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ANALYSIS OF AN SEIQRVS EPIDEMIC DYNAMICS FOR INFECTIOUS VIRAL DISEASE: QUARANTINE AS A CONTROL STRATEGY

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Abstract. An epidemic infectious disease model consists of six compartments viz. Susceptible, Exposed, Infected, Quarantine, Recovered, and Virus with nonlinear saturation incidence rate is proposed to know the viral disease dynamics. There exist two biological equilibrium points for the model system. The system's local and global stability is done through Lyapunov's direct method about equilibrium points. The sensitivity analysis has been performed for the basic reproduction number and equilibrium points through the normalized forward sensitivity index. Sensitivity analysis shows that virus growth and quarantine rates are more sensitive parameters. In support of mathematical conclusions, numerical experimentation has been shown.

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1. Introduction

By the word outbreak, the eruption of an infectious disease that infects the verbal population of an area is known. The fusion of ecology multiplexes, the rapid development of conditions, and different pathogens' exposures assure infectious diseases proceed to cause significant provocations for a foreseeable future [2, 12]. Immunity from infection offers human preservation for a period of time from successive infection by the same disease. One can achieve immunity until lifetime for infections such as measles, rubella, mumps, and chickenpox, i.e., some infections induce lifetime immunity. The level of immunity declines with time for other equivalent diseases. This weakening of immunity happens because of immunogenic variations in the transmission or depletion of antibodies over a

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period of time. The transformation from full to partial immunity can occur for distinct periods, like a few weeks with rotavirus and norovirus, for months with influenza [1, 2, 9, 12, 18].

Quarantine (of suspected to an infection) has historically been among the oldest public health prevention mechanisms for disseminating transmissible diseases. These interventions effectively enforced from the 13th-century plague outbreak to the 20th-century influenza epidemics. Further, this strategy was used intensely to battle the propagation of several emerging and reoccurring animal and human infections, including foot-and-mouth disease, severe acute respiratory syndrome (SARS), and the swine influenza pandemic 2009, and SARS-COV-2 in 2019. The 2003 SARS outbreaks were a significant example of a novel disease effectively controlled by isolation and quarantine. In SARS-COV-2 2019, we have also seen the magical effect of isolation and quarantine in preventing infection's spreading widely [3, 5, 10, 12, 13, 15, 17]. The incidence rate performs a significant part in the modeling of infectious diseases. Some factors, such as the population's response and lifestyle, may have an indirect or direct influence on the incidence of new infections, and the increase and fall in epidemics are likely [8, 12, 14, 21].

We discussed in this work the virus's impact on infected persons. Also, the transmission of infectious diseases can be the factor for the rise of viruses in the vulnerable population. The growth of the virus mortality rate is among infection management practices. We shall explore the model system to illustrate the virus's growth rate and mortality in the vulnerable population [7, 8, 16].

2. Model development

In this part, we will model the infection dynamics with six compartment as Susceptible(N_S), Exposed(N_E), Infected(N_I), Quarantine(N_Q), Recovered(N_R), Virus(N_V). Schematic flow of the disease dynamics is given in the figure (1) and the following ODEs govern the proposed mathematical model:

$$\frac{dN_S}{dt} = \Lambda - \frac{\beta N_S N_V}{1 + aN_V} - \mu_1 N_S + \theta N_R, \tag{1}$$

$$\frac{dN_E}{dt} = \frac{\beta N_S N_V}{1 + aN_V} - \rho N_E - \xi N_E - \mu_1 N_E, \tag{2}$$

$$\frac{dN_I}{dt} = \xi N_E - \mu_2 N_I - \gamma_1 N_I, \tag{3}$$

$$\frac{dN_Q}{dt} = \rho N_E - \mu_2 N_Q - \gamma_2 N_Q, \tag{4}$$

$$\frac{dN_R}{dt} = \gamma_1 N_I + \gamma_2 N_Q - \mu_1 N_R - \theta N_R, \tag{5}$$

$$\frac{dN_V}{dt} = r_1 N_I + r_2 N_E - \mu_3 N_V, \tag{6}$$

with the positive initial population: $N_S(0) > 0,\ N_E(0) > 0,\ N_I(0) > 0,\ N_Q(0) > 0,\ N_R(0) > 0,\ N_V(0) > 0.$

