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FUZZY SOLUTIONS OF ABEL DIFFERENTIAL EQUATIONS USING RESIDUAL POWER SERIES METHOD

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Abstract. In this article, we find the approximate solutions of Abel differential equation (ADE) with uncertainty using residual power series (RPS) method. This method helps to calculate the sequence of solutions of ADE. Finally, numerical illustrations demonstrate the applicability of the method.

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1. Introduction

Differential equations attract a great deal of interest in modeling various problems exist in physics and computer processes. Almost all specialists have frequently used crisp ordinary differential equations to bring most reviews of the problems recognizable. The purpose of this paper is to extend the RPS method to find numerical approximation of fuzzy ADE (IVP). This approach is using to solve problems in the field of engineering and science with power series solutions. We consider the following nonlinear fuzzy ADE:

$$\tilde{g}'(t) = P\tilde{g}^3(t) + Q\tilde{g}^2(t) + R\tilde{g}(t) + S, \quad t > 0 \tag{1}$$

with the fuzzy initial condition

$$\tilde{g}(0) = \tilde{g}_0, \tag{2}$$

where $\tilde{g}^3(t) \neq 0$, P, Q, R and $S \in \Re$, $\tilde{g}_0$ is an arbitrary fuzzy number, and $\tilde{g}(t)$ is an unknown fuzzy function of the crisp variable t. However, assume IVP (1) and (2) each $t > 0$ has a unique fuzzy solution. R_F denotes the set of all fuzzy numbers defined in R. The model helps develop existing systems for processing power series formulae by adding a specific selective structural constraint [4, 7,

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11, 12, 13, 14]. Again, we ask for many attributes to explain and review specific methods for dealing with the various problems that arise in healing [1, 6, 9, 10]. Numerical solutions are found to demonstrate the functionality and performance of the RPS method.

2. Preliminaries

The important definitions and related properties of fuzzy calculus are in this part.

Definition 2.1. [5] Let u is a fuzzy number iff $[u]^r$ is compact convex subset of $\Re$ for $r \in [0,1]$ and $[u]^1 \neq \phi$. If u is a fuzzy number, then $[u]^r = [u_1(r),\ u_2(r)]$, for each $s \in [u]^r$, $r \in [0,1]$, where $u_1(r)$ =min$\{s\}$, $u_2(r)$ =max$\{s\}$ and $[u]^r$ is called $r-$cut representation form.

Theorem 2.2. [5] *Let* $u_1,\ u_2 : [0,1] \to \Re$ *satisfy the below conditions:*

(1) u_1 *is a bounded non decreasing function,*
(2) u_2 *is a bounded non increasing function,*
(3) $u_1(1) \leq u_2(1)$,
(4) $lim_{r\to k^-} u_1(r) = u_1(k)$ *and* $lim_{r\to k^-} u_2(r) = u_2(k),\ k \in (0,1]$,
(5) $lim_{r\to 0^+} u_1(r) = u_1(0)$ *and* $lim_{r\to 0^+} u_2(r) = u_2(0)$.

Then $u : \Re \to [0,1]$*, defined by* $u(s) = sup\{r|u_1(r) \leq s \leq u_2(r)\}$ *is a fuzzy number with parameter* $[u_1(r), u_2(r)]$.

Definition 2.3. [5] If u and v are two fuzzy numbers, for each $r \in [0,1]$, we've

(1) $[u+v]^r = [u]^r + [v]^r = [u_{1r} + v_{1r},\ u_{2r} + v_{2r}]$,
(2) $[\lambda u]^r = \lambda[u]^r = [min\{\lambda u_{1r},\ \lambda u_{2r}\},\ max\{\lambda u_{1r},\ \lambda u_{2r}\}]$,
(3) $[uv]^r = [u]^r[v]^r = [min\{u_{1r}v_{1r},\ u_{1r}v_{2r},\ u_{2r}v_{1r},\ u_{2r}v_{2r}\},$ $max\{u_{1r}v_{1r},\ u_{1r}v_{2r},\ u_{2r}v_{1r},\ u_{2r}v_{2r}\}]$,
(4) $u = v$ if $[u]^r = [v]^r$ if and only if $u_{1r} = v_{1r}$ and $u_{2r} = v_{2r}$, collection of all fuzzy numbers with addition and scalar multiplication is a convex cone.

Definition 2.4. [8] Let $u,\ v$ and $w \in \Re_F$, such that $u = v + w$; then w is called the Hukuhara differentiable of u and v, denoted by $u \ominus v$. Let $u \ominus v \neq u + (-1)v = u - v$ is Hukuhara differentiable, then $[u \ominus v]^r = [u_{1r} - v_{1r},\ u_{2r} - v_{2r}]$.

