} \\
&= \bigwedge_{(a,uv)\in \mathbf{S}_x} \{\max\{\delta_A(a), \delta_A(v)\}\} \\
&\geq \bigwedge_{(a,uv)\in \mathbf{S}_x} \{\delta_A(auv)\} \\
&\geq \bigwedge_{(a,uv)\in \mathbf{S}_x} \{\delta_A(x)\} \\
&= \delta_A(x).
\end{aligned}
$$

For any two cases we obtain $A \circ \mathcal{S} \circ A \subseteq A$. (2.3) is clear by hypothesis.

(2)⇒(1). Firstly, we prove that A is a tripolar fuzzy subsemigroup of $\mathbf{S}$. Let $x, y \in S$. By the hypothesis of (2.1), we obtain that

$$\begin{aligned} \mu_A(xy) &\leq (\mu_A \circ \mu_A)(xy) \\ &= \bigvee_{(a,b)\in \mathbf{S}_{xy}} \{\min\{\mu_A(a), \mu_A(b)\}\} \\ &\leq \min\{\mu_A(x), \mu_A(y)\}, \end{aligned}$$

$$\begin{aligned} \lambda_A(xy) &\geq (\lambda_A \circ \lambda_A)(xy) \\ &= \bigwedge_{(a,b)\in \mathbf{S}_{xy}} \{\max\{\lambda_A(a), \lambda_A(b)\}\} \\ &\geq \max\{\lambda_A(x), \lambda_A(y)\}, \end{aligned}$$

and

$$\begin{aligned} \delta_A(xy) &\geq (\delta_A \circ \delta_A)(xy) \\ &= \bigwedge_{(a,b)\in \mathbf{S}_{xy}} \{\max\{\delta_A(a), \delta_A(b)\}\} \\ &\geq \max\{\delta_A(x), \delta_A(y)\}. \end{aligned}$$

Therefore A is a tripolar fuzzy subsemigroup of $\mathbf{S}$, as needed. Secondly, we let $x, y, z \in S$. By the hypothesis of (2.2), we obtain

$$\begin{aligned} \mu_A(xyz) &\leq (\mu_A \circ 1_S^+ \circ \mu_A)(xyz) \\ &= \bigvee_{(a,b)\in \mathbf{S}_{xyz}} \{\min\{\mu_A(a), (1_S^+ \circ \mu_A)(b)\}\} \\ &\leq \min\{\mu_A(x), (1_S^+ \circ \mu_A)(yz)\} \\ &= \min\{\mu_A(x), \bigvee_{(u,v)\in \mathbf{S}_{yz}} \{\min\{1_S^+(u), \mu_A(v)\}\}\} \\ &\leq \min\{\mu_A(x), \min\{1_S^+(y), \mu_A(z)\}\} \\ &= \min\{\mu_A(x), \mu_A(z)\}, \end{aligned}$$

$$\begin{aligned} \lambda_A(xyz) &\geq (\lambda_A \circ 0_S \circ \lambda_A)(xyz) \\ &= \bigwedge_{(a,b)\in \mathbf{S}_{xyz}} \{\max\{\lambda_A(a), (0_S \circ \lambda_A)(b)\}\} \\ &\geq \max\{\lambda_A(x), (0_S \circ \lambda_A)(yz)\} \\ &= \max\{\lambda_A(x), \bigwedge_{(u,v)\in \mathbf{S}_{yz}} \{\max\{0_S(u), \lambda_A(v)\}\}\} \\ &\geq \max\{\lambda_A(x), \max\{0_S(y), \lambda_A(z)\}\} \\ &= \max\{\lambda_A(x), \lambda_A(z)\}, \end{aligned}$$

$$\delta_A(xyz) \geq (\delta_A \circ 1_S^- \circ \delta_A)(xyz)$$

$$\begin{aligned}
&= \bigwedge_{(a,b)\in \mathbf{S}_{xyz}} \{\max\{\delta_A(a), (1_S^- \circ \delta_A)(b)\}\} \\
&\geq \max\{\delta_A(x), (1_S^- \circ \delta_A)(yz)\} \\
&= \max\{\delta_A(x), \bigwedge_{(u,v)\in \mathbf{S}_{yz}} \{\max\{1_S^-(u), \delta_A(v)\}\}\} \\
&\geq \max\{\delta_A(x), \max\{1_S^-(y), \delta_A(z)\}\} \\
&= \max\{\delta_A(x), \delta_A(z)\}.
\end{aligned}$$

By the firstly and the secondly, we have A is a tripolar fuzzy bi-ideal of $\mathbf{S}$. □

We, now give a relation between their tripolar fuzzy ideals.

Theorem 3.14. *Let $\mathbf{S}$ be an ordered semigroups. Every tripolar fuzzy right ideals of $\mathbf{S}$ are tripolar fuzzy quasi-ideals of $\mathbf{S}$.*

Proof. Let $A = (\mu_A, \lambda_A, \delta_A)$ be a tripolar fuzzy right ideal of $\mathbf{S}$. Since $(A \circ \mathcal{S}) \cap (\mathcal{S} \circ A) \subseteq A \circ \mathcal{S} \subseteq A$. By Lemma 3.10, A is a tripolar fuzzy quasi-ideal of $\mathbf{S}$. □

In a similar way to Theorem 3.14, we obtain the following theorem.

