J. Korean Soc. Ind. Appl. Math. Vol.27, No.4, 296–310, 2023 https://doi.org/10.12941/jksiam.2023.27.296

Γ-CONVERGENCE FOR AN OPTIMAL DESIGN PROBLEM WITH VARIABLE EXPONENT

HAMDI ZORGATI[1,2]

[1]DEPARTMENT OF MATHEMATICS AND STATISTICS, COLLEGE OF SCIENCES, IMAM MOHAMMAD IBN SAUD ISLAMIC UNIVERSITY (IMSIU), 90950, RIYADH 11623 SAUDI ARABIA

[2]DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TUNIS EL MANAR, FACULTY OF SCIENCES OF TUNIS, TUNISIA.

Email address: hzorgati@imamu.edu.sa

ABSTRACT. In this paper, we derive the Γ-limit of functionals pertaining to some optimal material distribution problems that involve a variable exponent, as the exponent goes to infinity. In addition, we prove a relaxation result for supremal optimal design functionals with respect to the weak-$*$ $L^\infty(\Omega;[0,1]) \times W^{1,p_0}(\Omega;\mathbb{R}^m)$ weak topology.

1. INTRODUCTION

In this paper, we study the asymptotic behavior of a sequence of functionals related to optimal design problems and defined on Sobolev spaces with variable exponent in space, as the exponent goes to infinity. It is a sequel of a previous work dealing with the constant exponent case [1].

Variable exponent spaces are connected with variational integrals verifying non standard growth and coercivity conditions [2]. Such functionals are used in the modeling of electrorheological fluids [3, 4, 5] and thermorheological fluids [6] and in image processing [7, 8]. The energy functionals considered correspond to a two-phase mixture of different properties, such as stiffness or electric resistivity, in different regions of the domain under consideration. The optimal design models, describe the optimal distribution of such a mixture with respect to some specific criterion. Minimizing such energies makes it possible to improve the mechanical or electrical performance by optimizing the distribution of these properties.

In order to perform our analysis, we use Γ-convergence techniques (see section 2). These techniques were already used in the study of optimal design models in the works [9, 10] in the context of a dimension reduction process for thin films.

In this work, we consider a sequence of optimal design models described by functionals of the form

$$J(\chi, u) = ||\chi W_1(\nabla u) + (1-\chi) W_2(\nabla u)||_{\mathbf{p}_n(.)},$$

Received June 27 2023; Revised November 2 2023; Published online December 25 2023.
2000 *Mathematics Subject Classification.* 74B20, 35E99, 35M10, 49J45.
Key words and phrases. Optimal design, Variable exponent, Γ-convergence, Relaxation, Variational methods.

where $\chi(x) \in \{0,1\}$ denotes the characteristic function of the first phase, ∇u the gradient of u and W_i, $i = 1, 2$ models the energy density of the ith phase. The norm $||.||_{\mathbf{p}_n(\cdot)}$ is the Luxembourg norm associated with the Lebesgue spaces with variable exponents $L^{\mathbf{p}_n(\cdot)}(\Omega; \mathbb{R}^m)$ (see Section 2.2), defined by

$$L^{\mathbf{p}_n(\cdot)}(\Omega; \mathbb{R}^m) = \{u : \Omega \longrightarrow \mathbb{R}^m, \text{ measurable, such that } \int_\Omega |u(x)|^{\mathbf{p}_n(x)} dx < \infty\}.$$

Then, we proceed with an asymptotic analysis when the exponent $\mathbf{p}_n(.)$ of the energy density goes to infinity in a sense specified below.

We obtain a limit energy of supremal kind that can model, for example, dielectric breakdown for double phase composites (see [11, 12] and the references therein) or some simplified models of polycrystal plasticity (see [13]). In the last two references, analogous asymptotic analyses using Γ-convergence techniques for functionals involving a single phase elastic density can be found. See also [14, 15], where the authors obtain limit models under some differential constraints, involving supremal functions and $\mathcal{A}$-quasiconvex envelopes. We also mention [16] where the authors obtain an L^p approximation and a lower semicontinuity result for supremal functionals.

In [17], the authors present an analogous analysis as in [14], generalizing to the variable exponent case. We mention that in these works, the authors make use of the technique of Young measures which we do not use in our analysis.

Let $p_0 > 1$. Suppose that Ω is a regular domain in $\mathbb{R}^N$ with $|\Omega| < +\infty$. Let $(\mathbf{p}_n) = (\mathbf{p}_n(x))$ be a sequence of Lipschitz, positive, continuous functions defined on Ω and satisfying

$$\mathbf{p}_n^- \to +\infty \text{ and } \lim_{n\to+\infty} \frac{\mathbf{p}_n^+}{\mathbf{p}_n^-} = 1,$$

where $\mathbf{p}_n^- = \inf \mathbf{p}_n$ and $\mathbf{p}_n^+ = \sup \mathbf{p}_n < \infty$. Notice that the last hypothesis imply that

$$\mathbf{p}_n^+ \leq \beta \mathbf{p}_n^-,$$

for some $\beta > 1$.