Definition 2.5. [2] Let g is strongly differentiable at $t_0 \in [a,b]$ and $g : [a,b] \to \Re_F$ such that

(1) For each $h > 0$, the Hukuhara differences $g(t_0+h) \ominus g(t_0),\ g(t_0) \ominus g(t_0-h)$ and
$$lim_{h\to 0^+} \frac{g(t_0+h) \ominus g(t_0)}{h} = lim_{h\to 0^+} \frac{g(t_0) \ominus g(t_0-h)}{h} = g'(t_0) \qquad (3)$$
(2) For each $h > 0$, the Hukuhara differences $g(t_0) \ominus g(t_0+h),\ g(t_0-h) \ominus g(t_0)$ and
$$lim_{h\to 0^+} \frac{g(t_0) \ominus g(t_0+h)}{-h} = lim_{h\to 0^+} \frac{g(t_0-h) \ominus g(t_0)}{-h} = g'(t_0). \qquad (4)$$

Theorem 2.6. [3] *For each* $r \in [0,1]$, $g : [a,b] \to \Re_F$ *and* $[g(t)]^r = [g_{1r}(t),\ g_{2r}(t)]$. *Such that* g_{1r} *and* g_{2r} *are differentiable functions on* $[a,b]$

(1) *If* g *is* $(1)-$*differentiable on* $[a,b]$ *then* $[g'(t)]^r = [g'_{1r}(t),\ g'_{2r}(t)]$,
(2) *If* g *is* $(2)-$*differentiable on* $[a,b]$ *then* $[g'(t)]^r = [g'_{2r}(t),\ g'_{1r}(t)]$.

Theorem 2.7. [3] *Let* $g : [a,b] \to \Re_F$ *be a fuzzy-valued function. For fixed* $t_0 \in [a,b]$ *and* $\epsilon > 0$ *if there exist* $\delta > 0$ *such that* $|t - t_0| < \delta$ *which implies* $d(g(t),\ g(t_0)) < \epsilon$, *then we say that* g *is continuous at* t_0.

3. Abel differential equation

Consider the nonlinear first order Abel type fuzzy differential equations,

$$\tilde{g}'(t) = P\tilde{g}^3(t) + Q\tilde{g}^2(t) + R\tilde{g}(t) + S, \quad t > 0. \tag{5}$$

with the fuzzy initial condition

$$\tilde{g}(0) = \tilde{g}_0, \tag{6}$$

where $P \neq 0$, P, Q, R and $S \in \Re$, $\tilde{g}(t) = [0,T] \to \Re_F$ and $\tilde{g}_0 \in \Re_F$. To construct the section of fuzzy Abel differential equation (FADE) (5) based on the type of differentiability and fuzzy initial condition (6), we consider the $r-$cut level representation of $\tilde{g}'(t)$, $\tilde{g}^3(t)$, $\tilde{g}^2(t)$, $\tilde{g}(t)$ and $\tilde{g}(0)$ as $[g'_{1r}(t),\ g'_{2r}(t)]$, $[g^3_{1r}(t),\ g^3_{2r}(t)]$, $[g^2_{1r}(t),\ g^2_{2r}(t)]$, $[g_{1r}(t),\ g_{2r}(t)]$, $[g_{0,1r}(t),\ g_{0,2r}(t)]$, respectively. Consequently, the FADEs (5) and (6) should be written as follows:

$$[\tilde{g}'(t)]^r = P[\tilde{g}^3(t)]^r + Q[\tilde{g}^2(t)]^r + R[\tilde{g}(t)]^r + S, \quad t > 0. \tag{7}$$

with the initial condition

$$[\tilde{g}(0)]^r = [\tilde{g}_0]^r. \tag{8}$$

Now, the residual power series for solving initial value problems (5) and (6) in r-cut representation that converted to crisp systems of ODEs. To obtain the fuzzy solution $\tilde{g}(t)$ for the initial value problems (5) and (6), two cases are considered according to kinds of differentiability, where $\tilde{g}(t)$ is either $(1)-$ differentiable or $(2)-$ differentiable.

