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THE PRICING OF VULNERABLE FOREIGN EXCHANGE OPTIONS UNDER A MULTISCALE STOCHASTIC VOLATILITY MODEL†

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Abstract. Foreign exchange options are derivative financial instruments that can exchange one currency for another at a prescribed exchange rate on a specified date. In this study, we examine the analytic formulas for vulnerable foreign exchange options based on multi-scale stochastic volatility driven by two diffusion processes: a fast mean-reverting process and a slow mean-reverting process. In particular, we take advantage of the asymptotic analysis and the technique of the Mellin transform on the partial differential equation (PDE) with respect to the option price, to derive approximated prices that are combined with a leading order price and two correction term prices. To verify the price accuracy of the approximated solutions, we utilize the Monte Carlo method. Furthermore, in the numerical experiments, we investigate the behaviors of the vulnerable foreign exchange options prices in terms of model parameters and the sensitivities of the stochastic volatility factors to the option price.

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1. Introduction and Literature Review

Vulnerable foreign exchange options are derivatives that combine standard vulnerable and foreign exchange options. Many options traded in over-the-counter (OTC) markets are exposed to default risks, so it is less likely that an option writer fulfills their contractual obligations. After the global financial

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crisis in 2007-2008, given the growing concerns about contingent claims with default risks, counterparty credit risks have been considered when pricing options are traded OTC. The *vulnerable option* refers to a financial instrument that considers the default risk of the counterparty (or the option writer) and was first proposed by Johnson and Stulz [20] in 1987. Since then, Klein [26] found an analytic pricing formula that deals with other liabilities of the option writer in the capital structure, the correlation between the option's underlying asset, and the default risk of the counterparty. In contrast, Hung and Liu [15] and Yang et al. [30] examined the pricing of vulnerable options as the market is incomplete.

Unlike general vanilla options, vulnerable options have more complicated stochastic dynamics representing the underlying asset process and the process of the option writer's market value. In other words, determining the fair price of vulnerable options is equivalent to solving the (1+2)-dimensional partial differential equation (PDE) with one time variable and two state variables. Several studies have been conducted on the pricing of vulnerable options using PDEs. Yoon and Kim [32] first studied the price of vulnerable European vanilla options by solving the given (1+2)-dimensional PDEs by utilizing an integral transform method. Applications of such vulnerable options in other types of financial problems, except for the pricing of European vanilla options, have been studied by [16], [17], [18], [22], [21], and [19].

Foreign Exchange Options are derivatives that trade the right to exchange one underlying asset for another. In particular, as a popular option used in foreign exchange transactions, it is usually traded by individuals, institutions, and governments to hedge risk from exchange rates effectively. According to 2019 Triennial Central Bank Survey of FX and OTC markets [2], the trade volume of foreign exchange option is roughly $6.6 trillion. Therefore, based on its usefulness, there have been many studies on foreign exchange (exotic) options. For example, Lindset [27] presented the price of the American exchange option under the assumption that the underlying assets follow jump-diffusion processes. Antonelli et al. [1] dealt with the pricing of exchange options under a stochastic volatility model. Kim and Koo [24] obtained closed-form solutions for foreign exchange options with credit risk. Recently, Kim et al. [23] derived closed-form solutions of exchange options considering early counterparty credit risks by adding the nature of credit risk to the existing exchange options.

As financial markets become increasingly complicated, the Black-Scholes model [3] has been widely used for pricing and hedging financial instruments because of its theoretical simplicity and practical usefulness. However, the assumption of a standard Black-Scholes model for underlying asset prices cannot reflect the empirical evidence in the financial market. In particular, many exogenous variables and extraordinary volatility behavior have had a significant impact on the market after the global financial crisis in 2007-2008, as shown in Choi et al. [7], and market participants trading financial derivatives have become interested in models that can forecast the behaviors of financial assets in the market exactly and effectively. Thus, if we consider that the volatility of an underlying asset follows more exogenous stochastic phenomena and then fluctuates very quickly throughout the lifetime of financial derivatives, it can be seen that the volatility is well modelled as a fast mean-reverting stochastic process, and the stochastic volatility (SV) model is well known for explaining their dynamics and reflecting real situations in financial markets. The Heston model (cf. Heston [14]) and the fast mean-reverting SV model proposed by Fouque et al. [11] have become representative SV models designed to capture the environment of the mean-reversion of volatility in the financial market. However, the structure of SV models is very complicated. To deal with the complex structures of the model dynamics, Fouque et al. [11] derived PDEs for option prices, and then took advantage of the singular-regular perturbation method on the PDEs because it is difficult to solve the PDEs directly. Applications of the perturbation method in PDEs enable us to find analytic solutions for option prices very easily and effectively. Since Fouque et al. [8, 10, 12] first introduced the pricing of various options by exploiting the singular perturbation method, many mathematicians have begun to focus on the diverse option pricing formula reflecting financial economics situations. Jeon et al. [19] studied the values of vulnerable options under a multi-scale stochastic volatility model utilizing a singular-regular perturbation approach, and Kim et al. [22] presented a pricing formula for the external barrier option under a (pure) stochastic volatility model using asymptotic analysis and the method of images to handle the boundary conditions. Developing the European vanilla options, Wang [29] derived the pricing formula for vulnerable basket warrants under stochastic volatility models, and especially, Choi et al. [6] studied the properties of the implied volatilities of foreign exchange markets such as EUD/USD, GBP/USD, and AUD/USD under a hybrid stochastic and local volatility (SLV) model by the same asymptotic techniques.

In this study, we deal with the **vulnerable foreign exchange options under a multi-scale stochastic volatility (MSV)**. According to Jeon et al. [19] and Fouque et al. [11], it is clear that the MSV model was analyzed and showed better calibration results, based on the empirical data such as S&P 500, especially. We incorporate stochastic volatility models into foreign exchange options with default risk, where the multi-scale stochastic volatility consists of stochastic volatility terms driven by fast-mean-reverting diffusion and slow-varying diffusion processes. We note that the two diffusions are governed by Ornstein-Uhlenbeck processes, based on the empirical fact that volatility converges to the long-run average. To find the analytic solutions for the exchange option prices with MSV, we use a multi-scale analysis (or singular-regular perturbation method). In other words, by applying the multi-scale analysis to the given PDE problems for the option value and commuting property, we obtain the approximated formulas for the option prices. Furthermore, using the Monte Carlo method, we examine the pricing accuracy of the approximated options and investigate the sensitivities of the stochastic volatility factors on the option values.

The remainder of this paper is organized as follows. In Section 2, we construct the model dynamics for vulnerable exchange options under MSV and introduce a (multi) stochastic volatility model. In Section 3, from the asymptotic analysis, we obtain the approximated option prices composed of one leading order price and two correction order prices, and theoretically provide the pricing accuracy for the option values. Section 4 verifies the accuracy of the approximated solutions of the option price by comparing it with the solution obtained from the Monte Carlo method and examines the price impact of MSV on the vulnerable exchange options with regard to the option parameters. Finally, concluding remarks are presented in Section 5.