Theorem 3.15. *Let $\mathbf{S}$ be an ordered semigroups. Every tripolar fuzzy left ideals of $\mathbf{S}$ are tripolar fuzzy quasi-ideals of $\mathbf{S}$.*

Combining, Theorem 3.13 and 3.14, we obtain the following result.

Corollary 3.16. *Let $\mathbf{S}$ be an ordered semigroups. Every tripolar fuzzy ideals of $\mathbf{S}$ are tripolar fuzzy quasi-ideals of $\mathbf{S}$.*

The converse of Corollary 3.16, in general, is not true, this is shown by Example 3.7.

Theorem 3.17. *Let $\mathbf{S}$ be an ordered semigroup. Every tripolar fuzzy quasi-ideals of $\mathbf{S}$ are tripolar fuzzy bi-ideals of $\mathbf{S}$.*

Proof. Let $A = (\mu_A, \lambda_A, \delta_A)$ be a tripolar fuzzy quasi-ideal of $\mathbf{S}$. Since $A \circ A \subseteq A \circ \mathcal{S}$ and $A \circ A \subseteq \mathcal{S} \circ A$ and then $A \circ A \subseteq (A \circ \mathcal{S}) \cap (\mathcal{S} \circ A) \subseteq A$, and since $A \circ \mathcal{S} \circ A \subseteq A \circ \mathcal{S} \circ \mathcal{S} \subseteq A \circ \mathcal{S}$ and $A \circ \mathcal{S} \circ A \subseteq \mathcal{S} \circ \mathcal{S} \circ A \subseteq \mathcal{S} \circ A$ and then $A \circ \mathcal{S} \circ A \subseteq (A \circ \mathcal{S}) \cap (\mathcal{S} \circ A) \subseteq A$. By Lemma 3.13, we obtain A is a tripolar fuzzy bi-ideal of $\mathbf{S}$. □

The converse of Theorem 3.17 is not true, in general as the following example.

Example 3.18. Let $A = (\mu_A, \lambda_A, \delta_A)$ be as Example 3.1. Then it is not difficult to show that A is not a tripolar fuzzy quasi-ideal of $\mathbf{S}$. Indeed, we have $\mathbf{S}_b = \{(d, c) \in S \times S \mid b \leq d \circ c\}$ and then

$$\bigvee_{(d,c)\in \mathbf{S}_b} \{\min\{\mu_A(c), \mu_A(d)\}\} = \min\{\mu_A(c), \mu_A(d)\} = \mu_A(c) > \mu_A(b).$$

We now have a question arising from the previously illustrated results: how do the concepts of tripolar fuzzy quasi-ideals and tripolar fuzzy bi-ideals coincide?

Let us consider a class of ordered semigroups called regular ordered semigroups. Let $\mathbf{S}$ be an ordered semigroup. The element $a \in S$ is said to be *regular* if there exists element $x \in S$ such that $a \leq axa$. An ordered semigroup $\mathbf{S}$ is regular if every element in S is regular. We address our question by the following theorem.

Theorem 3.19. *Let* $\mathbf{S}$ *be a regular ordered semigroup. Every tripolar fuzzy bi-ideals are tripolar fuzzy quasi-ideals of* $\mathbf{S}$.

Proof. Let $A = (\mu_A, \lambda_A, \delta_A)$ be a tripolar fuzzy bi-ideal of $\mathbf{S}$ and $a \in S$. Since $\mathbf{S}$ is regular ordered semigroup, there exists $x \in S$ such that $a \leq axa \leq ax(axa) = axa(xa)$, and then $S_a \neq \emptyset$, we have that

$$\begin{aligned}(\mu_A \circ 1_S^+)(a) &= \bigvee_{(u,v)\in S_a} \{\min\{\mu_A(u), 1_S^+(v)\}\} \leq \min\{\mu_A(axa), 1_S^+(xa)\} \\ &= \mu_A(axa) \leq \mu_A(a).\end{aligned}$$

Similarly, we also obtain that $(1_S^+ \circ \mu_A)(a) \leq \mu_A(a)$. It follows that

$$\left[(\mu_A \circ 1_S^+) \cap (1_S^+ \circ \mu_A)\right](a) = \min\{(\mu_A \circ 1_S^+)(a), (1_S^+ \circ \mu_A)(a)\} \leq \mu_A(a),$$

and

$$\begin{aligned}(\lambda_A \circ 0_S)(a) &= \bigwedge_{(u,v)\in S_a} \{\max\{\lambda_A(u), 0_S(v)\}\} \geq \max\{\lambda_A(axa), 0_S(xa)\} \\ &= \lambda_A(axa) \geq \lambda_A(a).\end{aligned}$$