Case 1: If $\tilde{g}(t)$ is $(1)-$ differentiable, then initial value problems (5) and (6) can be converted into the following system:

$$\begin{aligned} g'_{1r}(t) &= Pg^3_{1r}(t) + Qg^2_{1r}(t) + Rg_{1r}(t) + S, \\ g'_{2r}(t) &= Pg^3_{2r}(t) + Qg^2_{2r}(t) + Rg_{2r}(t) + S, \end{aligned} \tag{9}$$

with the initial condition

$$\begin{aligned} g_{1r}(0) &= g_{0,1r}, \\ g_{2r}(0) &= g_{0,2r}, \end{aligned} \tag{10}$$

Case 2: If $\tilde{g}(t)$ is $(2)-$ differentiable, then initial value problems (5) and (6) can be converted into the following system:

$$\begin{aligned} g'_{1r}(t) &= Pg_{2r}^3(t) + Qg_{2r}^2(t) + Rg_{2r}(t) + S, \\ g'_{2r}(t) &= Pg_{1r}^3(t) + Qg_{1r}^2(t) + Rg_{1r}(t) + S, \end{aligned} \tag{11}$$

with the initial condition

$$\begin{aligned} g_{1r}(0) &= g_{0,1r}, \\ g_{2r}(0) &= g_{0,2r}, \end{aligned} \tag{12}$$

4. The Residual Power Series method for the fuzzy Abel differential equation

In this section, we obtain the (1)-differentiable solution for the fuzzy Abel differential equations (9) and (10) by employing the procedures of residual power series method. Further, same procedure can be followed (2)-differentiable, we assume that $\tilde{g}(t)$ is (1)- differentiable, therefore the solutions of equations (11) and (12) at $t_0 = 0$ have the following forms:

$$\begin{aligned} g_{1r}(t) &= \textstyle\sum_{k=0}^{\infty} p_k t^k, \\ g_{2r}(t) &= \textstyle\sum_{k=0}^{\infty} q_k t^k. \end{aligned} \tag{13}$$

By using the initial conditions $g_{1r}(0) = g_{0,1r} = p_0$ and $g_{2r}(0) = g_{0,2r} = q_0$ as initial approximation, the expression of (13) can be written as:

$$\begin{aligned} g_{1r}(t) &= g_{0,1r} + \textstyle\sum_{k=1}^{\infty} p_k t^k, \\ g_{2r}(t) &= g_{0,2r} + \textstyle\sum_{k=1}^{\infty} q_k t^k. \end{aligned} \tag{14}$$

Consequently, the $i^{th}-$ truncated series solutions of $g_{1r}(t)$ and $g_{2r}(t)$ can be written as:

$$\begin{aligned} g_{i,1r}(t) &= g_{0,1r} + \textstyle\sum_{k=1}^{i} p_k t^k, \\ g_{i,2r}(t) &= g_{0,2r} + \textstyle\sum_{k=1}^{i} q_k t^k. \end{aligned} \tag{15}$$

According to the residual power series approach, the $i^{th}-$ residual functions of system (9) and (10) are defined by

$$\begin{aligned} Res_{i,1r}(t) &= g'_{1r}(t) - Pg_{1r}^3(t) - Qg_{1r}^2(t) - Rg_{1r}(t) - S, \\ Res_{i,2r}(t) &= g'_{2r}(t) - Pg_{2r}^3(t) - Qg_{2r}^2(t) - Rg_{2r}(t) - S. \end{aligned} \tag{16}$$

where the $\infty^{th}-$ residual functions are given by

$$\begin{aligned} & Res_{\infty,1r}(t) \\ & = lim_{i\to\infty} Res_{i,1r}(t) = g'_{1r}(t) - Pg_{1r}^3(t) - Qg_{1r}^2(t) - Rg_{1r}(t) - S, \\ & Res_{\infty,2r}(t) \\ & = lim_{i\to\infty} Res_{i,2r}(t) = g'_{2r}(t) - Pg_{2r}^3(t) - Qg_{2r}^2(t) - Rg_{2r}(t) - S, \end{aligned} \tag{17}$$

As in residual power series, put $Res_{\infty,ir}(t) = 0$ for each $t \in [0, R]$, R is radius of convergence and $i = \{1, 2\}$, which are infinitely differentiable functions at $t = 0$. Then we get $\frac{d^{k-1}}{dt^{k-1}} Res_{\infty,ir}(0) = \frac{d^{k-1}}{dt^{k-1}} Res_{k,ir}(0) = 0$, for $k = 1, 2, 3, \ldots j$. The residual power series p_n and q_n , $n \geq 1$. To find the coefficients p_1 and q_1 , substitute $g_{1,1r}(t) = g_{0,1r} + p_1 t$ and $g_{1,2r}(t) = g_{0,2r} + q_1 t$ to apply the residual functions, $Res_{1,1r}(t)$ and $Res_{1,2r}(t)$, at $i = 1$ of (16) we get:

$Res_{1,1r}(t) = g'_{1,1r}(t) - Pg^3_{1,1r}(t) - Qg^2_{1,1r}(t) - Rg_{1,1r}(t) - S,$
$= (g_{0,1r} + p_1t)' - P(g_{0,1r} + p_1t)^3 - Q(g_{0,1r} + p_1t)^2 - R(g_{0,1r} + p_1t) - S,$
$= p_1 - P(g_{0,1r} + p_1t)^3 - Q(g_{0,1r} + p_1t)^2 - R(g_{0,1r} + p_1t) - S.$