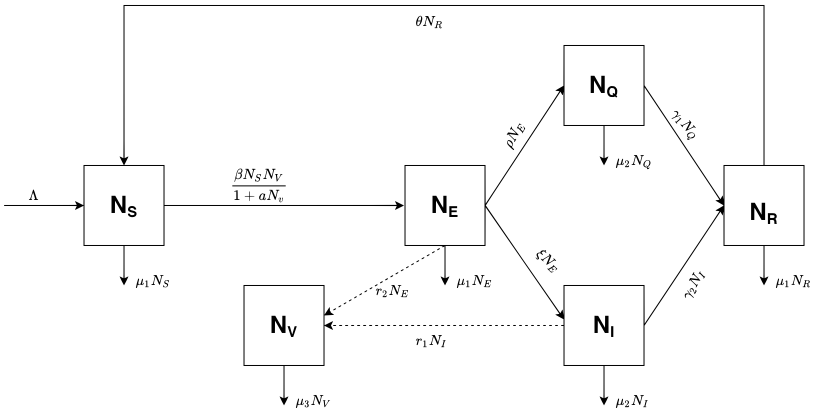

FIGURE 1. The diagrammatic flow of the proposed model

TABLE 1. Parameters descriptions for the proposed model

Parameter	Parameter's Description	Dimension
Λ	Recruitment rate of susceptible	per day
β	Transmission coefficient of exposed individuals	per day
$\frac{1}{a}$	Infected individual's Half-saturation constant	—-
μ_1	Natural mortality rate of population	per day
μ_2	Mortality rate of population due to infection	per day
μ_3	Natural mortality rate of virus	per day
θ	Immunity waning rate of recovered class	per day
ξ	Transmission rate from exposed to infected class	per day
ρ	Quarantine rate for exposed class	per day
γ_1	Recovery rate for infected class	per day
γ_2	Recovery rate for quarantine class	per day
r_1	Birth rate of virus from infected class	per day
r_2	Birth rate of virus from exposed class	per day

Total population N at time t is given by $N = N_S + N_E + N_I + N_Q + N_R$. The definitions of parameters are given in Table 1.

3. Positivity and Boundedness

This part has some lemmas for boundedness and positivity of the system solution (1)-(6), which are:

Lemma 3.1. *For non-negative initial conditions the solution of the proposed system (1)-(6) is non-negative for $t \geq 0$.*

Proof. Let $(N_S(t), N_E(t), N_I(t), N_Q(t), N_R(t), N_V(t))$ be the solution with non-negative initial population. For $t \geq 0$, the equation (1) becomes:

$$\frac{dN_S}{dt} \geq -(\mu_1 + \frac{\beta}{a})N_S \quad \Longrightarrow N_S(t) \geq N_S(0)e^{-(\mu_1+\frac{\beta}{a})t} > 0. \tag{7}$$

Also for $t \geq 0$, the equation (2) reduces to

$$\frac{dN_E}{dt} \geq -(\mu_1 + \xi + \rho)N_E, \quad \Longrightarrow N_E(t) \geq N_E(0)e^{-(\mu_1+\xi+\rho)t} > 0. \tag{8}$$

And for $t \geq 0$, the equation (3) reduces to

$$\frac{dN_I}{dt} \geq -(\mu_2 + \gamma_1)N_I, \quad \Longrightarrow N_I(t) \geq N_I(0)e^{-(\mu_2+\gamma_1)t} > 0. \tag{9}$$

Also for $t \geq 0$, the equation (4), state that

$$\frac{dN_Q}{dt} \geq -(\mu_2 + \gamma_2)N_Q, \quad \Longrightarrow N_Q(t) \geq N_Q(0)e^{-(\mu_2+\gamma_2)t} > 0. \tag{10}$$

Similarily for $t \geq 0$, the equation (5), state that

$$\frac{dN_R}{dt} \geq -(\mu_1 + \gamma_2 + \theta)N_R, \quad \Longrightarrow N_R(t) \geq N_R(0)e^{-(\mu_1+\gamma_2+\theta)t} > 0. \tag{11}$$

Finally for $t \geq 0$, the equation (6), state that

$$\frac{dN_V}{dt} \geq -\mu_3 N_V, \quad \Longrightarrow N_V(t) \geq N_V(0)e^{-\mu_3 t} > 0. \tag{12}$$

Hence the population remains positive for the system (1)-(6), i.e., $N_S(t) \geq 0, N_E(t) \geq 0, N_I(t) \geq 0, N_Q(t) \geq 0, N_R(t) \geq 0, N_V(t) \geq 0$ for all $t \geq 0$. □

Lemma 3.2. *If the initial population is positive then the system solution (1)-(6) is bounded uniformly in* Ω*, where*

$$\Omega = \{(N_S(t), N_E(t), N_I(t), N_Q(t), N_R(t)) : 0 \leq (N_S(t) + N_E(t) + N_I(t) + N_Q(t) + N_R(t)) \leq \tfrac{\Lambda}{\mu}\Big\}.$$

Proof. Assuming that $N_S(t) + N_E(t) + N_I(t) + N_Q(t) + N_R(t) = N(t)$. Now differentiating $N(t)$, then from the system (1)-(6):

$$\frac{dN(t)}{dt} = \Lambda - \mu(N_S(t) + N_E(t) + N_I(t) + N_Q(t) + N_R(t)) = \Lambda - \mu N,$$

here $\mu = \min(\mu_1, \mu_2, \mu_3)$, which gives that $N(t) = N(0)e^{-\mu t} + \frac{\Lambda}{\mu}$, as $t \to \infty$, $N(t) \to \frac{\Lambda}{\mu}$. Clearly the system (1)-(6) is bounded above for its each population. Since initially all population are positive hence system is bounded below by zero. Therefore the bounded biological feasible region Ω is given by

$$\Omega = \{(N_S(t), N_E(t), N_I(t), N_Q(t), N_R(t)) : 0 \leq (N_S(t) + N_E(t) + N_I(t) + N_Q(t) + N_R(t)) \leq \tfrac{\Lambda}{\mu}\Big\}.$$

□

4. Stability behaviour

In this section, we will evaluate the basic reproduction number R_0, all the equilibrium points for the proposed system, and analyze the local and global stability for each of the equilibrium points [11, 20].