Similarly, we also obtain that $(0_S \circ \lambda_A)(a) \geq \lambda_A(a)$. It follows that

$$\left[(\lambda_A \circ 0_S) \cup (0_S \circ \lambda_A)\right](a) = \max\{(\lambda_A \circ 0_S)(a), (0_S \circ \lambda_A)(a)\} \geq \lambda_A(a),$$

and

$$\begin{aligned}(\delta_A \circ 1_S^-)(a) &= \bigwedge_{(u,v)\in S_a} \{\max\{\delta_A(u), 1_S^-(v)\}\} \geq \max\{\delta_A(axa), 1_S^-(xa)\} \\ &= \delta_A(axa) \geq \delta_A(a).\end{aligned}$$

Similarly, we also obtain that $(1_S^- \circ \delta_A)(a) \geq \delta_A(a)$. It follows that

$$\left[(\delta_A \circ 1_S^-) \cup (1_S^- \circ \delta_A)\right](a) = \max\{(\delta_A \circ 1_S^-)(a), (1_S^- \circ \delta_A)(a)\} \geq \delta_A(a).$$

Therefore A is a tripolar fuzzy quasi-ideal in $\mathbf{S}$. □

The final major result that we illustrate concerns tripolar fuzzy right, left, and quasi-ideals of an ordered semigroup. That is, we aim to prove that for any tripolar fuzzy quasi-ideal, it is minimal of a tripolar right and left ideal. We also aim to prove the converse. Firstly, we illustrate that for any tripolar quasi-ideal, it is a minimum of a tripolar fuzzy right and left ideal. To do so, we provide some elements as follows:

Lemma 3.20. *Let* $\mathbf{S}$ *be an ordered semigroup and* $A = (\mu_A, \lambda_A, \delta_A)$ *a tripolar fuzzy subset of* S. *Then the following conditions hold. For each* $x, y \in S$

(1) $(1_S^+ \circ \mu_A)(xy) \geq \mu_A(y)$,
(2) $(0_S \circ \lambda_A)(xy) \leq \lambda_A(y)$,
(3) $(1_S^- \circ \delta_A)(xy) \leq \lambda_A(y)$.

Proof. Let $A = (\mu_A, \lambda_A, \delta_A)$ be a tripolar fuzzy subset of S and $x, y \in S$.

(1) Since $(x, y) \in \mathbf{S}_{xy}$, this implies that $\mathbf{S}_{xy} \neq \emptyset$, we obtain

$$\begin{aligned} (1_S^+ \circ \mu_A)(xy) &= \bigvee_{(a,b)\in \mathbf{S}_{xy}} \{\min\{1_S^+(a), \mu_A(b)\}\} \\ &\geq \min\{1_S^+(x), \mu_A(y)\} \\ &= \mu_A(y). \end{aligned}$$

(2) Since $(x, y) \in \mathbf{S}_{xy}$, this implies that $\mathbf{S}_{xy} \neq \emptyset$, we obtain

$$\begin{aligned} (0_S \circ \mu_A)(xy) &= \bigwedge_{(a,b)\in \mathbf{S}_{xy}} \{\max\{0_S(a), \lambda_A(b)\}\} \\ &\leq \max\{0_S(x), \lambda_A(y)\} \\ &= \lambda_A(y). \end{aligned}$$

(3) Since $(x, y) \in \mathbf{S}_{xy}$, this implies that $\mathbf{S}_{xy} \neq \emptyset$, we obtain

$$\begin{aligned} (1_S^- \circ \mu_A)(xy) &= \bigwedge_{(a,b)\in \mathbf{S}_{xy}} \{\max\{1_S^-(a), \delta_A(b)\}\} \\ &\leq \max\{1_S^-(x), \delta_A(y)\} \\ &= \delta_A(y). \end{aligned}$$

□

By a similar method of Lemma 3.20, we obtain the following result.

Lemma 3.21. *Let* $\mathbf{S}$ *be an ordered semigroup and* $A = (\mu_A, \lambda_A, \delta_A)$ *a tripolar fuzzy subset of* S. *Then the following conditions hold. For each* $x, y \in S$

(1) $(\mu_A \circ 1_S^+)(xy) \geq \mu_A(y)$,
(2) $(\lambda_A \circ 0_S)(xy) \leq \lambda_A(y)$,
(3) $(\delta_A \circ 1_S^-)(xy) \leq \lambda_A(y)$.

Lemma 3.22. *Let* $\mathbf{S}$ *be an ordered semigroup and* $A = (\mu_A, \lambda_A, \delta_A)$ *a tripolar fuzzy subset of* S. *Then the following conditions hold. For each* $x, y \in S$

(1) $(1_S^+ \circ \mu_A)(xy) \geq (1_S^+ \circ \mu_A)(y)$,
(2) $(0_S \circ \lambda_A)(xy) \leq (0_S \circ \lambda_A)(y)$,
(3) $(1_S^- \circ \lambda_A)(xy) \leq (1_S^- \circ \lambda_A)(y)$.

Proof. Let $A = (\mu_A, \lambda_A, \delta_A)$ be a tripolar fuzzy subset of S and $x, y \in S$.