$Res_{1,2r}(t) = g'_{1,2r}(t) - Pg^3_{1,2r}(t) - Qg^2_{1,2r}(t) - Rg_{1,2r}(t) - S,$
$= (g_{0,2r} + q_1t)' - P(g_{0,2r} + q_1t)^3 - Q(g_{0,2r} + q_1t)^2 - R(g_{0,2r} + q_1t) - S,$
$= q_1 - P(g_{0,2r} + q_1t)^3 - Q(g_{0,2r} + q_1t)^2 - R(g_{0,2r} + q_1t) - S.$
Using $Res_{1,1r}(0) = 0$ and $Res_{1,2r}(0) = 0$ in (14) we get results to
$p_1 = P\tilde{g}^3_{0,1r} + Q\tilde{g}^2_{0,1r} + R\tilde{g}_{0,1r} + S$ and $q_1 = P\tilde{g}^3_{0,2r} + Q\tilde{g}^2_{0,2r} + R\tilde{g}_{0,2r} + S$.
Then the first approximations are:

$$\begin{aligned} g_{1,1r}(t) &= p_0 + \left(Pp_0^3 + Qp_0^2 + Rp_0 + S\right)t, \\ g_{1,2r}(t) &= q_0 + \left(Pq_0^3 + Qq_0^2 + Rq_0 + S\right)t. \end{aligned} \tag{18}$$

For $i = 2$, the second approximations are:

$$\begin{aligned} g_{2,1r}(t) &= p_0 + \left(Pp_0^3 + Qp_0^2 + Rp_0 + S\right)t + p_2t^2, \\ g_{2,2r}(t) &= q_0 + \left(Pq_0^3 + Qq_0^2 + Rq_0 + S\right)t + q_2t^2. \end{aligned} \tag{19}$$

The residual functions, $Res_{2,1r}(t)$ and $Res_{2,2r}(t)$ of (16) such that

$Res_{2,1r}(t) = g'_{2,1r}(t) - Pg^3_{2,1r}(t) - Qg^2_{2,1r}(t) - Rg_{2,1r}(t) - S,$
$= \left(\left(Pp_0^3 + Qp_0^2 + Rp_0 + S\right) + 2p_2t\right) - P\left(p_0 + \left(Pp_0^3 + Qp_0^2 + Rp_0 + S\right)t + p_2t^2\right)^3$
$-Q\left(p_0 + \left(Pp_0^3 + Qp_0^2 + Rp_0 + S\right)t + p_2t^2\right)^2$
$-R\left(p_0 + \left(Pp_0^3 + Qp_0^2 + Rp_0 + S\right)t + p_2t^2\right) - S,$

$Res_{2,2r}(t) = g'_{2,2r}(t) - Pg^3_{2,2r}(t) - Qg^2_{2,2r}(t) - Rg_{2,2r}(t) - S,$
$= \left(\left(Pq_0^3 + Qq_0^2 + Rq_0 + S\right) + 2q_2t\right) - P\left(q_0 + \left(Pq_0^3 + Qq_0^2 + Rq_0 + S\right)t + q_2t^2\right)^3$
$-Q\left(q_0 + \left(Pq_0^3 + Qq_0^2 + Rq_0 + S\right)t + q_2t^2\right)^2$
$-R\left(q_0 + \left(Pq_0^3 + Qq_0^2 + Rq_0 + S\right)t + q_2t^2\right) - S.$

Now, differentiality both sides of $Res_{2,1r}(t)$ and $Res_{2,2r}(t)$ we get

$\frac{d}{dt}Res_{2,1r}(t) = \frac{d}{dt}\left[g'_{2,1r}(t) - Pg^3_{2,1r}(t) - Qg^2_{2,1r}(t) - Rg_{2,1r}(t) - S\right],$

$= \frac{d}{dt}\left[\left(\left(Pp_0^3 + Qp_0^2 + Rp_0 + S\right) + 2p_2t\right)\right]$

$-\frac{d}{dt}\left[P\left(p_0 + \left(Pp_0^3 + Qp_0^2 + Rp_0 + S\right)t + p_2t^2\right)^3\right]$

$-\frac{d}{dt}\left[Q\left(p_0 + \left(Pp_0^3 + Qp_0^2 + Rp_0 + S\right)t + p_2t^2\right)^2\right]$

$-\frac{d}{dt}\left[R\left(p_0 + \left(Pp_0^3 + Qp_0^2 + Rp_0 + S\right)t + p_2t^2\right)\right] - \frac{d}{dt}[S],$