4.1. Equilibrium point. System has two equilibrium points, namely disease free $E^0 = (S^0, 0, 0, 0, 0)$, here $S^0 = \frac{\lambda}{\mu_1}$ and endemic, also known as interior, $\bar{E} = (N_S^*,\ N_E^*,\ N_I^*,\ N_Q^*,\ N_R^*,\ N_V^*)$, here

$$\begin{aligned}
N_E^* &= \frac{\mu_3(\mu_2+\gamma_1)R_0}{a[\xi r_1 + (\mu_2+\gamma_1)r_2] + \mu_3(\mu_2+\gamma_1)},\\
N_S^* &= \frac{\Lambda}{\mu_1} + \left[\frac{(\mu_2+\gamma_1)\gamma_2\rho + \theta(\mu_2+\gamma_2)\gamma_1\xi}{(\mu_2+\gamma_1)(\mu_2+\gamma_2)(\mu_1+\theta)} - \frac{S^0\beta(\xi r_1 + r_2(\mu_2+\gamma_1))}{R_0(\mu_2+\gamma_1)\mu_3}\right]\frac{N_E^*}{\mu_1},\\
N_I^* &= (\frac{\xi}{\mu_2+\gamma_1})N_E^*, \qquad N_Q^* = (\frac{\rho}{\mu_2+\gamma_2})N_E^*,\\
N_R^* &= (\frac{1}{\mu_1+\theta})\left[\frac{\gamma_1\xi}{\mu_2+\gamma_1} + \frac{\gamma_2\rho}{\mu_2+\gamma_2}\right]N_E^*, \quad N_V^* = (\frac{1}{\mu_3})\left[\frac{r_1\xi}{\mu_2+\gamma_1} + r_2\right]N_E^*,
\end{aligned}$$

where R_0 is the basic reproduction number. This is defined as expected secondary cases due to single infected individuals in a fully susceptible population and determined using Next Generation Matrix similarly as in [6, 19]. For the system R_0 is given by

$$R_0 = \frac{\beta[\xi r_1 + (\mu_2+\gamma_1)r_2]S^0}{(\xi+\rho+\mu_1)(\gamma_1+\mu_2)\mu_3} = \frac{\beta\Lambda[\xi r_1 + r_2(\mu_2+\gamma_1)]}{\mu_1\mu_3(\xi+\rho+\mu_1)(\mu_2+\gamma_1)}. \tag{13}$$

The existence condition for endemic equilibrium is $R_0 > 1$, it is clear from (13).

4.2. Local stability of disease-free and endemic equilibrium. Analytical findings can be discussed by looking at the limiting framework for the system (1)-(6) where a total population $N = \frac{\Lambda}{\mu_1}$ is considered to remain constant, the limiting system is governed by set of equations:

$$\frac{dN_S}{dt} = \Lambda(1+\frac{\theta}{\mu_1}) - (\mu_1+\theta)N_S - \frac{\beta N_S N_V}{1+aN_V} - \theta N_E - \theta N_I - \theta N_Q, \tag{14}$$

$$\frac{dN_E}{dt} = \frac{\beta N_S N_V}{1+aN_V} - \xi N_E - \rho N_E - \mu_1 N_E, \tag{15}$$

$$\frac{dN_I}{dt} = \xi N_E - \mu_2 N_I - \gamma_1 N_I, \tag{16}$$

$$\frac{dN_Q}{dt} = \rho N_E - \mu_2 N_Q - \gamma_2 N_Q, \tag{17}$$

$$\frac{dN_V}{dt} = r_1 N_I + r_2 N_E - \mu_3 N_V. \tag{18}$$

with the same set of initial conditions.

4.2.1. Local stability of disease-free equilibrium. The variational matrix at DFE is:

$$J_0 = \begin{bmatrix} -(\mu_1+\theta) & -\theta & -\theta & -\theta & \frac{-\beta\Lambda}{\mu_1} \\ 0 & -(\xi+\rho+\mu_1) & 0 & 0 & \frac{\beta\Lambda}{\mu_1} \\ 0 & \xi & -(\mu_2+\gamma_1) & 0 & 0 \\ 0 & \rho & 0 & -(\mu_2+\gamma_2) & 0 \\ 0 & r_2 & r_1 & 0 & -\mu_3 \end{bmatrix}.$$

and corresponding characteristic equation is:

$$(\mu_2+\gamma_2+\lambda)(\mu_1+\theta+\lambda)[(\mu_2+\gamma_1+\lambda)(-\mu_3-\lambda)(\xi+\rho+\mu_1+\lambda) + \frac{\Lambda}{\mu_1}(\xi r_1 + r_2\lambda + r_2\mu_2 + r_2\gamma_1)\beta] = 0. \tag{19}$$

root of above equations are $-\theta-\mu_1$, $-\gamma_2-\mu_2$ and the solution of equation