(1) If $S_y = \emptyset$, then $(1_A^+ \circ \mu_A)(xy) \geq 0 = (1_A^+ \circ \mu_A)(y)$. If $S_y \neq \emptyset$, it follows that $y \leq uv$ for all $(u,v) \in S_y$ and then $xy \leq xuv$. Then

$$\begin{aligned}
(1_S^+ \circ \mu_A)(xy) &= \bigvee_{(a,b)\in S_{xy}} \{\min\{1_S^+(a), \mu_A(b)\}\} \\
&\geq \min\{1_S^+(xu), \mu_A(v)\} \\
&\geq \min\{1_S^+(u), \mu_A(v)\} \\
&= \bigvee_{(u,v)\in S_y} \{\min\{1_S^+(u), \mu_A(v)\}\} \\
&= (1^+ \circ \mu_A)(y).
\end{aligned}$$

(2) If $S_y = \emptyset$, then $(0_A \circ \lambda_A)(xy) \leq 1 = (0_A \circ \lambda_A)(y)$. If $S_y \neq \emptyset$, it follows that $y \leq uv$ for all $(u,v) \in S_y$ and then $xy \leq xuv$. Then

$$\begin{aligned}
(0_S \circ \lambda_A)(xy) &= \bigwedge_{(a,b)\in S_{xy}} \{\max\{0_S(a), \lambda_A(b)\}\} \\
&\leq \max\{0_S(xu), \lambda_A(v)\} \\
&= \max\{0_S(u), \lambda_A(v)\} \\
&= \bigwedge_{(u,v)\in \mathbf{S}_y} \{\max\{0_S(u), \lambda_A(v)\}\} \\
&= (0_S \circ \lambda_A)(y).
\end{aligned}$$

(3) If $S_y = \emptyset$, then $(1_A^- \circ \delta_A)(xy) \leq 0 = (1_A^- \circ \delta_A)(y)$. If $S_y \neq \emptyset$, it follows that $y \leq uv$ for all $(u,v) \in S_y$ and then $xy \leq xuv$. Then and

$$\begin{aligned}
(1_S^- \circ \delta_A)(xy) &= \bigwedge_{(a,b)\in S_{xy}} \{\max\{1_S^-(a), \delta_A(b)\}\} \\
&\leq \max\{1_S^-(xu), \delta_A(v)\} \\
&= \max\{1_S^-(u), \delta_A(v)\} \\
&= \bigwedge_{(u,v)\in \mathbf{S}_y} \{\max\{1_S^-(u), \delta_A(v)\}\} \\
&= (1_S^- \circ \delta_A(y).
\end{aligned}$$

□

In a similar way to Lemma 3.22, we obtain the following lemmas.

Lemma 3.23. *Let* $\mathbf{S}$ *be an ordered semigroup and* $A = (\mu_A, \lambda_A, \delta_A)$ *a tripolar fuzzy subset of* S. *Then the following conditions hold. For each* $x, y \in S$

(1) $(\mu_A \circ 1_S^+)(xy) \geq (\mu_A \circ 1_S^+)(x)$,
(2) $(\lambda_A \circ 0_S)(xy) \leq (\lambda_A \circ 0_S)(x)$,
(3) $(\delta_A \circ 1_S^-)(xy) \leq (\lambda_A \circ 1_S^-)(x)$.

It is easy to verify the following lemmas.

Lemma 3.24. *Let* $\mathbf{S}$ *be an ordered semigroup and* $x, y \in S$. *Let* $A = (\mu_A, \lambda_A, \delta_A)$ *be a tripolar fuzzy subset of* S *and* $x \le y$. *Then the following conditions hold. For each* $x, y \in S$

(1) $(\mu_A \circ 1_S^+)(x) \ge (\mu_A \circ 1_S^+)(y)$,
(2) $(\lambda_A \circ 0_S)(x) \le (\lambda_A \circ 0_S)(y)$,
(3) $(\delta_A \circ 1_S^-)(x) \le (\delta_A \circ 1_S^-)(y)$.

Proof. Let $A = (\mu_A, \lambda_A, \delta_A)$ be a tripolar fuzzy subset of S and $x \le y$ for all $x, y \in S$. It is easy to verify that $\mathbf{S}_y \subseteq \mathbf{S}_x$.
(1) If $\mathbf{S}_y = \emptyset$, we obtain $(\mu_A \circ 1_S^+)(x) \ge 0 = (\mu_A \circ 1_S^+)(y)$. If $\mathbf{S}_y \neq \emptyset$, it follows that $y \le uv$ for all $(u, v) \in \mathbf{S}_y$ and we obtain

$$\begin{aligned} (\mu_A \circ 1_S^+)(y) &= \bigvee_{(u,v)\in \mathbf{S}_y} \{\min\{\mu_A(u), 1_S^+(v)\}\} \\ &\le \bigvee_{(u,v)\in \mathbf{S}_x} \{\min\{\mu_A(u), 1_S^+(v)\}\} \\ &= (\mu_A \circ 1_S^+)(x). \end{aligned}$$