$\frac{d}{dt}Res_{2,1r}(t) = 2p_2 - 3P\left(p_0 + \left(Pp_0^3 + Qp_0^2 + Rp_0 + S\right)t + p_2t^2\right)^2$

$\times\left(\left(Pp_0^3 + Qp_0^2 + Rp_0 + S\right) + 2p_2t\right)$

$-2Q\left(p_0 + \left(Pp_0^3 + Qp_0^2 + Rp_0 + S\right)t + p_2t^2\right)\left(\left(Pp_0^3 + Qp_0^2 + Rp_0 + S\right) + 2p_2t\right)$

$-R\left(\left(Pp_0^3 + Qp_0^2 + Rp_0 + S\right) + 2p_2t\right),$

$\frac{d}{dt}Res_{2,2r}(t) = \frac{d}{dt}\left[g'_{2,2r}(t) - Pg_{2,2r}^3(t) - Qg_{2,2r}^2(t) - Rg_{2,2r}(t) - S\right],$

$= \frac{d}{dt}\left[\left(\left(Pq_0^3 + Qq_0^2 + Rq_0 + S\right) + 2q_2t\right)\right]$

$-\frac{d}{dt}\left[P\left(q_0 + \left(Pq_0^3 + Qq_0^2 + Rq_0 + S\right)t + q_2t^2\right)^3\right]$

$-\frac{d}{dt}\left[Q\left(q_0 + \left(Pq_0^3 + Qq_0^2 + Rq_0 + S\right)t + q_2t^2\right)^2\right]$

$-\frac{d}{dt}\left[R\left(q_0 + \left(Pq_0^3 + Qq_0^2 + Rq_0 + S\right)t + q_2t^2\right)\right] - \frac{d}{dt}\left[S\right],$

$\frac{d}{dt}Res_{2,2r}(t) = 2q_2 - 3P\left(q_0 + \left(Pq_0^3 + Qq_0^2 + Rq_0 + S\right)t + q_2t^2\right)^2$

$\times\left(\left(Pq_0^3 + Qq_0^2 + Rq_0 + S\right) + 2q_2t\right)$

$-2Q\left(q_0 + \left(Pq_0^3 + Qq_0^2 + Rq_0 + S\right)t + q_2t^2\right)\left(\left(Pq_0^3 + Qq_0^2 + Rq_0 + S\right) + 2q_2t\right)$

$-R\left(\left(Pq_0^3 + Qq_0^2 + Rq_0 + S\right) + 2q_2t\right),$

by using $\frac{d}{dt}Res_{2,1r}(0) = 0$ and $\frac{d}{dt}Res_{2,2r}(0) = 0$, it can be deduced to the residual functions

$$\begin{array}{l} p_2 = \frac{3}{2}Pp_0^2p_1 + Qp_0p_1 + \frac{1}{2}Rp_1, \\ q_2 = \frac{3}{2}Pq_0^2q_1 + Qq_0q_1 + \frac{1}{2}Rq_1. \end{array} \tag{20}$$

Then the second approximations are:

$$\begin{array}{l} g_{2,1r}(t) = p_0 + \left(Pp_0^3 + Qp_0^2 + Rp_0 + S\right)t + \left(\frac{3}{2}Pp_0^2p_1 + Qp_0p_1 + \frac{1}{2}Rp_1\right)t^2, \\ g_{2,2r}(t) = q_0 + \left(Pq_0^3 + Qq_0^2 + Rq_0 + S\right)t + \left(\frac{3}{2}Pq_0^2q_1 + Qq_0q_1 + \frac{1}{2}Rq_1\right)t^2. \end{array} \tag{21}$$

For $i = 3$, the third approximations are $g_{3,1r}(t)$ and $g_{3,2r}(t)$ into the residual functions, $Res_{3,1r}(t)$ and $Res_{3,2r}(t)$ of (16) utilized the residual power series $\frac{d^2}{dt^2}Res_{3,1r}(0) = 0$ and $\frac{d^2}{dt^2}Res_{3,2r}(0) = 0$. Then we get the third coefficients given by

$$\begin{array}{l} p_3 = P\left(p_2p_0^2 + p_1^2p_0\right) + \frac{1}{3}Q\left(2p_0p_2 + p_1^2\right) + \frac{1}{3}Rp_2, \\ q_3 = P\left(q_2q_0^2 + q_1^2q_0\right) + \frac{1}{3}Q\left(2q_0q_2 + q_1^2\right) + \frac{1}{3}Rq_2. \end{array} \tag{22}$$

For $i = 4$, the fourth approximations are $g_{4,1r}(t)$ and $g_{4,2r}(t)$ into the residual functions, $Res_{4,1r}(t)$ and $Res_{4,2r}(t)$ of (16) utilized the residual power series