2. The model

In this section, we construct mathematical circumstances for vulnerable options on foreign exchange markets under multiscale stochastic volatility. Let a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t|_{0\le t\le T}, \mathbb{P}^*)$, where Ω is a set of outcomes, $\mathcal{F}$ is a set of events ($\sigma-$ field), $\mathcal{F}_t$ is a filtration generated by the Brownian motion, and $\mathbb{P}^*$ is a martingale measure (or risk-neutral measure). Next, under the equivalent martingale measure $\mathbb{P}^*$, we establish the stochastic dynamics for the domestic currency S_1 and the foreign currency S_2 by

$$\begin{aligned} \mathrm{d}S_1(t) &= rS_1(t)\,\mathrm{d}t + f(Y(t), Z(t))S_1(t)\,\mathrm{d}W_1^*(t), \\ \mathrm{d}S_2(t) &= rS_2(t)\,\mathrm{d}t + \sigma_F S_2(t)\,\mathrm{d}W_2^*(t), \end{aligned} \tag{1}$$

respectively. Here, r is the risk-free interest rate with a constant value, and σ_F is the constant volatility of foreign currency $S_2(t)$. According to Fouque et al. [12], the stochastic volatility $f(Y(t), Z(t))$ of the domestic currency $S_1(t)$ is driven by two factors: the fast reverting factor Y and the slowly varying factor Z, where the function f is a positive smooth function such that $\int_{\mathbb{R}} f^2(\cdot, z)dz < \infty$. In market model (1), the fast-mean-reverting process (Ornstein-Uhlenbeck process, OU process) $Y(t)$ is governed by

$$\mathrm{d}Y(t) = \left[\kappa(m - Y(t)) - \frac{u\sqrt{2}}{\sqrt{\epsilon}}\Lambda_y(Y(t), Z(t))\right]\mathrm{d}t + \frac{u\sqrt{2}}{\sqrt{\epsilon}}\mathrm{d}W_y^*(t), \tag{2}$$

where $\kappa := \epsilon^{-1}$ is the fast reverting rate that the process $Y(t)$ converges to the mean level m, and $\frac{u\sqrt{2}}{\sqrt{\epsilon}}$ is the variance of $Y(t)$, and the function $\Lambda_y(Y(t), Z(t))$ is the market price of volatility risk which is assumed to be a constant in this paper. Also, we consider the slowly varying diffusion $Z(t)$

$$\mathrm{d}Z(t) = \left[\delta c(Z(t)) - \sqrt{\delta} g(Z(t))\Lambda_z(Y(t), Z(t))\right]\mathrm{d}t + \sqrt{\delta} g(Z(t))\mathrm{d}W_z^*(t), \tag{3}$$

where δ is a small time-scale parameter such that $0 < \epsilon < \delta < \sqrt{\epsilon} \ll 1$, $c(\cdot)$ and $g(\cdot)$ are smooth functions, and $\Lambda_z(Y(t), Z(t))$ is the market price of the volatility risk which is also assumed to be a constant. Finally, because we focus on the **vulnerable** options in the foreign exchange market with a multi-scale stochastic volatility model, we consider the dynamics of the market value of the option's issuer (that is, the writer) $V(t)$ as follows:

$$\mathrm{d}V(t) = rV(t)\,\mathrm{d}t + \sigma_V V(t)\,\mathrm{d}W_3^*(t) \tag{4}$$

where σ_V denotes a constant volatility for V.

In the dynamics (1)-(4), the instantaneous correlations between Brownian motions $W_1^*(t)$, $W_2^*(t)$, $W_3^*(t)$, $W_y^*(t)$, and $W_z^*(t)$ are expressed by

$$\begin{aligned}
&\mathrm{d}\langle W_1^*, W_2^*\rangle_t = \rho_{12}\mathrm{d}t,\ \mathrm{d}\langle W_1^*, W_3^*\rangle_t = \rho_{13}\mathrm{d}t,\ \mathrm{d}\langle W_1^*, W_y^*\rangle_t = \rho_{1y}\mathrm{d}t,\\
&\mathrm{d}\langle W_1^*, W_z^*\rangle_t = \rho_{1z}\mathrm{d}t,\ \mathrm{d}\langle W_2^*, W_3^*\rangle_t = \rho_{23}\mathrm{d}t,\ \mathrm{d}\langle W_2^*, W_y^*\rangle_t = \rho_{2y}\mathrm{d}t,\\
&\mathrm{d}\langle W_2^*, W_z^*\rangle_t = \rho_{2z}\mathrm{d}t,\ \mathrm{d}\langle W_3^*, W_y^*\rangle_t = \rho_{3y}\mathrm{d}t,\ \mathrm{d}\langle W_3^*, W_z^*\rangle_t = \rho_{3z}\mathrm{d}t,\\
&\mathrm{d}\langle W_y^*, W_z^*\rangle_t = \rho_{yz}\mathrm{d}t,
\end{aligned}$$

where the correlations ρ_{12}, ρ_{13}, ρ_{1y}, ρ_{1z}, ρ_{23}, ρ_{2y}, ρ_{2z}, $\rho_{3y}\,\rho_{3z}$, $\rho_{yz} \in [-1, 1]$ and we assume that $\rho_{2y} = \rho_{2z} = \rho_{3y} = \rho_{3z} = 0$.

Then, under the equivalent martingale measure $\mathbb{P}^*$, the fair option's price $P^{\epsilon,\delta} = P^{\epsilon,\delta}(t, s_1, s_2, v, y, z)$ at the current time $t(\geq 0)$ is a conditional expectation of the discounted payoff. i.e.,

$$P^{\epsilon,\delta} = \mathbb{E}^*\left[e^{-r(T-t)}h(S_1(T), S_2(T), V(T), Y(T), Z(T))\,\middle|\,\mathcal{F}_t\right], \tag{5}$$

where $\mathbb{E}^*[\cdot]$ means the expectation under the equivalent martingale measure $\mathbb{P}^*$, the information up to the current time ($\mathcal{F}_t$) is given by $\mathcal{F}_t = \{S_1(t) = s_1, S_2(t) = s_2, V(t) = v, Y(t) = y, Z(t) = z\}$, and the final condition (or payoff function) $h = h(S_1(T), S_2(T), V(T), Y(T), Z(T))$ is given by

$$h = (S_1(T) - S_2(T))^+\left(\mathbf{1}_{\{V(T)>D^*\}} + \frac{1-\alpha}{D}V(T)\mathbf{1}_{\{V(T)\leq D^*\}}\right), \tag{6}$$

as observed in [23]. The payoff function (6) consists of two parts: (i) The first part $(S_1(T)-S_2(T))^+$ demonstrates the standard foreign exchange options. The buyer of the options abandons currency S_2 and receives another currency S_1 at maturity time T. (ii) The second part $\left(\mathbf{1}_{\{V(T)>D^*\}} + \frac{1-\alpha}{D}V(T)\mathbf{1}_{\{V(T)\leq D^*\}}\right)$ proposed by Johnson and Stulz [20] describes the credit risks of the option writer, where α is the deadweight cost related to the bankruptcy of the counterparty, D denotes the total liabilities given by D^*, and D^* represents the critical level (or default boundary) for the default. If the market value of the option writer, $V(T)$, is greater than the critical level D^*, the standard profit $(S_1 - S_2)^+$ at maturity T. Otherwise, counterparty credit risk occurs and only the $\frac{1-\alpha}{D}V(T)(S_1(T) - S_2(T))^+$ is paid out.