(2) If $\mathbf{S}_y = \emptyset$, we obtain $(\lambda_A \circ 0_S)(x) \le 1 = (\lambda_A \circ 0_S)(y)$. If $\mathbf{S}_y \neq \emptyset$, it follows that $y \le uv$ for all $(u, v) \in \mathbf{S}_y$ and we obtain

$$\begin{aligned} (\lambda_A \circ 0_S)(y) &= \bigwedge_{(u,v)\in \mathbf{S}_y} \{\max\{\lambda_A(u), 0_S(v)\}\} \\ &\ge \bigwedge_{(u,v)\in \mathbf{S}_x} \{\max\{\lambda_A(u), 0_S(v)\}\} \\ &= (\lambda_A \circ 0_S)(x). \end{aligned}$$

(3) If $\mathbf{S}_y = \emptyset$, we obtain $(\delta_A \circ 1_S^-)(x) \le 0 = (\delta_A \circ 1_S^-)(y)$. If $\mathbf{S}_y \neq \emptyset$, it follows that $y \le uv$ for all $(u, v) \in \mathbf{S}_y$ and we obtain

$$\begin{aligned} (\delta_A \circ 1_S^-)(y) &= \bigwedge_{(u,v)\in \mathbf{S}_y} \{\max\{\delta_A(u), 1_S^-(v)\}\} \\ &\ge \bigwedge_{(u,v)\in \mathbf{S}_x} \{\max\{\delta_A(u), 1_S^-(v)\}\} \\ &= (\delta_A \circ 1_S^-)(x). \end{aligned}$$

□

In a similar way to Lemma 3.24, we obtain the following lemma.

Lemma 3.25. *Let* $\mathbf{S}$ *be an ordered semigroup and* $x, y \in S$. *Let* $A = (\mu_A, \lambda_A, \delta_A)$ *be a tripolar fuzzy subset of* S *and* $x \le y$. *Then the following conditions hold. For each* $x, y \in S$

(1) $(1_S^+ \circ \mu_A)(x) \ge (1_S^+ \circ \mu_A)(y)$,
(2) $(0_S \circ \lambda_A)(x) \le (0_S \circ \lambda_A)(y)$,
(3) $(1_S^- \circ \delta_A)(x) \le (1_S^- \circ \delta_A)(y)$.

Lemma 3.26. *Let* **S** *be an ordered semigroup and* $A = (\mu_A, \lambda_A, \delta_A)$ *a tripolar fuzzy subset of* S*. Then the following conditions hold. For each* $x, y \in S$

(1) $\left[\mu_A \cup (1_S^+ \circ \mu_A)\right](xy) \geq \left[\mu_A \cup (1_S^+ \circ \mu_A)\right](y)$,
(2) $\left[\lambda_A \cap (0_S \circ \lambda_A)\right](xy) \leq \left[\lambda_A \cap (0_S \circ \lambda_A)\right](y)$,
(3) $\left[\delta_A \cap (1_S^- \circ \delta_A)\right](xy) \leq \left[\delta_A \cap (1_S^- \circ \delta_A)\right](y)$.

Proof. Let $A = (\mu_A, \lambda_A, \delta_A)$ be a tripolar fuzzy subset of S and $x, y \in S$.
(1) Since $1_S^+ \circ \mu_A \subseteq \mu_A \cup (1_S^+ \circ \mu_A)$, we have

$$\left[\mu_A \cup (1_S^+ \circ \mu_A)\right](xy) \geq (1_S^+ \circ \mu_A)(xy),$$

and by Lemma 3.20, we have that $(1_S^+ \circ \mu_A)(xy) \geq \mu_A(y)$, and by Lemma 3.22, we have $(1_S^+ \circ \mu_A)(xy) \geq (1_S^+ \circ \mu_A)(y)$, so we obtain

$$(1_S^+ \circ \mu_A)(xy) \geq \max\{\mu_A(y), (1_S^+ \circ \mu_A)(y)\} = \left[\mu_A \cup (1_S^+ \circ \mu_A)\right](y).$$

Therefore $\left[\mu_A \cup (1_S^+ \circ \mu_A)\right](xy) \geq \left[\mu_A \cup (1_S^+ \circ \mu_A)\right](y)$.
(2) Since $0_S \circ \lambda_A \supseteq \lambda_A \cap (0_S \circ \lambda_A)$, we have

$$\left[\lambda_A \cap (0_S \circ \lambda_A)\right](xy) \leq (0_S \circ \lambda_A)(xy),$$

and by Lemma 3.20, we have that $(0_S \circ \lambda_A)(xy) \leq \lambda_A(y)$. By Lemma 3.22, we have $(0_S \circ \lambda_A)(xy) \leq (0_S \circ \lambda_A)(y)$, so we obtain

$$(0_S \circ \lambda_A)(xy) \leq \min\{\lambda_A(y), (0_S \circ \lambda_A)(y)\} = \left[\lambda_A \cap (0_S \circ \lambda_A)\right](y).$$