$\frac{d^3}{dt^3}Res_{4,1r}(0) = 0$ and $\frac{d^3}{dt^3}Res_{4,2r}(0) = 0$. Then we get the third coefficients given by

$$\begin{aligned} p_4 &= \tfrac{1}{4}P\left(p_1^3 + 6p_0p_1p_2 + 3p_0^2p_3\right) + \tfrac{1}{2}Q\left(p_1p_2 + p_0p_3\right) + \tfrac{1}{4}Rp_3, \\ q_4 &= \tfrac{1}{4}P\left(q_1^3 + 6q_0q_1q_2 + 3q_0^2q_3\right) + \tfrac{1}{2}Q\left(q_1q_2 + q_0q_3\right) + \tfrac{1}{4}Rq_3. \end{aligned} \tag{23}$$

By continuing the same procedure upto arbitrary order $i = n$ using residual power series facts $\frac{d^{(n-1)}}{dt^{(n-1)}}Res_{n,1r}(0) = 0$ and $\frac{d^{(n-1)}}{dt^{(n-1)}}Res_{n,2r}(0) = 0$, it can be deduced that the residual functions p_n and q_n. Similarly, $\tilde{g}(t)$ is $(2)-$ solution for the $(2)-$ differentiable fuzzy Abel differential equation (11) and (12) can be obtained.

5. Numerical Examples

Example 5.1. Consider the following Abel initial value problem,

$$\tilde{g'}(t) - 3\tilde{g}(t)^3 + \tilde{g}(t) = 0, \quad t > 0, \tag{24}$$

with the fuzzy initial condition

$$[\tilde{g}(0)]^r = [\tfrac{7}{24} + \tfrac{1}{24}r,\ \tfrac{101}{300} - \tfrac{1}{300}r], \quad r \in [0,1]. \tag{25}$$

In particular for $r = 1$, the solution of (24) with crisp initial condition $\tilde{g}(0) = \frac{1}{3}$ as follows:

$$\tilde{g}(t) = \tfrac{1}{\sqrt{6e^{2t}+3}}. \tag{26}$$

we represent the parametric forms of (24) as follows:

$$\begin{aligned} g'_{1r}(t) &= 3g_{1r}(t)^3 - g_{1r}(t), \\ g'_{2r}(t) &= 3g_{2r}(t)^3 - g_{2r}(t), \end{aligned} \tag{27}$$

with the fuzzy initial condition

$$\begin{aligned} g_{1r}(0) &= \tfrac{7}{24} + \tfrac{1}{24}r, \\ g_{2r}(0) &= \tfrac{101}{300} - \tfrac{1}{300}r. \end{aligned} \tag{28}$$

By using the initial conditions $g_{1r}(0) = g_{0,1r} = p_0$ and $g_{2r}(0) = g_{0,2r} = q_0$ as initial approximations, the expression of (28) can be written as $g_{1r}(0) = \frac{7}{24} + \frac{1}{24}r$ and $g_{2r}(0) = \frac{101}{300} - \frac{1}{300}r$, the residual power series solutions $g'_{1r}(t)$ and $g'_{2r}(t)$ of system (27) can be written as:

$$\begin{aligned} g_{1r}(t) &= \tfrac{7}{24} + \tfrac{1}{24}r + p_1t + p_2t^2 + \ldots + p_it^i + \ldots, \\ g_{2r}(t) &= \tfrac{101}{300} - \tfrac{1}{300}r + q_1t + q_2t^2 + \ldots + q_it^i + \ldots. \end{aligned} \tag{29}$$

By utilizing the residual power series $\frac{d^{(i-1)}}{dt^{(i-1)}}Res_{i,1r}(0) = 0$ and $\frac{d^{(i-1)}}{dt^{(i-1)}}Res_{i,2r}(0) = 0$, for $i = 1, 2\ldots$, the terms of p_i and q_i are:

$p_0 = \frac{7}{24} + \frac{1}{24}r,$

$p_1 = \frac{1}{4608}(r+7)\left(r^2 + 14r - 143\right),$

$p_2 = \frac{1}{589824}(r-1)(r+15)(r+7)\left(r^2 + 14r - 143\right),$

$p_3 = \frac{1}{339738624}\left(5r^4+140r^3+702r^2-3892r-13339\right)(r+7)\left(r^2+14r-143\right),$

$\vdots$

and

$q_0 = \frac{101}{300} - \frac{1}{300}r,$

$q_1 = \frac{1}{9000000}(101-r)\left(r^2-202r-19799\right),$

$q_2 = \frac{1}{180000000000}(101-r)\left(r^2-202r-19799\right)(1-r)(201-r),$

$q_3 = \frac{1}{3240000000000000}$

$(101-r)\left(r^2-202r-19799\right)\left(r^4-404r^3+37206r^2+726796r-80763599\right),$

$\vdots$

and so on.