3. Derivation of Pricing Formula

By applying the Feynman-Kac formula to the expectation representation (5), the price function $P(t, s_1, s_2, v, y, z)$ satisfies the following partial differential equation (PDE):

$$\begin{aligned}
&\frac{1}{\epsilon}\Big((m-y)P_y+u^2P_{yy}\Big)+\frac{1}{\sqrt{\epsilon}}\Big(-u\sqrt{2}\Lambda_y(y,z)P_y+u\sqrt{2}\rho_{1y}f(y,z)s_1P_{s_1y}\Big)\\
&-rP+P_t+rs_1P_{s_1}+rs_2P_{s_2}+rvP_v+\frac{f^2(y,z)}{2}s_1^2P_{s_1s_1}+\frac{\sigma_F^2}{2}s_2^2P_{s_2s_2}+\frac{\sigma_V^2}{2}v^2P_{vv}\\
&+\sigma_F\rho_{12}f(y,z)s_1s_2P_{s_1s_2}+\sigma_V\rho_{13}f(y,z)s_1vP_{s_1v}+\sigma_F\sigma_V\rho_{23}s_2vP_{s_2v}\\
&\sqrt{\delta}\left(-g(z)\Lambda_z(y,z)P_z+g(z)\rho_{1z}f(y,z)s_1P_{s_1z}\right)\\
&+\delta\left(c(z)P_z+\frac{g^2(z)}{2}P_{zz}\right)+\frac{ug(z)\sqrt{2\delta}}{\sqrt{\epsilon}}\rho_{yz}P_{yz}=0
\end{aligned} \tag{7}$$

with the final condition $P(T,s_1,s_2,v,y,z)=(s_1-s_2)^+\left(\mathbf{1}_{\{v>D^*\}}+\frac{1-\alpha}{D}v\mathbf{1}_{\{v\le D^*\}}\right)$ on the domain $\{(t,s_1,s_2,v,y,z): t\in[0,T), s_1,s_2,v\in[0,\infty), y,z\in(-\infty,\infty)\}$. Because the PDE (7) consists of terms of order $\frac{1}{\epsilon}$, $\frac{1}{\sqrt{\epsilon}}$, 1, $\sqrt{\delta}$, δ, and $\sqrt{\frac{\delta}{\epsilon}}$, we define the differential operators $\mathcal{L}_0$, $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{M}_1$, $\mathcal{M}_2$, and $\mathcal{M}_3$ as follows:

$$\begin{aligned}
\mathcal{L}_0=&(m-y)\frac{\partial}{\partial y}+u^2\frac{\partial^2}{\partial y^2},\\
\mathcal{L}_1=&-u\sqrt{2}\Lambda_y(y,z)\frac{\partial}{\partial y}+u\sqrt{2}\rho_{1y}f(y,z)s_1\frac{\partial^2}{\partial s_1\partial y},\\
\mathcal{L}_2=&\frac{\partial}{\partial t}+rs_1\frac{\partial}{\partial s_1}+rs_2\frac{\partial}{\partial s_2}+rv\frac{\partial}{\partial v}+\frac{1}{2}f^2(y,z)s_1^2\frac{\partial^2}{\partial s_1^2}+\frac{1}{2}\sigma_F^2s_2^2\frac{\partial^2}{\partial s_2^2}+\frac{1}{2}\sigma_V^2v^2\frac{\partial^2}{\partial v^2},\\
&+\sigma_F\rho_{12}f(y,z)s_1s_2\frac{\partial^2}{\partial s_1\partial s_2}+\sigma_V\rho_{13}f(y,z)s_1v\frac{\partial^2}{\partial s_1\partial v}+\sigma_F\sigma_V\rho_{23}s_2v\frac{\partial^2}{\partial s_2\partial v}-r\cdot,\\
\mathcal{M}_1=&-g(z)\Lambda_z(y,z)\frac{\partial}{\partial z}+\rho_{1z}f(y,z)g(z)s_1\frac{\partial^2}{\partial s_1\partial z},\\
\mathcal{M}_2=&c(z)\frac{\partial}{\partial z}+\frac{g^2(z)}{2}\frac{\partial^2}{\partial z^2},\quad \mathcal{M}_3=\sqrt{2}u\rho_{yz}g(z)\frac{\partial^2}{\partial y\partial z},
\end{aligned}$$

where $\mathcal{L}_0$ is related to the infinitesimal generator of the OU process Y, $\mathcal{L}_1$ consists of the mixed partial derivatives s_1 and y, $\mathcal{L}_2$ is considered as the (1+3) Black-Scholes operator (with one time variable and three state variables) for vulnerable foreign exchange options at the volatility level $f(y,z)$, and $\mathcal{M}_1$ contains the derivatives s_1 and z, and $\mathcal{M}_2$ is related to the infinitesimal generator of Z, $\mathcal{M}_3$ consists of the derivatives y and z.

From the operator notations $\mathcal{L}_0$, $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{M}_1$, $\mathcal{M}_2$, and $\mathcal{M}_3$, the formulated PDE (7) can be rewritten as

$$\begin{cases}\mathcal{L}^{\epsilon,\delta}P^{\epsilon,\delta}(t,s_1,s_2,v,y,z)=0 & \text{in } \mathcal{D},\\ P^{\epsilon,\delta}(T,s_1,s_2,v,y,z)=h(s_1,s_2,v,y,z) & \text{on } \{t=T\},\end{cases} \tag{8}$$

where $\mathcal{L}^{\epsilon,\delta}:=\frac{1}{\epsilon}\mathcal{L}_0+\frac{1}{\sqrt{\epsilon}}\mathcal{L}_1+\mathcal{L}_2+\sqrt{\delta}\mathcal{M}_1+\delta\mathcal{M}_2+\sqrt{\frac{\delta}{\epsilon}}\mathcal{M}_3$ and $\mathcal{D}:=[0,T)\times\mathbb{R}_{\ge0}\times\mathbb{R}_{\ge0}\times\mathbb{R}_{\ge0}\times\mathbb{R}\times\mathbb{R}$. To solve the PDE (8), as in Fouque et al. [12], we expand the price function $P^{\epsilon,\delta}$ with respect to the small parameter $\sqrt{\delta}$. i.e.,

$$P^{\epsilon,\delta}(t,s_1,s_2,v,y,z)=\sum_{n=0}^{\infty}\delta^{n/2}P_n^{\epsilon}(t,s_1,s_2,v,y,z). \tag{9}$$

for $0 < \epsilon < \delta < \sqrt{\epsilon} \ll 1$. Substituting (9) into (8), the following relations is obtained:

$$\begin{aligned}
&\left(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2\right) P_0^\epsilon = 0,\\
&\left(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2\right) P_1^\epsilon + \left(\mathcal{M}_1 + \frac{1}{\sqrt{\epsilon}}\mathcal{M}_3\right) P_0^\epsilon = 0,\\
&\left(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2\right) P_2^\epsilon + \mathcal{M}_1 P_1^\epsilon + \mathcal{M}_2 P_0^\epsilon + \frac{1}{\sqrt{\epsilon}}\mathcal{M}_3 P_1^\epsilon = 0,\\
&\cdots .
\end{aligned} \tag{10}$$

Next, we also expand P_n^ϵ in power of $\sqrt{\epsilon}$ as

$$P_n^\epsilon(t, s_1, s_2, v, y, z) = \sum_{i=0}^{\infty} \epsilon^{i/2} P_{n,i}(t, s_1, s_2, v, y, z) \tag{11}$$

for any $n = 0, 1, 2, \cdots$. Then, the equation $\left(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2\right) P_0^\epsilon = 0$ implies that $P_{0,0}$ and $P_{0,1}$ are independent of the volatility level y (detailed proof is presented in [12] or [19]). Also, from the PDE (8) and expansion of $P^{\epsilon,\delta}$, we obtain the following sequential PDEs for $P_{0,i}$ $(i \geq 0)$.