Therefore $\left[\lambda_A \cap (0_S \circ \lambda_A)\right](xy) \leq \left[\lambda_A \cap (0_S \circ \lambda_A)\right](y)$.
(3) Since $1_S^- \circ \delta_A \supseteq \delta_A \cap (1_S^- \circ \delta_A)$, we have

$$\left[\delta_A \cap (1_S^- \circ \delta_A)\right](xy) \leq (1_S^- \circ \delta_A)(xy),$$

and by Lemma 3.20, we obtain $(1_S^- \circ \delta_A)(xy) \leq \delta_A(y)$, and by Lemma 3.22, we have $(1_S^- \circ \delta_A)(xy) \leq (1_S^- \circ \delta_A)(y)$, so we obtain

$$(1_S^- \circ \delta_A)(xy) \leq \min\{\delta_A(y), (1_S^- \circ \delta_A)(y)\} = \left[\delta_A \cap (1_S^- \circ \delta_A)\right](y).$$

Therefore $\left[\delta_A \cap (1_S^- \circ \delta_A)\right](xy) \leq \left[\delta_A \cap (1_S^- \circ \delta_A)\right](y)$. □

In a similar way to Lemma 3.26, we obtain the following result.

Lemma 3.27. *Let* **S** *be an ordered semigroup and* $A = (\mu_A, \lambda_A, \delta_A)$ *a tripolar fuzzy subset of* S*. Then the following conditions hold. For each* $x, y \in S$

(1) $\left[\mu_A \cup (\mu_A \circ 1_S^+)\right](xy) \geq \left[\mu_A \cup (\mu_A \circ 1_S^+)\right](x)$,
(2) $\left[\lambda_A \cap (\lambda_A \circ 0_S)\right](xy) \leq \left[\lambda_A \cap (\lambda_A \circ 0_S)\right](x)$,
(3) $\left[\delta_A \cap (\delta_A \circ 1_S^-)\right](xy) \leq \left[\delta_A \cap (\delta_A \circ 1_S^-)\right](x)$.

By Lemma 3.26 and Lemma 3.27, we have the following lemma.

Lemma 3.28. *Let* **S** *be an ordered semigroup and* $A = (\mu_A, \lambda_A, \delta_A)$ *a tripolar fuzzy subset of* S *such that for all* $x, y \in S$ *if* $x \leq y$*, we have* $\mu_A(x) \geq \mu_A(y)$*,* $\lambda_A(x) \leq \lambda_A(y)$ *and* $\delta_A(x) \leq \delta_A(y)$*. Then the following statements hold.*

(1) $A \cup (A \circ \mathcal{S})$ *is a tripolar fuzzy right ideal of* S*.*
(2) $A \cup (\mathcal{S} \circ A)$ *is a tripolar fuzzy left ideal of* S*.*

Proof. Let $A = (\mu_A, \lambda_A, \delta_A)$ be a tripolar fuzzy subset of S and $x, y \in S$. Fristly, by Lemma 3.27, we have

$$\left[\mu_A \cup (\mu_A \circ 1_S^+)\right](xy) \geq \left[\mu_A \cup (\mu_A \circ 1_S^+)\right](x),$$
$$\left[\lambda_A \cap (\lambda_A \circ 0_S)\right](xy) \leq \left[\lambda_A \cap (\lambda_A \circ 0_S)\right](x),$$

and

$$\left[\delta_A \cap (\delta_A \circ 1_S^-)\right](xy) \leq \left[\delta_A \cap (\delta_A \circ 1_S^-)\right](x).$$

Secondly, we let $x \leq y$. By Lemma 3.24, we have $(\mu_A \circ 1_S^+)(x) \geq (\mu_A \circ 1_S^+)(y)$, $(\lambda_A \circ 0_S)(x) \leq (\lambda_A \circ 0_S)(y)$, $(\delta_A \circ 1_S^-)(x) \leq (\delta_A \circ 1_S^-)(y)$, and by hypothesis, $\mu_A(x) \geq \mu_A(y)$, $\lambda_A(x) \leq \lambda_A(y)$ and $\delta_A(x) \leq \delta_A(y)$. Then

$$\begin{aligned}
(\mu_A \cup (\mu_A \circ 1_S^+))(x) &= \max\{\mu_A(x), (\mu_A \circ 1_S^+)(x)\} \\
&\geq \max\{\mu_A(y), (\mu_A \circ 1_S^+)(y)\} \\
&= (\mu_A \cup (\mu_A \circ 1_S^+))(y),
\end{aligned}$$

$$\begin{aligned}
(\lambda_A \cap (\lambda_A \circ 0_S))(x) &= \min\{\lambda_A(x), (\lambda_A \circ 0_S)(x)\} \\
&\leq \min\{\lambda_A(y), (\lambda_A \circ 0_S)(y)\} \\
&= (\lambda_A \cap (\lambda_A \circ 0_S))(y),
\end{aligned}$$

and

$$\begin{aligned}
(\delta_A \cap (\delta_A \circ 1_S^-))(x) &= \min\{\delta_A(x), (\delta_A \circ 1_S^-)(x)\} \\
&\leq \min\{\delta_A(y), (\delta_A \circ 1_S^-)(y)\} \\
&= (\delta_A \cap (\delta_A \circ 1_S^-))(y).
\end{aligned}$$

Therefore $A \cup (A \circ \mathcal{S})$ is a tripolar fuzzy right ideal of S.