If $r = 1$, then the residual power series solution becomes

$$g(t) = \frac{1}{3} - \frac{2}{9}t + \frac{4}{81}t^3 - \frac{2}{243}t^4 + \ldots\ldots \quad (30)$$

The numerical results of Example 1 for various t in $[0,1]$ is shown in Table 1 and Fig 1.

TABLE 1. Value of g(t)

t	Exact solution	RPSM solution	Absolute Error
0	[0.333333333333333]	[0.333333333333333]	[0]
0.1	[0.311159544858791]	[0.311159670781893]	[1.26 $\times 10^{-7}$]
0.2	[0.289266951196178]	[0.289270781893004]	[123 $\times 10^{-6}$]
0.3	[0.267905825643850]	[0.267933333333333]	[2.75 $\times 10^{-5}$]
0.4	[0.247285140856786]	[0.247394238683128]	[1.09 $\times 10^{-4}$]
0.5	[0.227568600277633]	[0.227880658436214]	[52 $\times 10^{-4}$]
0.6	[0.208874799971873]	[0.209600000000000]	[7.25 $\times 10^{-4}$]
0.7	[0.191280479364827]	[0.192739917695473]	[1.46 $\times 10^{-3}$]
0.8	[0.174825723730926]	[0.177468312757202]	[2.64 $\times 10^{-3}$]
0.9	[0.159520099726950]	[0.163933333333333]	[161 $\times 10^{-3}$]
1.0	[0.145348934559835]	[0.152263374485597]	[6.91 $\times 10^{-3}$]

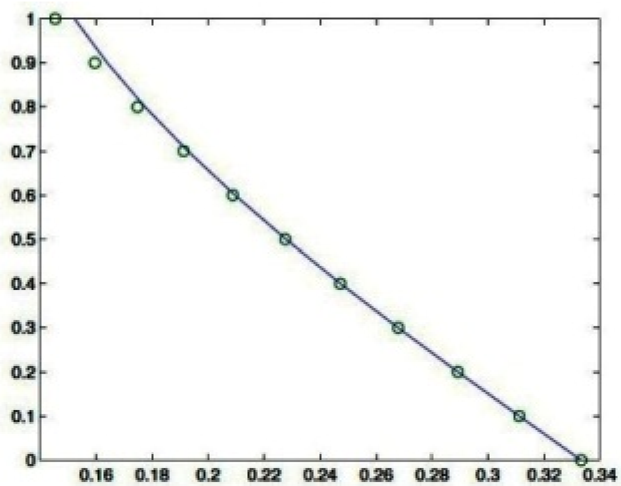

Fig. 1 Value of f(x)

Example 5.2. Consider the following Abel initial value problem,

$$\tilde{g}'(t) + \tilde{g}(t)^3 - \tilde{g}(t) = 0, \quad t > 0, \tag{31}$$

with the fuzzy initial condition

$$[\tilde{g}(0)]^r = [\tfrac{7}{24} + \tfrac{1}{24}r,\ \tfrac{101}{300} - \tfrac{1}{300}r], \quad r \in [0,1]. \tag{32}$$

In particular for $r = 1$, the solution of (31) with crisp initial condition $\tilde{g}(0) = \frac{1}{3}$ can be found as:

$$\tilde{g}(t) = \frac{e^t}{\sqrt{e^{2t}+8}}. \tag{33}$$

we represent the parametric forms of (31) as follows:

$$\begin{aligned} g'_{1r}(t) &= g_{1r}(t) - g_{1r}(t)^3, \\ g'_{2r}(t) &= g_{1r}(t) - g_{2r}(t)^3. \end{aligned} \tag{34}$$

with the fuzzy initial condition

$$\begin{aligned} g_{1r}(0) &= \tfrac{7}{24} + \tfrac{1}{24}r, \\ g_{2r}(0) &= \tfrac{101}{300} - \tfrac{1}{300}r. \end{aligned} \tag{35}$$

By using the initial conditions $g_{1r}(0) = g_{0,1r} = p_0$ and $g_{2r}(0) = g_{0,2r} = q_0$ as initial approximations. Then, the expression of (32) can be written as $g_{1r}(0) = \frac{7}{24} + \frac{1}{24}r$ and $g_{2r}(0) = \frac{101}{300} - \frac{1}{300}r$, the residual power series solutions $g'_{1r}(t)$ and $g'_{2r}(t)$ of system (241) can be written as:

$$\begin{aligned} g_{1r}(t) &= \tfrac{7}{24} + \tfrac{1}{24}r + p_1 t + p_2 t^2 + \ldots + p_i t^i + \ldots, \\ g_{2r}(t) &= \tfrac{101}{300} - \tfrac{1}{300}r + q_1 t + q_2 t^2 + \ldots + q_i t^i + \ldots. \end{aligned} \tag{36}$$