$$\langle \mathcal{L}_2 \rangle P_{0,i}(t, s_1, s_2, v, z) = 0 \quad \text{in } \mathcal{D} \tag{12}$$

with the terminal conditions are $P_{0,0}(T, s_1, s_2, v, z) = h(s_1, s_2, v)$ and

$$P_{0,i}(T, s_1, s_2, v, y, z) = 0$$

for any $i \geq 1$. In (12), the expected operator $\langle \mathcal{L}_2 \rangle$ is

$$\begin{aligned}
\langle \mathcal{L}_2 \rangle = {} & \frac{\partial}{\partial t} + r s_1 \frac{\partial}{\partial s_1} + r s_2 \frac{\partial}{\partial s_2} + r v \frac{\partial}{\partial v} + \frac{1}{2}\bar{\sigma}_1^2(z) s_1^2 \frac{\partial^2}{\partial s_1^2} + \frac{1}{2}\sigma_F^2 s_2^2 \frac{\partial^2}{\partial s_2^2} + \frac{1}{2}\sigma_V^2 v^2 \frac{\partial^2}{\partial v^2}\\
& + \bar{\rho}_{12}(z)\bar{\sigma}_1(z)\sigma_F s_1 s_2 \frac{\partial^2}{\partial s_1 \partial s_2} + \bar{\rho}_{13}(z)\bar{\sigma}_1(z)\sigma_V s_1 v \frac{\partial^2}{\partial s_1 \partial v} + \sigma_F \sigma_V \rho_{23} s_2 v \frac{\partial^2}{\partial s_2 \partial v} - r\cdot,
\end{aligned}$$

$$\bar{\sigma}_1(z) = \sqrt{\langle f^2(y,z)\rangle},\ \bar{\rho}_{12}(z) = \rho_{12}\frac{\langle f(y,z)\rangle}{\bar{\sigma}_1(z)},\ \bar{\rho}_{13}(z) = \rho_{13}\frac{\langle f(y,z)\rangle}{\bar{\sigma}_1(z)},$$

and $\langle \cdot \rangle$ denotes the expectation under the invariant distribution of Y. That is, $\langle w \rangle = \frac{1}{2\pi i}\int_{\mathbb{R}} w(z)\exp(-\frac{z-m}{2u^2})\mathrm{d}z$ for any real function w. Also, we note that the inequalities hold: $\bar{\rho}_{12}(z) \leq \rho_{12}$ and $\bar{\rho}_{13}(z) \leq \rho_{13}$ from the Cauchy-Schwartz inequality.

As shown in (12), the leading order term $P_{0,0}(t, s_1, s_2, v, z)$ is a solution of the PDE

$$\langle \mathcal{L}_2 \rangle P_{0,0}(t, s_1, s_2, v, z) = 0$$

with the final condition $P_{0,0}(T, s_1, s_2, v, z) = h(s_1, s_2, v)$. The following theorem represents the solution of the PDE:

Theorem 3.1. *From the PDE* (12) *for* $i = 0$*, the pricing formula for the vulnerable foreign exchange option* $P_{0,0}(t, s_1, s_2, v, z)$ *is given by*

$$\begin{aligned} P_{0,0}(t, s_1, s_2, v, z) &= s_1\Phi_2(a_1, a_2, \theta_3) - s_2\Phi_2(b_1, b_2, \theta_3) \\ &\quad + \frac{1-\alpha}{D} v \left(s_1 e^{(\theta_1+\theta_2)(T-t)}\Phi_2(c_1, c_2, -\theta_3) - s_2 e^{\theta_2(T-t)}\Phi_2(d_1, d_2, -\theta_3) \right), \end{aligned} \tag{13}$$

where

$$\begin{aligned} a_1 &= \frac{1}{\hat{\sigma}\sqrt{T-t}} \ln\frac{s_1}{s_2} + \frac{\hat{\sigma}}{2}\sqrt{T-t}, \\ a_2 &= \frac{1}{\sigma_V\sqrt{T-t}} \ln\frac{v}{D^*} + \left(\frac{\sigma_V}{2} + \frac{\bar{\rho}_{13}\bar{\sigma}_1\sigma_V + r}{\sigma_V}\right)\sqrt{T-t}, \\ b_1 &= \frac{1}{\hat{\sigma}\sqrt{T-t}} \ln\frac{s_1}{s_2} - \frac{\hat{\sigma}}{2}\sqrt{T-t}, \\ b_2 &= \frac{1}{\sigma_V\sqrt{T-t}} \ln\frac{v}{D^*} + \left(\frac{\rho_{23}\sigma_F\sigma_V + r}{\sigma_V} - \frac{\sigma_V}{2}\right)\sqrt{T-t}, \\ c_1 &= \frac{1}{\hat{\sigma}\sqrt{T-t}} \ln\frac{s_1}{s_2} + \left(\frac{\bar{\rho}_{13}\bar{\sigma}_1\sigma_V - \rho_{23}\sigma_F\sigma_V}{\hat{\sigma}} + \frac{\hat{\sigma}}{2}\right)\sqrt{T-t}, \\ c_2 &= -\frac{1}{\sigma_V\sqrt{T-t}} \ln\frac{v}{D^*} - \left(\frac{\sigma_V}{2} + \frac{\bar{\rho}_{13}\bar{\sigma}_1\sigma_V + r}{\sigma_V}\right)\sqrt{T-t}, \\ d_1 &= \frac{1}{\hat{\sigma}\sqrt{T-t}} \ln\frac{s_1}{s_2} + \left(\frac{\bar{\rho}_{13}\bar{\sigma}_1\sigma_V - \rho_{23}\sigma_F\sigma_V}{\hat{\sigma}} - \frac{\hat{\sigma}}{2}\right)\sqrt{T-t}, \\ d_2 &= -\frac{1}{\sigma_V\sqrt{T-t}} \ln\frac{v}{D^*} - \left(\frac{\rho_{23}\sigma_F\sigma_V + r}{\sigma_V} + \frac{\sigma_V}{2}\right)\sqrt{T-t}, \end{aligned}$$

with $\hat{\sigma} = \sqrt{\bar{\sigma}_1^2 + \sigma_F^2 - 2\bar{\rho}_{12}\bar{\sigma}_1\sigma_F}$, $\theta_1 = \bar{\rho}_{13}\bar{\sigma}_1\sigma_V - \rho_{23}\sigma_F\sigma_V$, $\theta_2 = r + \rho_{23}\sigma_F\sigma_V$, *and* $\theta_3 = \frac{\theta_1}{\hat{\sigma}\sigma_V}$. *and* $\Phi_2(n_1, n_2, \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{n_1}\int_{-\infty}^{n_2} e^{-\frac{1}{2(1-\rho^2)}(x^2-2xy\rho+y^2)}\mathrm{d}y\mathrm{d}x$.

Proof. As in (12), the leading order price $P_{0,0}(t, s_1, s_2, v, z)$ satisfies the following parabolic PDE:

$$\begin{aligned} \langle\mathcal{L}_2\rangle P_{0,0}(t, s_1, s_2, v, z) &= 0, \\ P_{0,0}(T, s_1, s_2, v, z) &= h(s_1, s_2, v) \end{aligned}$$

which is a fundamental result of Kim and Koo [24]. They obtained the closed-form solution of the above vulnerable exchange options (PDE) using the Margrabe formula [28] and double Mellin transforms. Thus, we omit the detailed proof. □

Next, we derive explicit formulas for fast time-scale correction and slow time-scale correction terms.