(2) By similar method of (1), we can prove (2).

□

Now, we introduce a useful lemma given by Kehayopulu and Tsinglis [6].

Lemma 3.29. *If a, b, c are real numbers, then*

(1) $\min\{a, \max\{b, c\}\} = \max\{\min\{a, b\}, \min\{a, c\}\}$,
(2) $\max\{a, \min\{b, c\}\} = \min\{\max\{a, b\}, \max\{a, c\}\}$.

By the above lemma, we obtain a similar result in terms of elements in the set of tripolar fuzzy subsets of S.

Lemma 3.30. *Let $A = (\mu_A, \lambda_A, \delta_A)$, $B = (\mu_B, \lambda_B, \delta_B)$ and $C = (\mu_C, \lambda_C, \delta_C)$ be tripolar fuzzy subsets of an ordered semigroup S. Then $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.*

Proof. Let $x \in S$. Then we have that

$$\begin{aligned}
\left[\mu_A \cap (\mu_B \cup \mu_C)\right](x) &= \min\{\mu_A(x), (\mu_B \cup \mu_C)(x)\} \\
&= \min\{\mu_A(x), \max\{\mu_B(x), \mu_C(x)\}\} \\
&= \max\{\min\{\mu_A(x), \mu_B(x)\}, \min\{\mu_A(x), \mu_C(x)\}\}
\end{aligned}$$

$$\begin{aligned} &= \max\{(\mu_A \cap \mu_B)(x), (\mu_A \cap \mu_C)(x)\} \\ &= [(\mu_A \cap \mu_B) \cup (\mu_A \cap \mu_C)](x), \end{aligned}$$

$$\begin{aligned} [\lambda_A \cup (\lambda_B \cap \lambda_C)](x) &= \max\{\lambda_A(x), (\lambda_B \cap \lambda_C)(x)\} \\ &= \max\{\lambda_A(x), \min\{\lambda_B(x), \lambda_C(x)\} \\ &= \min\{\max\{\lambda_A(x), \lambda_B(x)\}, \max\{\lambda_A(x), \lambda_C(x)\}\} \\ &= \min\{(\lambda_A \cup \lambda_B)(x), (\lambda_A \cup \lambda_C)(x)\} \\ &= [(\lambda_A \cup \lambda_B) \cap (\lambda_A \cup \lambda_C)](x). \end{aligned}$$

Similar method we obtain that $[\delta_A \cup (\delta_B \cap \delta_C)](x) = [(\delta_A \cup \delta_B) \cap (\delta_A \cup \delta_C)](x)$. Therefore $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. □

Corollary 3.31. *Let* $\mathbf{S}$ *be an ordered semigroup, then the set of all tripolar fuzzy subsets of* S *is a distributive lattice.*

This will allow us to prove our theorem.

Theorem 3.32. *Let* $\mathbf{S}$ *be an ordered semigroup and* $A = (\mu_A, \lambda_A, \delta_A)$ *a tripolar fuzzy subset of* S*. Then the following condition are equivalent.*

(1) *A is a tripolar fuzzy quasi-ideal of* $\mathbf{S}$*.*
(2) $A = B \cap C$ *for some tripolar fuzzy left ideal* $B = (\mu_B, \lambda_B, \delta_B)$ *and tripolar fuzzy right ideal* $C = (\mu_C, \lambda_C, \delta_C)$ *of* $\mathbf{S}$*.*

Proof. (1)⇒(2). Assume that (1) holds. By Corollary 3.31, we have

$$(A \cup (A \circ \mathcal{S}) \cap (A \cup (\mathcal{S} \circ A)) = A \cup ((\mathcal{S} \circ A)) \cap A) \cup ((A \circ \mathcal{S}) \cap A) \cup ((\mathcal{S} \circ A) \cap (\mathcal{S} \circ A)),$$

since $A \subseteq A \cup (A \circ \mathcal{S})$ and $A \subseteq A \cup (\mathcal{S} \circ A)$, we obtain

$$\begin{aligned} A &\subseteq (A \cup (A \circ \mathcal{S}) \cap (A \cup (\mathcal{S} \circ A)) \\ &= A \cup ((\mathcal{S} \circ A)) \cap A) \cup ((A \circ \mathcal{S}) \cap A) \cup ((\mathcal{S} \circ A) \cap (\mathcal{S} \circ A)) \\ &\subseteq A \cup A \cup A \cup A \\ &= A. \end{aligned}$$

This implies that $A = (A \cup (A \circ \mathcal{S})) \cap (A \cup (\mathcal{S} \circ A))$. The prove is completed, since $A \cup (A \circ \mathcal{S})$ and $A \cup (\mathcal{S} \circ A)$ is a tripolar fuzzy right ideal of $\mathbf{S}$ and a tripolar fuzzy left ideal of $\mathbf{S}$, respectively, by Lemma 3.28.