By utilizing the residual power series $\frac{d^{(i-1)}}{dt^{(i-1)}} Res_{i,1r}(0) = 0$ and $\frac{d^{(i-1)}}{dt^{(i-1)}} Res_{i,2r}(0) = 0$, for $i = 1,\ 2\ldots$, the terms of p_i and q_i are:

$p_0 = \frac{7}{24} + \frac{1}{24}r,$
$p_1 = \frac{1}{13824}\left(r+7\right)\left(17-r\right)\left(r+31\right),$

$p_2 = \frac{1}{5308416}\left(r+7\right)\left(r-17\right)\left(r+31\right)\left(r^2+14r-143\right),$

$p_3 = \frac{1}{9172942848}\left(5r^4+140r^3-834r^2-25396r+9701\right)\left(r+7\right)\left(17-r\right)\left(r+31\right),$
$\vdots$
and
$q_0 = \frac{101}{300} - \frac{1}{300}r,$

$q_1 = \frac{1}{27000000}\left(r-101\right)\left(r+199\right)\left(r-401\right),$

$q_2 = \frac{1}{1620000000000}\left(401-r\right)\left(r-101\right)\left(r+199\right)\left(r^2-202r-19799\right),$

$q_3 = \frac{1}{87480000000000000}$

$\left(r-101\right)\left(r+199\right)\left(r-401\right)\left(r^4-404r^3-10794r^2+10422796r-90411599\right),$
$\vdots$
and so on.
If $r = 1$, then the residual power series solution

$$g(t) = \frac{1}{3} + \frac{8}{27}t + \frac{8}{81}t^2 - \frac{16}{2187}t^3 - \frac{440}{19683}t^4 + \ldots\ldots \tag{37}$$

The numerical results of Example 2 for various t in $[0,1]$ is shown in Table 2 and Fig 2.

Table 2. Value of g(t)

t	Exact solution	RPSM solution	Absolute Error
0	[0.333333333333333]	[0.333333333333333]	[0]
0.1	[0.363940973764747]	[0.363941065894427]	[9.21×10^{-8}]
0.2	[0.396446001535819]	[0.396448915307626]	[2.91×10^{-6}]
0.3	[0.430710791531335]	[0.430732510288066]	[2.17×10^{-5}]
0.4	[0.466524705168791]	[0.466613829192704]	[8.91×10^{-5}]
0.5	[0.503598763165705]	[0.503861200020322]	[2.62×10^{-4}]
0.6	[0.541565802609583]	[0.542189300411523]	[6.23×10^{-4}]
0.7	[0.579987893064792]	[0.581259157648732]	[1.27×10^{-3}]
0.8	[0.618372060898173]	[0.620678148656201]	[2.31×10^{-3}]
0.9	[0.656194098474177]	[0.660000000000000]	[121×10^{-3}]
1.0	[0.692928605869062]	[0.698724787888025]	[310×10^{-3}]

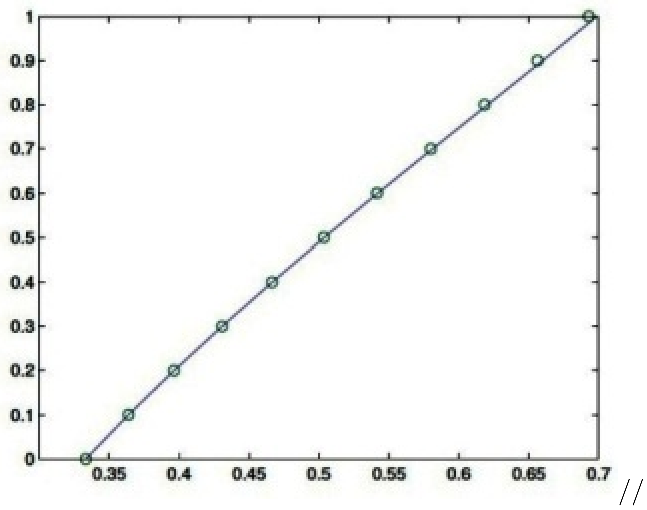

Fig. 2 Value of f(x)

6. Conclusions

In this paper, the RPS has been used to examine the convergence analysis to fuzzy ADE. This technique may be applied immediately by evaluating different starting predictions without being generally separated or modified. Simulation findings have demonstrated the efficiency and dependability of the current method. According to the findings, the RPS technique is extremely efficient and powerful for solving nonlinear fuzzy Abel differential equations with less computations and effort.

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Data availability : Not applicable

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