Theorem 3.2. *The fast time scale correction price* $P_{0,1}^{\epsilon} = \sqrt{\epsilon}P_{0,1}$ *is given by*

$$P_{0,1}^{\epsilon} = -(T-t)\mathcal{A}^{\epsilon}P_{0,0}(t, s_1, s_2, v, z) \tag{14}$$

where $\mathcal{A}^\epsilon = V^\epsilon_{2,1}D_1^2 + V^\epsilon_{2,2}D_1D_2 + V^\epsilon_{2,1}D_1D_v + D_1\left[V^\epsilon_{3,1}D_1^2 + V^\epsilon_{3,2}D_1D_2 + V^\epsilon_{3,3}D_1D_v\right] + 2V^\epsilon_{3,1}D_1^2 + V^\epsilon_{3,2}D_1D_2 + V^\epsilon_{3,3}D_1D_v$. *Here, the group parameters in the source term* $\mathcal{A}^\epsilon$ *are defined as*

$$V^\epsilon_{2,1} = -\frac{u\sqrt{\epsilon}}{\sqrt{2}}\left\langle \Lambda_y(y,z)\frac{\partial\psi_1}{\partial y}(y,z)\right\rangle,\ V^\epsilon_{2,2} = -u\sqrt{2\epsilon}\sigma_F\rho_{12}\left\langle \Lambda_y(y,z)\frac{\partial\psi_2}{\partial y}(y,z)\right\rangle,$$
$$V^\epsilon_{2,3} = -u\sqrt{2\epsilon}\sigma_V\rho_{13}\left\langle \Lambda_y(y,z)\frac{\partial\psi_2}{\partial y}(y,z)\right\rangle, V^\epsilon_{3,1} = \frac{u\sqrt{\epsilon}}{\sqrt{2}}\rho_{1y}\left\langle f(y,z)\frac{\partial\psi_1}{\partial y}(y,z)\right\rangle,$$
$$V^\epsilon_{3,2} = u\sqrt{2\epsilon}\sigma_F\rho_{12}\rho_{1y}\left\langle f(y,z)\frac{\partial\psi_2}{\partial y}(y,z)\right\rangle,\ V^\epsilon_{3,3} = u\sqrt{2\epsilon}\sigma_V\rho_{13}\rho_{1y}\left\langle f(y,z)\frac{\partial\psi_2}{\partial y}(y,z)\right\rangle,$$

and $D_1^n = s_1^n\frac{\partial^n}{\partial s_1^n}$, $D_2^n = s_2^n\frac{\partial^n}{\partial s_2^n}$, *and* $D_v^n = v^n\frac{\partial^n}{\partial v^n}$. *In* $\mathcal{A}^\epsilon$, *functions* $\psi_1 = \psi_1(y,z)$ *and* $\psi_2 = \psi_2(y,z)$ *are solutions of the Poisson equations* $\mathcal{L}_0^{-1}\left(f^2 - \langle f^2\rangle\right)$ *and* $\mathcal{L}_0^{-1}\left(f - \langle f\rangle\right)$, *respectively.*

Proof. From the first equation in (10), we have $P_{0,2} = -\mathcal{L}_0^{-1}(\mathcal{L}_2 - \langle\mathcal{L}_2\rangle)P_{0,0}$ and

$$\langle\mathcal{L}_2\rangle P_{0,1} = -\langle\mathcal{L}_1 P_{0,2}\rangle. \tag{15}$$

By substituting the $P_{0,2}$ into (15), we obtain the equation

$$\begin{aligned}
\langle\mathcal{L}_2\rangle P^\epsilon_{0,1} = -u\sqrt{2\epsilon}\Big[&\frac{1}{2}\left\langle \Lambda_y(y,z)\frac{\partial\psi_1}{\partial y}(y,z)\right\rangle s_1^2\frac{\partial^2 P_{0,0}}{\partial s_1^2}\\
&+\sigma_F\rho_{12}\left\langle \Lambda_y(y,z)\frac{\partial\psi_2}{\partial y}(y,z)\right\rangle s_1s_2\frac{\partial^2 P_{0,0}}{\partial s_1\partial s_2} + \sigma_V\rho_{13}\left\langle \Lambda_y(y,z)\frac{\partial\psi_2}{\partial y}(y,z)\right\rangle s_1 v\frac{\partial^2 P_{0,0}}{\partial s_1\partial v}\Big]\\
&+u\sqrt{2\epsilon}\rho_{1y}\Big[\left\langle f(y,z)\frac{\partial\psi_1}{\partial y}(y,z)\right\rangle s_1^2\frac{\partial^2 P_{0,0}}{\partial s_1^2} + \frac{1}{2}\left\langle f(y,z)\frac{\partial\psi_1}{\partial y}(y,z)\right\rangle s_1^3\frac{\partial^3 P_{0,0}}{\partial s_1^3}\\
&+\sigma_F\rho_{12}\left\langle f(y,z)\frac{\partial\psi_2}{\partial y}(y,z)\right\rangle s_1s_2\frac{\partial^2 P_{0,0}}{\partial s_1\partial s_2} + \sigma_F\rho_{12}\left\langle f(y,z)\frac{\partial\psi_2}{\partial y}(y,z)\right\rangle s_1^2s_2\frac{\partial^3 P_{0,0}}{\partial s_1^2\partial s_2}\\
&+\sigma_V\rho_{13}\left\langle f(y,z)\frac{\partial\psi_2}{\partial y}(y,z)\right\rangle s_1 v\frac{\partial^2 P_{0,0}}{\partial s_1\partial v} + \sigma_V\rho_{13}\left\langle f(y,z)\frac{\partial\psi_2}{\partial y}(y,z)\right\rangle s_1^2 v\frac{\partial^3 P_{0,0}}{\partial s_1^2\partial v}\Big]\\
&:= \mathcal{A}^\epsilon P_{0,0}.
\end{aligned}$$

Next, applying the commuting property [22] to the above equation and since $P_{0,0}$ does not depend on y, the following relation can be obtained: $\langle\mathcal{L}_2\rangle\left(-(T-t)\mathcal{A}^\epsilon P_{0,0}\right) = \mathcal{A}^\epsilon P_{0,0} - (T-t)\mathcal{A}^\epsilon\left(\langle\mathcal{L}_2\rangle P_{0,0}\right) = \mathcal{A}^\epsilon P_{0,0}$. Hence, one can find the explicit form of $P^\epsilon_{0,1}$, which is the fast time-scale correction term we wanted. □

Now, to derive the slow correction term $P^\delta_{1,0} = \sqrt{\delta}P_{1,0}$, we consider the second equation of (10). As in [12], by expanding P^ϵ_1 in power of the small parameter $\sqrt{\epsilon}$ from (11), we know that $P_{1,1}$ is independent of the fast mean-reverting diffusion y.

Theorem 3.3. *The fast slow-scale correction price* $P^\delta_{1,0} = \sqrt{\delta}P_{1,0}$ *is given by*

$$P^\delta_{1,0} = \frac{1}{2}(T-t)^2\mathcal{A}^\delta P_{0,0}(t,s_1,s_2,v,z) \tag{16}$$

where $\mathcal{A}^\delta = V^\delta_{0,1}D_1^2 + V^\delta_{0,2}D_1D_2 + V^\delta_{0,3}D_1D_v + D_1\left(V^\delta_{1,1}D_1^2 + V^\delta_{1,2}D_1D_2 + V^\delta_{1,3}D_1D_v\right)$. *Here, the group parameters in a source term* $\mathcal{A}^\delta$ *are defined as*

$$\begin{aligned}
V^\delta_{0,1} &= -\sqrt{\delta}g(z)\langle\Lambda_z(y,z)\rangle\frac{\partial\bar{\sigma}_1}{\partial z}\bar{\sigma}_1,\\
V^\delta_{0,2} &= -\sqrt{\delta}g(z)\langle\Lambda_z(y,z)\rangle\frac{\partial\bar{\sigma}_1}{\partial z}\bar{\rho}_{12}\sigma_F + \frac{\partial\bar{\rho}_{12}}{\partial z}\bar{\sigma}_1\sigma_F,\\
V^\delta_{0,3} &= -\sqrt{\delta}g(z)\langle\Lambda_z(y,z)\rangle\frac{\partial\bar{\sigma}_1}{\partial z}\bar{\rho}_{13}\sigma_V + \frac{\partial\bar{\rho}_{13}}{\partial z}\bar{\sigma}_1\sigma_V,\\
V^\delta_{1,1} &= \sqrt{\delta}\rho_{1z}g(z)\langle f(y,z)\rangle\frac{\partial\bar{\sigma}_1}{\partial z}\bar{\sigma}_1,\\
V^\delta_{1,2} &= \sqrt{\delta}\rho_{1z}g(z)\langle f(y,z)\rangle\frac{\partial\bar{\sigma}_1}{\partial z}\bar{\rho}_{12}\sigma_F + \frac{\partial\bar{\rho}_{12}}{\partial z}\bar{\sigma}_1\sigma_F,\\
V^\delta_{1,3} &= \sqrt{\delta}\rho_{1z}g(z)\langle f(y,z)\rangle\frac{\partial\bar{\sigma}_1}{\partial z}\bar{\rho}_{13}\sigma_V + \frac{\partial\bar{\rho}_{13}}{\partial z}\bar{\sigma}_1\sigma_V.
\end{aligned}$$