$$\lambda^3 + \lambda^2 C_1 + \lambda C_2 + C_3 = 0,$$

where
$C_1 = \gamma_1+\mu_2+\mu_1+\rho+\xi+\mu_3$,
$C_2 = \xi\mu_3+\mu_1\mu_3+\rho\mu_3+\mu_2\mu_3+\gamma_1\mu_3+\xi\mu_2+\xi\gamma_1+\rho\mu_2+\rho\gamma_1+\mu_1\mu_2+\mu_1\gamma_1-\frac{-\beta\Lambda r_2}{\mu_1}$,
$C_3 = \xi\mu_2\mu_3+\xi\mu_3\gamma_1+\rho\mu_2\mu_3+\rho\gamma_1\mu_3+\mu_1\mu_2\mu_3+\mu_1\gamma_1\mu_3-\frac{\beta\Lambda\xi r_1}{\mu_1}-\frac{\beta\Lambda r_2\mu_2}{\mu_1}-\frac{\beta\Lambda r_2\gamma_1}{\mu_1}$,
since the given two roots are negative and to know the nature of remaing three roots, we will apply the Routh-Hurwtiz Criterion for the cubic equation, i.e., $C_1>0$, $C_3>0$ and $C_1C_2>C_3$ Hence, we have
$C_1C_2 - C_3 = (\gamma_1+\xi+\mu_1+\mu_2+\mu_3+\rho)(\mu_2\mu_3+\gamma_1\mu_3+\mu_2\xi+\xi\gamma_1+\rho\mu_2+\rho\gamma_1+\mu_1\mu_2+\mu_1\gamma_1)+(\xi+\mu_1+\mu_3+\rho)(\mu_3\xi+\rho\mu_3+\mu_1\mu_3+\frac{\beta\Lambda r_2}{\mu_1})+\frac{\Lambda\beta\xi r_1}{\mu_1}>0$,
when $R_0<1$. Clearly, $C_1>0$, $C_2>0$ and $(C_1C_2-C_3)>0$. Hence, Routh-Hurwitz criteria verify that the remaining three roots of characteristic equation have negative real part. Since, all five roots of the characteristics equation are either negative or having negative real part if $R_0<1$, hence the DFE (E^0) for the system (14)-(18) is locally asymptotically stable if $R_0<1$. Also the similar approach, Routh-Hurwitz criteria gives the local stability condition for endemic equilibrium as $R_0>1$.

4.3. Global stability dynamics of equilibrium points. We are discussing global stability of the endemic and disease free equilibrium.

4.3.1. Global stability of disease-free equilibrium. We will follow the method developed by Castillo-Chavez et.al., [2] for the global stability analysis of disease-free equilibrium. Let $X=(N_S)$ and $Z=(N_E,\ N_I,\ N_Q,\ N_V)$, therefore $U_0=(X^0,\ 0)$, will represent the DFE where $X^0=\frac{\Lambda}{\mu_1}$. Now, rewrite the system (14)-(18) as

$$\frac{dX}{dt} = F(X,\ Z), \tag{20}$$

$$\frac{dZ}{dt} = G(X,\ Z). \tag{21}$$

Clearly system given by (20) becomes from equation (14)

$$\frac{dX}{dt} = F(X,\ Z) = (1+\frac{\theta}{\mu_1})\Lambda - (\theta+\mu_1)N_S - \frac{\beta N_S N_V}{aN_V+1} - \theta N_E - \theta N_I - \theta N_Q.$$

$$\text{At } Z=0,\ G(X,\ 0)=0,\ \text{and} \quad \frac{dX}{dt} = F(X,\ 0) = (1+\frac{\theta}{\mu_1})\Lambda - (\theta+\mu_1)X.$$

As $t \to \infty, X \to X^0$, Thus, $X = X^0 (= S^0)$ is globally asymptotically stable for the system given by (20). Now the system given by (20) becomes by using equations (15)-(18)

$$\begin{aligned} G(X,\ Z) &= \begin{bmatrix} -(\xi+\mu_1+\rho) & 0 & 0 & \beta S^0 \\ \xi & -(\mu_2+\gamma_1) & 0 & 0 \\ \rho & 0 & -(\mu_2+\gamma_2) & 0 \\ r_2 & r_1 & 0 & -\mu_3 \end{bmatrix} \begin{bmatrix} N_E \\ N_I \\ N_Q \\ N_V \end{bmatrix} \\ &\quad - \begin{bmatrix} \beta S^0 N_V - \frac{\beta N_S N_V}{1+aN_V} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \end{aligned} \tag{22}$$

on comparing the system (22) with $G(X,\ Z) = BZ - \widehat{G}(X,\ Z)$, we get

$$B = \begin{bmatrix} -(\xi+\mu_1+\rho) & 0 & 0 & \beta S^0 \\ \xi & -(\gamma_1+\mu_2) & 0 & 0 \\ \rho & 0 & -(\mu_2+\gamma_2) & 0 \\ r_2 & r_1 & 0 & -\mu_3 \end{bmatrix}$$

and $\widehat{G}(X,\ Z) = \begin{bmatrix} \beta S^0 N_V - \frac{\beta N_S N_V}{1+aN_V} \\ 0 \\ 0 \\ 0 \end{bmatrix}$, clearly $\widehat{G}(X,\ 0) = 0$.