(2)⇒(1). Let $x \in S$. If $S_x = \emptyset$, then

$$((\mu_A \circ 1_S^+) \cap (1_S^+ \circ \mu_A))(x) = \min\{(\mu_A \circ 1_S^+)(x), (1_S^+ \circ \mu_A)(x)\} = 0 \leq \mu_A(x),$$

$$((\lambda_A \circ 0_S) \cup (0_S \circ \lambda_A))(x) = \max\{(\lambda_A \circ 0_S)(x), (0_S \circ \lambda_A)(x)\} = 1 \geq \lambda_A(x),$$

and

$$((\delta_A \circ 1_S^-) \cup (1_S^- \circ \delta_A))(x) = \max\{(\delta_A \circ 1_S^-)(x), (1_S^- \circ \delta_A)(x)\} = 0 \geq \delta_A(x).$$

Therefore $(A \circ \mathcal{S}) \cap (\mathcal{S} \circ A) \subseteq A$. If $S_x \neq \emptyset$, then we let B and C be a tripolar fuzzy left ideal of $\mathbf{S}$ and a tripolar fuzzy right ideal of $\mathbf{S}$, respectively and we have

$$\begin{aligned}(\mu_A \circ 1_S^+)(x) &= \bigvee_{(a,b)\in S_x} \{\min\{\mu_A(a), 1_S^+(b)\}\} = \bigvee_{(a,b)\in S_x} \{\mu_A(a)\} \\ &= \bigvee_{(a,b)\in S_x} \{(\mu_B \cap \mu_C)(a)\} \leq \bigvee_{(a,b)\in S_x} \{\mu_B(a)\} \\ &= \mu_B(a) \leq \mu_B(ab) \leq \mu_B(x),\end{aligned}$$

and by the same method we have that $(\mu_A \circ 1_S^+)(x) \leq \mu_C(x)$ and then $(\mu_A \circ 1_S^+)(x) \leq (\mu_B \cap \mu_C)(x) = \mu_A(x)$,

$$\begin{aligned}(\lambda_A \circ 0_S)(x) &= \bigwedge_{(a,b)\in S_x} \{\max\{\lambda_A(a), 0_S(b)\}\} = \bigwedge_{(a,b)\in S_x} \{\lambda_A(a)\} \\ &= \bigwedge_{(a,b)\in S_x} \{(\lambda_B \cup \lambda_C)(a)\} \geq \bigwedge_{(a,b)\in S_x} \{\lambda_B(a)\} \\ &= \lambda_B(a) \geq \lambda_B(ab) \geq \lambda_B(x),\end{aligned}$$

and by the same method we have that $(\lambda_A \circ 0_S)(x) \geq \lambda_C(x)$ and then $(\lambda_A \circ 0_S)(x) \geq (\lambda_B \cup \lambda_C)(x) = \lambda_A(x)$, and

$$\begin{aligned}(\delta_A \circ 1_S^-)(x) &= \bigwedge_{(a,b)\in S_x} \{\max\{\delta_A(a), 1_S^-(b)\}\} = \bigwedge_{(a,b)\in S_x} \{\delta_A(a)\} \\ &= \bigwedge_{(a,b)\in S_x} \{(\delta_B \cup \delta_C)(a)\} \geq \bigwedge_{(a,b)\in S_x} \{\delta_B(a)\} \\ &= \delta_B(a) \geq \delta_B(ab) \geq \delta_B(x),\end{aligned}$$

and by the same method we have that $(\delta_A \circ 1_S^-)(x) \geq \delta_C(x)$ and then $(\delta_A \circ 1_S^-)(x) \geq (\delta_B \cup \delta_C)(x) = \delta_A(x)$. Therefore $A \circ \mathcal{S} \subseteq A$. By the same method, we have that $\mathcal{S} \circ A \subseteq A$. Altogether, we obtain $(A \circ \mathcal{S}) \cap (\mathcal{S} \circ A) \subseteq A$. Therefore A is a tripolar fuzzy quasi-ideal of $\mathbf{S}$. $\square$

4. Conclusion

In this present paper, we applied the concept of tripolar fuzzy sets to ordered semigroups. We introduced the notions of tripolar fuzzy subsemigroups, tripolar fuzzy ideals, tripolar fuzzy quasi-ideals, and tripolar fuzzy bi-ideals of ordered semigroups, and some of their algebraic properties and the relations between them are studied. Moreover, we proved that tripolar fuzzy bi-ideals and quasi-ideals coincide only in a particular class of ordered semigroups. Finally, we proved that every tripolar fuzzy quasi-ideal is the intersection of a tripolar fuzzy left ideal and a tripolar fuzzy right ideal. The notions presented in this paper can be applied to the theory of hyperstructures, ordered hyperstructures, semirings, hemirings, groups, BCI/BCK algebras, etc.

Conflicts of interest : The authors declare no conflict of interest.

Data availability : Not applicable

Acknowledgments : We would like to express our sincere gratitude to the anonymous referee for their valuable comments and suggestions, which significantly improved the quality of this paper.

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