Proof. From the second equation of the relations (10), we have $\langle\mathcal{L}_2\rangle P^\delta_{1,0} = -\sqrt{\delta}\langle\mathcal{M}_1\rangle P_{0,0}$. To compute the right-hand side $\sqrt{\delta}\langle\mathcal{M}_1\rangle P_{0,0}$, we consider the following three terms: $\frac{\partial P_{0,0}}{\partial\bar{\sigma}_1}$, $\frac{\partial P_{0,0}}{\partial\bar{\rho}_{12}}$, and $\frac{\partial P_{0,0}}{\partial\bar{\rho}_{13}}$. These can be obtained by differentiating both sides of the equation $\langle\mathcal{L}_2\rangle P_{0,0} = 0$ with respect to $\bar{\sigma}_1$, $\bar{\rho}_{12}$, and $\bar{\rho}_{13}$, respectively. Next, by using these Greeks, $\sqrt{\delta}\langle\mathcal{M}_1\rangle P_{0,0}$ becomes

$$\begin{aligned}
&\sqrt{\delta}\langle\mathcal{M}_1\rangle P_{0,0}\\
=&\sqrt{\delta}\langle -g(z)\Lambda_z(y,z)\frac{\partial}{\partial z} + g(z)\rho_{1z}f(y,z)s_1\frac{\partial^2}{\partial s_1\partial z}\rangle P_{0,0}\\
=&\sqrt{\delta}\left(-g(z)\langle\Lambda_z(y,z)\rangle + g(z)\rho_{1z}\langle f(y,z)\rangle s_1\frac{\partial}{\partial s_1}\right)\frac{\partial P_{0,0}}{\partial z},\\
=&\sqrt{\delta}\left(-g(z)\langle\Lambda_z(y,z)\rangle + g(z)\rho_{1z}\langle f(y,z)\rangle s_1\frac{\partial}{\partial s_1}\right)\left(\frac{\partial\bar{\sigma}_1}{\partial z}\frac{\partial P_{0,0}}{\partial\bar{\sigma}_1} + \frac{\partial\bar{\rho}_{12}}{\partial z}\frac{\partial P_{0,0}}{\partial\bar{\rho}_{12}} + \frac{\partial\bar{\rho}_{13}}{\partial z}\frac{\partial P_{0,0}}{\partial\bar{\rho}_{13}}\right).\\
=&\sqrt{\delta}(T-t)\left(-g(z)\langle\Lambda_z(y,z)\rangle + g(z)\rho_{1z}\langle f(y,z)\rangle s_1\frac{\partial}{\partial s_1}\right)\\
&\times\left[\frac{\partial\bar{\sigma}_1}{\partial z}\left(\bar{\sigma}_1 s_1^2\frac{\partial^2 P_{0,0}}{\partial s_1^2} + \bar{\rho}_{12}\sigma_F s_1 s_2\frac{\partial^2 P_{0,0}}{\partial s_1\partial s_2} + \bar{\rho}_{13}\sigma_V s_1 v\frac{\partial^2 P_{0,0}}{\partial s_1\partial v}\right)\right.\\
&\qquad\left. + \frac{\partial\bar{\rho}_{12}}{\partial z}\bar{\sigma}_1\sigma_F s_1 s_2\frac{\partial^2 P_{0,0}}{\partial s_1\partial s_2} + \frac{\partial\bar{\rho}_{13}}{\partial z}\bar{\sigma}_1\sigma_V s_1 v\frac{\partial^2 P_{0,0}}{\partial s_1\partial v}\right].
\end{aligned}$$

Now, letting $\sqrt{\delta}\langle\mathcal{M}_1\rangle = (T-t)\mathcal{A}^\delta$ and using the commuting property,

$$\begin{aligned}
\langle\mathcal{L}_2\rangle P^\delta_{1,0} =&\langle\mathcal{L}_2\rangle\left(\frac{1}{2}(T-t)^2\mathcal{A}^\delta P_{0,0}\right)\\
=&-(T-t)\mathcal{A}^\delta P_{0,0} + \frac{1}{2}(T-t)^2\langle\mathcal{L}_2\rangle\mathcal{A}^\delta P_{0,0}\\
=&-(T-t)\mathcal{A}^\delta P_{0,0} + \frac{1}{2}(T-t)^2\mathcal{A}^\delta\langle\mathcal{L}_2\rangle P_{0,0}.\\
=&-(T-t)\mathcal{A}^\delta P_{0,0} = -\sqrt{\delta}\langle\mathcal{M}_1\rangle P_{0,0}.
\end{aligned}$$

Thus, the slow scale correction price $P^\delta_{1,0}$ is given by $P^\delta_{1,0} = \frac{1}{2}(T-t)^2\mathcal{A}^\delta P_{0,0}$.

□

In Section 3, we obtain the leading-order price $P_{0,0}$ (Theorem 3.1.), the fast time scale correction term $P^{\epsilon}_{0,1}$ (Theorem 3.2.), and the slow time-scale correction term $P^{\delta}_{1,0}$ (Theorem 3.3.). We refer to $P^{\epsilon}_{0,1} + P^{\delta}_{1,0}$ a **correction term** and $\widetilde{P}^{\epsilon,\delta} := P_{0,0} + P^{\epsilon}_{0,1} + P^{\delta}_{1,0}$ a **corrected price**. Now, we present the difference between the true price $P^{\epsilon,\delta}$ and the corrected price $\widetilde{P}^{\epsilon,\delta}$ is less than a constant, depending on the parameters ϵ and δ as follows:

Theorem 3.4. *For any* $0 < \epsilon, \delta \leq 1$*, the accuracy of the price* $P^{\epsilon,\delta}$ *and the approximation price* $\widetilde{P}^{\epsilon,\delta}$ *is*

$$\left|P^{\epsilon,\delta}(t, s_1, s_2, v, y, z) - \widetilde{P}^{\epsilon,\delta}(t, s_1, s_2, v, z)\right| = \mathcal{O}(\epsilon^{5/4-} + \epsilon|\log\epsilon|\sqrt{\delta} + \delta\sqrt{\epsilon} + \delta^{3/2}) \quad (17)$$

where $\widetilde{P}^{\epsilon,\delta}(t, s_1, s_2, v, z) = P_{0,0} + P^{\epsilon}_{0,1} + P^{\delta}_{1,0}$*, and the notation* $\mathcal{O}(\epsilon^{5/4-}) = \mathcal{O}(\epsilon^{1+q/4})$ *for any* $q < 1$.