Thus both the conditions discussed in [2] are met, then the DFE (E^0) is globally asymptotically stable if $R_0 < 1$.

4.3.2. Global stability dynamics of Endemic Equilibrium. We will use Lyapunov's Direct Method to show the endemic equilibrium's global stability for the system (14)-(18). Let a positive definite function:

$$V_1 = \frac{1}{2}(D_1 N_S^2 + D_2 N_E^2 + D_3 N_I^2 + D_4 N_Q^2 + D_5 N_V^2), \tag{23}$$

Then using the system (14)-(18) in $\frac{dV_1}{dt}$, we get,

$$\begin{aligned} \frac{dV_1}{dt} &= (D_1 N_S)(\Lambda(1+\frac{\theta}{\mu}) - \mu_1 N_S - \theta N_S - \frac{\beta N_S N_V}{1+aN_V} - \theta N_E - \theta N_I - \theta N_Q) \\ &\quad +(D_2 N_E)(\frac{\beta N_S N_V}{1+aN_V} - \xi N_E - \rho N_E - \mu_1 N_E) \\ &\quad +(D_3 N_I)(\xi N_E - \mu_2 N_I - \gamma_1 N_I) + (D_4 N_Q)(\rho N_E - \mu_2 N_Q - \gamma_2 N_Q) \\ &\quad +(D_5 N_V)(r_1 N_I + r_2 N_E - \mu_3 N_V), \end{aligned} \tag{24}$$

$$\begin{aligned}\frac{dV_1}{dt} &= D_1(\Lambda N_S + \frac{\theta\Lambda N_S}{\mu_1} - \mu_1 N_S^2 - \theta N_S^2 - \theta N_S N_E - \theta N_S N_I - \theta N_S N_Q) \\ &+D_2(-\xi N_E^2 - \rho N_E^2 - \mu_1 N_E^2) + D_3(\xi N_E N_I - \mu_2 N_I^2 - \gamma_1 N_I^2) \qquad (25) \\ &+D_4(\rho N_E N_Q - \mu_2 N_Q^2 - \gamma_2 N_Q^2) + D_5(r_1 N_I N_V + r_2 N_E N_V - \mu_3 N_V^2),\end{aligned}$$

Using the inequality $\pm 2ab \leq (a^2 + b^2)$, we get,

$$\begin{aligned}\frac{dV_1}{dt} &\leq -[(\frac{a_{11}N_S^2}{3} - a_{12}N_S N_E + \frac{a_{22}N_E^2}{4}) + (\frac{a_{22}N_E^2}{4} - a_{23}N_E N_I + \frac{a_{33}N_I^2}{2}) \\ &+(\frac{a_{22}N_E^2}{4} - a_{24}N_E N_Q + \frac{a_{44}N_Q^2}{2}) + (\frac{a_{22}N_E^2}{4} - a_{25}N_E N_V + \frac{a_{55}N_V^2}{2}) \\ &+(\frac{a_{33}N_I^2}{2} - a_{35}N_I N_V + \frac{a_{55}N_V^2}{2}) + (\frac{a_{11}N_S^2}{2} - a_{13}N_S N_I + \frac{a_{33}N_I^2}{3}) \\ &+(\frac{a_{11}N_S^2}{2} - a_{14}N_S N_Q + \frac{a_{44}N_Q^2}{2})], \qquad (26)\end{aligned}$$

where $a_{11} = D_1(\mu_1 + \theta - \frac{\beta\Lambda}{2\mu_1})$, $a_{12} = D_1\theta$, $a_{13} = D_1\theta$, $a_{14} = D_1\theta$, $a_{22} = D_2(\mu_1 + \xi + \rho - \frac{\beta\Lambda}{2\mu_1})$, $a_{23} = D_3\xi$, $a_{24} = D_4\rho$, $a_{25} = D_5 r_2$, $a_{33} = D_3(\mu_2 + \gamma_1)$, $a_{35} = D_5 r_1$, $a_{44} = D_4(\mu_2 + \gamma_2)$, $a_{55} = D_5\mu_3$. The following conditions are observed by Lyapunov's direct method of stability for endemic equilibrium's global or non-linearly stability.

(1)$[\theta + \mu_1 - \frac{\Lambda\beta}{2\mu_1}][\mu_1 + \xi + \rho - \frac{\Lambda\beta}{2\mu_1}]D_2 > D_1\theta^2$,
(2)$[\xi + \mu_1 + \rho - \frac{\Lambda\beta}{2\mu_1}][\mu_2 + \gamma_1]D_2 > D_3\xi^2$,
(3)$[\xi + \mu_1 + \rho - \frac{\Lambda\beta}{2\mu_1}][\mu_2 + \gamma_2]D_2 > \rho^2 D_4$,
(4)$[\xi + \mu_1 + \rho - \frac{\Lambda\beta}{2\mu_1}][\mu_3]D_2 > D_5 r_2^2$,
(5)$[\gamma_1 + \mu_2][\mu_3]D_3 > r_1^2 D_5$,
(6)$[\theta + \mu_1 - \frac{\Lambda\beta}{2\mu_1}][\mu_2 + \gamma_1]D_3 > D_1\theta^2$,
(7)$[\theta + \mu_1 - \frac{\Lambda\beta}{2\mu_1}][\mu_2 + \gamma_2]D_4 > D_1\theta^2$.