Proof. This proof is similar to that of Theorem 2.5 in Fouque and Lorig [13]. However, as shown in (6), since the payoff function h is not continuously differentiable at $s_1 = s_2$, we utilize the payoff smoothing method described in [12]. Also, it is easily seen that $h \leq (S_1(T) - S_2(T))^+$. Then, the following inequality is satisfied:

$$(S(T) - 1)^+ \left(\mathbf{1}_{\{V(T)>D^*\}} + \frac{1-\alpha}{D} V(T)\mathbf{1}_{\{V(T)\leq D^*\}}\right) \leq (S(T) - 1)^+ \quad (18)$$

where the right hand side of (18) is a payoff function of the European call option with $K = 1$. Thus, we can apply Theorem 2.5 in [13] to (18) and then, the accuracy of the first-order approximation in (17) is established. □

4. Numerical Experiments and Implications

In this section, we investigate the price behavior of the vulnerable exchange option under a multi-scale stochastic volatility (MSV) model with respect to given model parameters. In particular, for the numerical experiments, we focus on the corrected option price $\widetilde{P}^{\epsilon,\delta}(= P_{0,0} + P^{\epsilon}_{0,1} + P^{\delta}_{1,0})$ mentioned in Section 3.

First, to verify the accuracy of the corrected price $\widetilde{P}^{\epsilon,\delta}$ numerically, we calculate the price of the exchange options with default risk under the MSV model (1)–(4) using the Monte Carlo method. Considering that volatility follows a fast mean-reverting Ornstein-Ulenbeck process as shown in Fouque et al. [9], we can denote the volatility as an exponent function. Then, the stochastic dynamics for assets S_1, S_2, the market value of option writer V, and mean-reverting processes

Y and Z given in (1)–(4) can be expressed by

$$
\begin{aligned}
\mathrm{d}S_1(t) &= rS_1(t)\,\mathrm{d}t + \underbrace{e^{Y(t)+Z(t)}}_{=\exp\mathrm{OU}} S_1(t)\,\mathrm{d}W_1^*(t), \\
\mathrm{d}S_2(t) &= rS_2(t)\,\mathrm{d}t + \sigma_F S_2(t)\,\mathrm{d}W_2^*(t), \\
\mathrm{d}V(t) &= rV(t)\,\mathrm{d}t + \sigma_V V(t)\,\mathrm{d}W_3^*(t), \\
\mathrm{d}Y(t) &= \frac{1}{\epsilon}(m_y - Y(t))\mathrm{d}t + \frac{u_y\sqrt{2}}{\sqrt{\epsilon}}\mathrm{d}W_y^*(t), \\
\mathrm{d}Z(t) &= \delta(m_z - Z(t))\mathrm{d}t + \sqrt{2\delta}u_z\mathrm{d}W_z^*(t),
\end{aligned}
\tag{19}
$$

respectively. In Table 1, we present the price difference between the Monte Carlo price P_{MC} and the corrected price $\widetilde{P}^{\epsilon,\delta}$ with respect to the fast mean reversion rate ϵ and the slow-mean reversion rate δ mentioned in Section 2, with the relationship between ϵ and δ satisfying $0 < \epsilon < \delta < \sqrt{\epsilon} \ll 1$. The parameters selected in the Monte Carlo simulation are $\alpha = 0.3$, $D = 100 = D^*$, $r = 0.01$, $\rho_{12} = 0.196 = \rho_{13}$, $\rho_{1y} = -0.5 = \rho_{1z}$, $\rho_{23} = 0.2$, $s_1 = 1$, $s_2 = 0.4$, $\sigma_F = 0.2 = \sigma_V$, $T - t = 0.5$, $m_y = 0.2$, $m_z = -2$, $u_y = 0.2 = u_z$, $y = 0$, $v = 120$ and $z = -2$, and the number of the sample paths are $10,000$.

ϵ	δ	P_{MC}	$\widetilde{P}^{\epsilon,\delta}$	$\lvert P_{\mathrm{MC}} - \widetilde{P}^{\epsilon,\delta}\rvert$	RE (%)
10^{-3}	10^{-2}	0.607 262	0.602 361	0.004 901	0.807 065
10^{-4}	10^{-3}	0.606 042	0.602 284	0.003 758	0.620 089
10^{-5}	10^{-4}	0.602 956	0.602 259	0.000 697	0.115 597
10^{-6}	10^{-5}	0.602 032	0.602 251	0.000 219	0.036 377

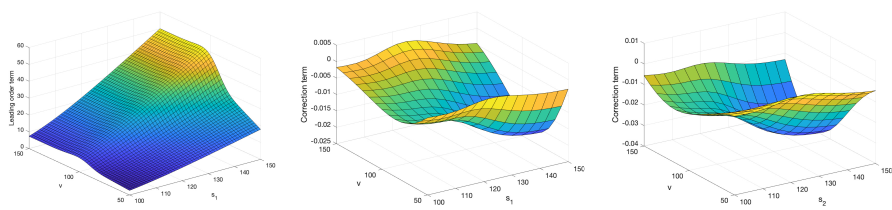
Table 1. The error comparison between the Monte-Carlo price for (19) (denoted by P_{MC}) and the corrected formula ($\widetilde{P}^{\epsilon,\delta} = \widetilde{P}^{\epsilon,\delta}(t, s_1, s_2, v, z)$) with respect to (ϵ, δ) pairs which comes from Choi et al. [5]. All the computations were performed using 1.6GHz Intel Core i5 CPU and 8GB memory. The parameters we choose are: $\alpha = 0.3$, $D = 100 = D^*$, $r = 0.01$, $\rho_{12} = 0.196 = \rho_{13}$, $\rho_{1y} = -0.5 = \rho_{1z}$, $\rho_{23} = 0.2$, $\rho_{2z} = 0$, $s_1 = 1$, $s_2 = 0.4$, $\sigma_F = 0.2 = \sigma_V$, $T - t = 0.5$, $m_y = 0.2$, $m_z = -2$, $u_y = 0.2 = u_z$, $y = 0$, $v = 120$ and $z = -2$.

As shown in Table 1, if the parameters ϵ and δ go to zero, then the price gap between $\widetilde{P}^{\epsilon,\delta}$ and P_{MC} and the corresponding relative error converge to zero quickly. Hence, one can observe that the option pricing formula is accurately derived if both the mean-reverting parameters are sufficiently small. Additionally, we carry out a Monte Carlo simulation under a fixed $\epsilon = 10^{-6}$ and $\delta = 10^{-5}$ to investigate the price difference and the corresponding relative error with regard to the number of simulations. As shown in Table 2, if the number of simulations increases, then the price difference $|P_{\mathrm{MC}} - \widetilde{P}^{\epsilon,\delta}|$ and the corresponding relative error are closer to zero. Therefore, from both Tables, we observe that the corrected price $\widetilde{P}^{\epsilon,\delta}$ presented in Section 3 becomes an accurate solution for vulnerable exchange options with the MSV.

# of simulations	P_{MC}	$\widetilde{P}^{\epsilon=10^{-6},\,\delta=10^{-5}}$	$\lvert P_{\mathrm{MC}} - \widetilde{P}^{\epsilon,\delta}\rvert$	RE (%)
10,000	0.602 032	0.602 251	0.000 219	0.036 377
20,000	0.602 369	0.602 251	0.000 118	0.019 512
30,000	0.602 191	0.602 251	0.000 060	0.009 954
50,000	0.602 228	0.602 251	0.000 023	0.003 873
70,000	0.602 230	0.602 251	0.000 021	0.003 518

TABLE 2. The error comparison between the Monte-Carlo price for (19) (denoted by P_{MC}) and the corrected formula ($\widetilde{P}^{\epsilon,\delta} = \widetilde{P}^{\epsilon,\delta}(t, s_1, s_2, v, z)$) with respect to the number of paths and the fixed $\epsilon (= 10^{-6})$ and $\delta (= 10^{-5})$. All the computations were performed using 1.6GHz Intel Core i5 CPU and 8GB memory. Adopted parameters are: $\alpha = 0.3$, $D = 100 = D^*$, $r = 0.01$, $\rho_{12} = 0.196 = \rho_{13}$, $\rho_{1y} = -0.5 = \rho_{1z}$, $\rho_{23} = 0.2$, $\rho_{2z} = 0$, $s_1 = 1$, $s_2 = 0.4$, $\sigma_F = 0.2 = \sigma_V$, $T - t = 0.5$, $m_y = 0.2$, $m_z = -2$, $u_y = 0.2 = u_z$, $y = 0$, $v = 120$, $\epsilon = 10^{-6}$, $\delta = 10^{-5}$ and $z = -2$.