Selecting again $D_1 = 1$, we observe

(i) $\frac{4\theta^2}{3(\mu_1+\theta-\frac{\beta\Lambda}{2\mu_1})(\mu_1+\xi+\rho-\frac{\Lambda\beta}{2\mu_1})} < D_2$,
(ii) $\frac{\theta^2}{(\mu_2+\gamma_1)(\mu_1+\theta-\frac{\beta\Lambda}{2\mu_1})} < D_3 < \frac{\theta^2(\mu_2+\gamma_1)}{\xi^2(\mu_1+\theta-\frac{\Lambda\beta}{2\mu_1})}$,
(iii) $\frac{2\theta^2}{3(\mu_2+\gamma_2)(\mu_1+\theta-\frac{\Lambda\beta}{2\mu_1})} < D_4 < \frac{2\theta^2(\mu_2+\gamma_2)}{3\rho^2(\mu_1+\theta-\frac{\beta\Lambda}{2\mu_1})}$,
(iv) $\frac{2\theta^2\mu_3}{3r^2(\mu_1+\theta-\frac{\Lambda\beta}{2\mu_1})} > D_5$,

where r stands simultaneously for r_1 or r_2. Finally global stability conditions for endemic equilibrium are:

(i) $[\theta + \mu_1 - \frac{\Lambda\beta}{2\mu_1}] > 0$.

(ii) $[\xi + \mu_1 + \rho - \frac{\Lambda\beta}{2\mu_1}] > 0$.
(iii) $[\theta + \mu_1 - \frac{\Lambda\beta}{2\mu_1}][\mu_1 + \xi + \rho - \frac{\Lambda\beta}{2\mu_1}] > [\theta^2]$.

5. Numerical Simulation

Earlier established results are illustrated numerically in this segment for the parameter's value given in the table 2.

TABLE 2. The of parameters values used for numerical simulation

Parameter	Value	Dimension
Recruitment rate of susceptible (Λ)	0.400	per day
Transmission coefficient of exposed class (β)	0.008	per day
Half-saturation constant of infected class ($1/a$)	10.00	——
Natural mortality rate of susceptible and exposed class (μ_1)	0.005	per day
Natural mortality rate of infectious and quarantine class (μ_2)	0.008	per day
Natural mortality rate of virus (μ_3)	0.800	per day
Immunity waning rate (θ)	0.010	per day
Infectious rate of exposed class (ξ)	0.100	per day
Quarantine rate of exposed class (ρ)	0.100	per day
Recovery rate of infected class (γ_1)	variable	per day
Recovery rate of quarantine class (γ_2)	0.040	per day
Virus' birth rate due to infectious class (r_1)	variable	per day
Virus' birth rate due to exposed class (r_2)	0.100	per day

(a) For the given set of parameters with virus birth rate and recovery rate for infected class $r_1 = 0.3$, $\gamma_1 = 0.03$, we obtained effective reproduction number $R_0 = 3.47112 > 1$ and endemic equilibrium $\bar{E}(30.3302, 2.84208, 7.47915, 5.92099, 3.15994)$. Also the global stability conditions,
$[\mu_1+\theta-\frac{\beta\Lambda}{2\mu_1}] > 0$, $[\mu_1+\xi+\rho-\frac{\beta\Lambda}{2\mu_1}] > 0$, and $[\mu_1+\theta-\frac{\beta\Lambda}{2\mu_1}][\mu_1+\xi+\rho-\frac{\beta\Lambda}{2\mu_1}] > [\theta^2]$, for the same set of parameters, are satisfied. So therefore the endemic equilibrium $\bar{E}$ is globally asymptotically stable (See Figure 3).

(b) For the given set of parameters with virus birth rate and recovery rate for infected class $r_1 = 0.15$, $\gamma_1 = 0.09$, we obtained effective reproduction number $R_0 = 0.987556 < 1$ and the DFE $E^0(80, 0, 0, 0, 0)$. So it is clear that DFE E^0 is globally asymptotically stable (See Figure 2).

(c) For the given set of parameters with virus birth rate and recovery rate for infected class $r_1 = 0.15$, $\gamma_1 = 0.09$, we obtained effective reproduction number $R_0 = 0.987556 < 1$ and the DFE $E^0(80, 0, 0, 0, 0)$. If we slightly lower the quarantine parameter from $\rho = 0.10$ to $\rho = 0.30$ DFE E^0 loses its stability and EE $\bar{E}$ become stable and this phenomena is shown in figure (4) So it is clear that quarantine behave as control strategy.

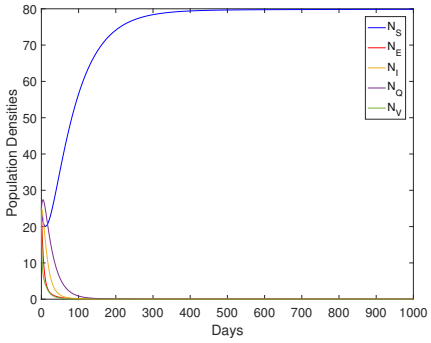

FIGURE 2. Population densities at virus rate $r_1 = 0.15$ and Quarantine rate $\rho = 0.10$

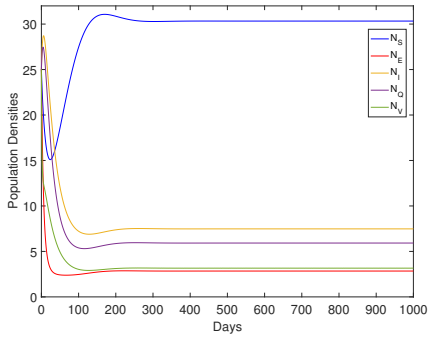

FIGURE 3. Population densities at virus rate $r_1 = 0.30$

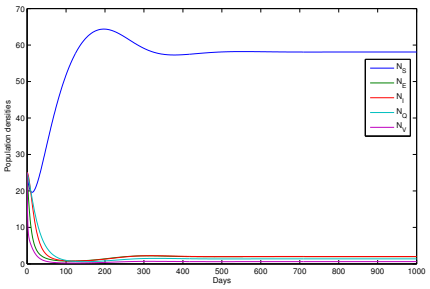

FIGURE 4. Population densities at quarantine rate $\rho = 0.30$

6. Sensitivity Analysis

In this section, sensitivity for R_0 and endemic steady state is analyzed for system parameters. We have derive the expression and evaluate the Normalized sensitivity indices for R_0 corresponding to parameters $\Lambda, \beta, \mu_1, \rho, \mu_2, \mu_3, \xi,\ r_1,$ $r_2, \gamma_1,$ and γ_2 similar as in [4]:

$$\begin{aligned}
\Upsilon_{\Lambda}^{R_0} &= \frac{\partial R_0}{\partial \Lambda}\frac{\Lambda}{R_0} = 1,\\
\Upsilon_{\mu_1}^{R_0} &= \frac{\partial R_0}{\partial \mu_1}\frac{\mu_1}{R_0} = \frac{-(\xi+\rho+2\mu_1)}{\xi+\rho+\mu_1},\\
\Upsilon_{\mu_2}^{R_0} &= \frac{\partial R_0}{\partial \mu_2}\frac{\mu_2}{R_0} = \frac{-\xi\ r_1\mu_2}{(\gamma_1+\mu_2)(\xi\ r_1+\ r_2\gamma_1+\ r_2\mu_2)},\\
\Upsilon_{\mu_3}^{R_0} &= \frac{\partial R_0}{\partial \mu_3}\frac{\mu_3}{R_0} = -1,\\
\Upsilon_{\beta}^{R_0} &= \frac{\partial R_0}{\partial \beta}\frac{\beta}{R_0} = \quad 1,\\
\Upsilon_{\rho}^{R_0} &= \frac{\partial R_0}{\partial \rho}\frac{\rho}{R_0} = \frac{-\rho}{\rho+\xi+\mu_1},\\
\Upsilon_{\gamma_1}^{R_0} &= \frac{\partial R_0}{\partial \gamma_1}\frac{\gamma_1}{R_0} = \frac{-\xi\ r_1\gamma_1}{(\gamma_1+\mu_2)(\xi\ r_1+\ r_2\gamma_1+\ r_2\mu_2)},\\
\Upsilon_{\xi}^{R_0} &= \frac{\partial R_0}{\partial \xi}\frac{\xi}{R_0} = \frac{\xi(\rho\ r_1+\ r_1\mu_1-\ r_2\gamma_1-\ r_2\mu_2)}{(\xi+\rho+\mu_1)(\xi\ r_1+\ r_2\gamma_1+\ r_2\mu_2)},\\
\Upsilon_{r_1}^{R_0} &= \frac{\partial R_0}{\partial\ r_1}\frac{r_1}{R_0} = \frac{r_1\xi}{\xi\ r_1+\ r_2\gamma_1+\ r_2\mu_2},\\
\Upsilon_{r_2}^{R_0} &= \frac{\partial R_0}{\partial\ r_2}\frac{r_2}{R_0} = \frac{r_2(\gamma_1+\mu_2)}{\xi\ r_1+\ r_2\gamma_1+\ r_2\mu_2}.
\end{aligned}$$

For a particular set of parametric values shown in Table 2, the sensitivity indices are given in Table 3. From the table 3, we explore that Λ and β are highly sensitive, $r_1,\ r_2, \rho, \xi, \mu_2$ are less sensitive and γ_1, μ_3, μ_1 are sensitive to R_0.