Figure 1(a) plots the price changes of the Black-Scholes price $P_{0,0}$ with respect to the asset S_1 and the market price of option writer V. Also, Figures 1(b) and 1(c) show the change in the correction term $P^{\epsilon}_{0,1} + P^{\delta}_{1,0}$ with respect to the domestic currency S_1, foreign currency S_2, and the price of the option's issuer, V. As shown in Figures 1(b) and 1(c), the values of the correction term tend to be negative, exhibiting a hump phenomenon as the market value of the option writer is near the critical default level $D = D^* = 100$. This implies that the price of the vulnerable exchange options under stochastic volatility tends to be undervalued compared to the Black-Scholes price $P_{0,0}$ described by (13), and the pricing impact of the correction term on the option prices is most sensitive when the market value of the option issuer gets closer to the critical default value. Additionally, one can observe that the more S_1 or S_2 increases, the greater the pricing sensitivity of the correction term. This means that as the price of the domestic currency or the value of the foreign currency increases, the stochastic volatility has a more significant influence on the option price.



(a) $s_2 = 100$ (b) $s_2 = 100$ (c) $s_1 = 150$

FIGURE 1. Surfaces of leading order term $P_{0,0}$ and correction term $P^{\epsilon}_{0,1} + P^{\delta}_{1,0}$. Selected parameters: $\alpha = 0.3$, $D = 100 = D^*$, $r = 0.01$, $\rho_{12} = 0.196 = \rho_{13}$, $\rho_{1y} = -0.5 = \rho_{1z}$, $\rho_{23} = 0.2$, $\rho_{2z} = 0$, $\sigma_F = 0.1 = \sigma_V$, $T - t = 1$, $m_y = 0.01$, $m_z = -2$, $u_y = \sqrt{2} = u_z$, $y = 0$, $v = 120$, $\epsilon = 10^{-6}$, $\delta = 10^{-5}$ and $z = -2$.

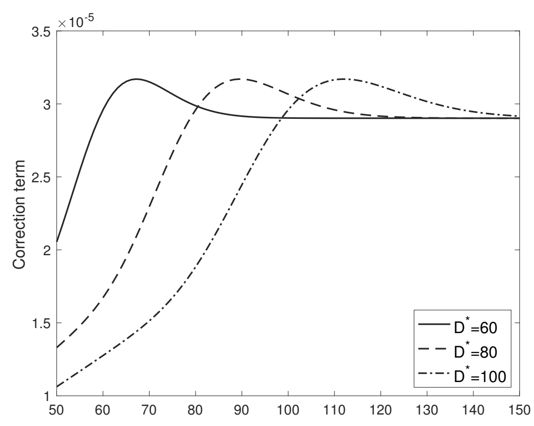
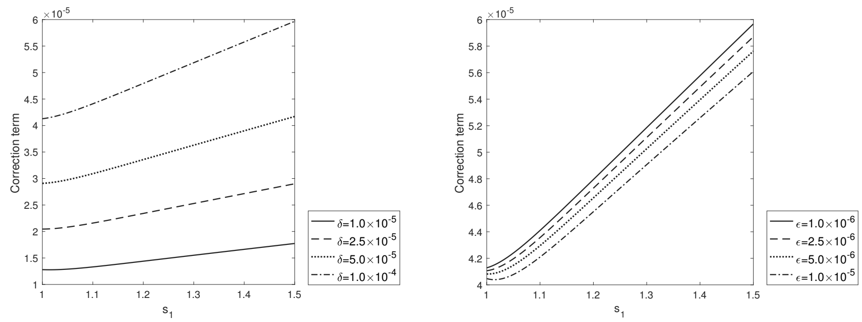


FIGURE 2. The impact of the critical level D^* on the correction term. Fixed parameters: $\alpha = 0.3$, $r = 0.01$, $\rho_{12} = 0.196 = \rho_{13}$, $\rho_{1y} = -0.5 = \rho_{1z}$, $\rho_{23} = 0.2$, $\rho_{2z} = 0$, $s_1 = 1$, $s_2 = 1$, $\sigma_F = 0.2 = \sigma_V$, $T - t = 0.5$, $m_y = 0.2$, $m_z = -2$, $u_y = 0.2 = u_z$, $y = 0$, $\epsilon = 10^{-6}$, $\delta = 10^{-5}$ and $z = -2$.

Figure 2 shows the price behavior of the correction term $P^{\epsilon}_{0,1} + P^{\delta}_{1,0}$ with respect to the market value V. For each D^*, we note that the value of the correction term increases dramatically against the market value V, but it gradually decreases again as V reaches the critical default level D^*, which ultimately converges to a certain constant level. Thus, we can observe that the correction term has a hump curve around the value of D^*, regardless of the market value of the option writer. Also, as V decreases, the price impact of the correction term becomes larger, but when V is larger, the price influence of the correction term is less significant. It can be observed that the more likely it is to have default risk, the more sensitive the effect of stochastic volatility on the option price.



(a) $\epsilon = 10^{-6}$ (b) $\delta = 10^{-4}$

Figure 3. The impact of the fast-mean-reverting factor ϵ and slow-reverting factor δ on the correction term. Selected parameters: $\alpha = 0.3$, $r = 0.01$, $\rho_{12} = 0.196 = \rho_{13}$, $\rho_{1y} = -0.5 = \rho_{1z}$, $\rho_{23} = 0.2$, $\rho_{2z} = 0$, $s_2 = 0.7$, $\sigma_F = 0.2 = \sigma_V$, $T - t = 0.5$, $m_y = 0.2$, $m_z = -2$, $u_y = 0.5 = u_z$, $y = 0$, $\epsilon = 10^{-6}$, $\delta = 10^{-5}$, $v = 100$, $D = 100 = D^*$, and $z = -2$.

Figure 3 displays the structure of the option price correction term depending on parameters ϵ and δ corresponding to the term of the fast factor and the term of the slow factor, respectively. We find that the value of the correction term is more sensitive to the small parameter δ than the small parameter ϵ. This implies that the slow scale correction has a more significant impact than the fast scale correction on the option price in terms of the underlying asset value.

5. Conclusion

This study is based on the model dynamics of the underlying assets (domestic currency, foreign currency, and market price of the option's issuer), and is driven by a perturbative form of multi-scale stochastic volatility with a fast mean reverting process and a slow varying process. The study approximates closed-form solutions, which consist of one leading order price and two correction order prices, derived for the vulnerable exchange option prices. By numerical computation and a comparison of the approximated option pricing formula, we observe that the corrections of the stochastic volatility have a significant effect on the option value, showing the hump shapes and the highest sensitivity, as the market value of the option writer is close to the critical default level. Furthermore, in the multi-scale stochastic volatility model, we observe that: a slow scale mean-reverting factor on stochastic volatility term has a more significant influence on the option price than a fast scale mean-reverting factor. In other words, the role of the slow factor of the multi-scale stochastic volatility is very crucial for option pricing along with the cases of stochastic volatility of Chen and Zhu [4] and Yoon [31], and the hybrid stochastic and the local volatility model of Kim et al. [25]. Finally, our study focuses on the necessity of studying a multi-scale SV model for the pricing of the option.

Conflicts of interest : The authors declare no conflict of interest.

Data availability : Not applicable

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