TABLE 3. The sensitivity indices, $\Upsilon_{x_j}^{R_0} = \frac{\partial R_0}{\partial x_j} \times \frac{x_j}{R_0}$, of R_0 corresponding parameters x_j for values given in Table 2 at $r_1 = 0.30$

Parameters (x_j)	Sensitivity index ($\Upsilon_{x_j}^{R_0}$)
Λ	1.0000
β	1.0000
μ_1	-1.0244
μ_2	-0.18686
μ_3	-1.0000
ξ	0.39977
ρ	-0.48781
γ_1	-0.70071
r_1	0.88757
r_2	0.11243

Similarly, we have calculated the normalized sensitivity indices for each state variable of the interior equilibrium $\bar{E} = (N_S^*,\ N_E^*,\ N_I^*,\ N_Q^*,\ N_R^*,\ N_V^*)$ corresponding to $\Lambda, \beta, \mu_1, \mu_2, \mu_3, \rho, a, \xi, \theta,\ r_1,\ r_2, \gamma_1,$ and γ_2. which are given in Table

4 and graphically presented in figure 5. From Table 4, we have explored that N_S^*,N_E^*,N_I^*,N_Q^*,N_R^*,N_V^* are highly sensitive to parameters $\Lambda, \beta, \mu_1, \mu_3, r_1$ and γ_1.

TABLE 4. The sensitivity indices, $\Upsilon_{x_j}^{y_i} = \frac{\partial y_i}{\partial x_j} \times \frac{x_j}{y_i}$, of endemic equilibrium's state variables, y_i, corresponding to parameters, x_j

x_j	$\Upsilon_{x_j}^{N_S^*}$	$\Upsilon_{x_j}^{N_E^*}$	$\Upsilon_{x_j}^{N_I^*}$	$\Upsilon_{x_j}^{N_Q^*}$	$\Upsilon_{x_j}^{N_R^*}$	$\Upsilon_{x_j}^{N_V^*}$
Λ	0.33728	1.40460	1.40460	1.40460	1.40460	1.40460
μ_1	-0.250206	-1.14359	-1.14359	-1.14359	-1.24217	-1.14359
μ_2	0.131032	-0.04563	-0.25616	-0.212301	-0.44916	-0.23249
μ_3	0.662712	-0.40468	-0.404675	-0.404675	-0.40468	-1.40468
γ_1	0.480969	-0.214446	-1.00392	-0.214446	-0.08007	-0.91516
γ_2	0.013869	0.057757	0.057757	-0.775576	0.17005	0.05776
β	-0.872125	0.53255	0.53255	0.53255	0.52155	0.53255
ξ	-0.263684	-0.320832	0.679168	-0.320832	0.15916	0.56674
ρ	0.328901	-0.661772	-0.661772	0.338228	-0.14632	-0.66177
r_1	-0.588206	0.359179	0.359179	0.359179	0.35918	1.24675
r_2	-0.074506	0.045496	0.045496	0.045496	0.0455	0.15792
θ	0.093956	0.391291	0.391291	0.391291	-0.33771	0.39129
a	0.209413	-0.127875	-0.127875	-0.127875	-0.11688	-0.127875

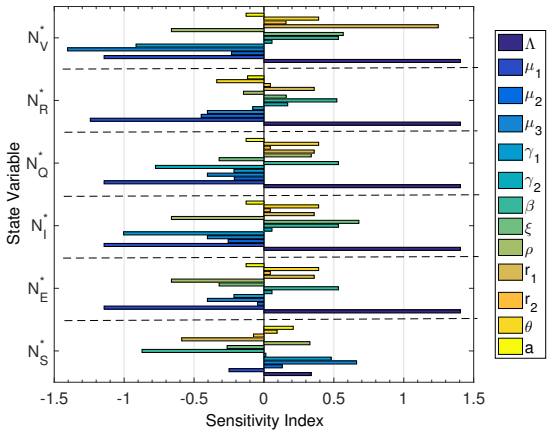

FIGURE 5. Sensitivity Indices for Endemic Equilibrium State Variables

7. Summary of the Work

We have proposed and analyzed a six compartmental epidemic model to explore the viral disease dynamics and found two biological equilibrium points: disease-free and endemic. Also derived an expression for most important threshold stability parameter i.e., basic reproduction number (R_0) and found that DFE (E^0) is locally stable if $R_0 < 1$ and EE ($\bar{E}$) is stable if ($R_0 < 1$). The global stability condition is derived for DFE (E^0) and EE ($\bar{E}$) using Lyapunov's Direct Method and found that DFE (E^0) local stability condition $R_0 < 1$ remain sufficient for global stability. But EE ($\bar{E}$) local stability condition does not remain sufficient for global stability. Sensitivity analysis is also performed using Normalized forward sensitivity index for reproduction number R_0 and EE ($\bar{E}$) to find the most sensitive parameters, which are given in Table 3 and 4 and also represented graphically in figure 5. It is also found from sensitivity analysis that virus growth rate r_1 is also more sensitive to threshold parameter R_0 and numerically shown that by changing the value of r_1 from 0.15 to 0.30 DFE (E^0) becomes unstable and EE ($\bar{E}$) becomes Stable (see in figures (2) and (3)). Numerically it is shown that by controlling quarantine parameter, the disease can be eradicated from the environment, i.e., quarantine can play as a control strategy in the absence of vaccination for any viral disease.

Conflicts of interest : The authors declare no conflict of interest.

Data availability : Not applicable

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