Consider the sequence of functionals I_n defined on $L^\infty(\Omega; [0,1]) \times L^{p_0}(\Omega; \mathbb{R}^m)$ by

$$I_n(\chi, u) = \begin{cases} \|\chi W_1(\nabla u) + (1-\chi) W_2(\nabla u)\|_{\mathbf{p}_n(.)} \text{ if } \begin{cases} \chi \in L^\infty(\Omega; \{0,1\}), \\ u \in W^{1,\mathbf{p}_n(\cdot)}(\Omega; \mathbb{R}^m), \end{cases} \\ +\infty \quad \text{otherwise,} \end{cases}$$

where W_i are continuous functions verifying that there exist $\alpha_i > 0$ and $\gamma_i > 0$ $i = 1, 2$, such that

$$W_i(A) \geq \alpha_i |A|^{\gamma_i}.$$

The functional I_n represents, for example, the elastic energy of a solid occupying the domain Ω and undergoing the deformation u, while χ represents the characteristic function of the first phase of stiffness.

Let $V : [0,1] \times \mathbb{M}^{m\times N} \longrightarrow \mathbb{R}$ defined by

$$V(\kappa, A) = \kappa W_1(A) + (1-\kappa)W_2(A). \tag{1.1}$$

Let I defined on $L^\infty(\Omega;[0,1]) \times L^{p_0}(\Omega;\mathbb{R}^m)$ by

$$I(\chi,u) = \begin{cases} ||V^\star(\chi,\nabla u)||_\infty & \text{if } u \in W^{1,\infty}(\Omega;\mathbb{R}^m), \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$V^\star(\kappa,A) := \lim_{p\to+\infty} \inf_{\theta,\varphi} \Big\{ \Big(\int_Q \big(V(\theta(x), A+\nabla\varphi(x))\big)^p dx \Big)^{\frac{1}{p}},$$
$$\varphi \in W^{1,p}_{\#}(Q;\mathbb{R}^m),\ \theta \in L^\infty(\Omega;\{0,1\}),\ \int_\Omega \theta(x)dx = \kappa \Big\}.$$

Our main result is the following Theorem.

Theorem 1.1. *The sequence of functionals I_n Γ-converges with respect to the $L^\infty(\Omega;[0,1])$ weak-$*$ $\times$ $W^{1,p_0}(\Omega;\mathbb{R}^m)$ weak topology to I as n goes to ∞.*

In the next section we present some brief preliminaries on the notions of Γ-convergence, Lebesgue-Sobolev spaces with variable exponent and cross quasiconvexity. The following section contains a relaxation result, Proposition 3.1, which is a consequence of the Γ-limit result obtained in [1]. Its proof is based on Theorem 6.1 where we recover the same result as in [1], by writing the limit functional in a different form, see the Appendix section. The proof of Theorem 1.1 is given in section 4. Section 5 is then devoted to some auxiliary results.

2. Preliminaries

2.1. Γ **convergence.** Let $(G_n)_n$ be a sequence of functionals defined on a topological space X with values in $\mathbb{R}\cup\{+\infty\}$. The Γ-lower limit and Γ-upper limit of $(G_n)_n$ are given by

$$\Gamma - \liminf G_n(x) := \sup_{U\in\mathcal{N}(x)} \liminf_{n\to\infty} \inf_{y\in U} G_n(y)$$

and

$$\Gamma - \limsup G_n(x) := \sup_{U\in\mathcal{N}(x)} \limsup_{n\to\infty} \inf_{y\in U} G_n(y),$$

where $\mathcal{N}(x)$ denotes the set of all neighborhoods of x in X. If there exist $G : X \to \mathbb{R}\cup\{+\infty\}$ such that $\Gamma - \liminf G_n = \Gamma - \limsup G_n = G$, then we say that $(G_n)_n$ Γ-converges to G and we write $G := \Gamma - \lim G_n$. When X is first countable we have the equivalent definition in terms of sequences, that is, $(G_n)_n$ is said to Γ-converge to the limit functional G with respect to the topology of X if and only if the following two conditions are satisfied for every $x \in X$:

$$\begin{cases} \forall\, x_n \to x, \liminf\limits_{n\to\infty} G_n(x_n) \geq G(x), \\ \exists\, x_n \to x, \limsup\limits_{n\to\infty} G_n(x_n) \leq G(x). \end{cases}$$

The main properties of Γ-convergence are first that, up to a subsequence, the Γ-limit always exists and second that if a sequence of almost minimizers stays in a compact subset of X, then the limits of any converging subsequence minimize the Γ-limit. In particular we have that, if G is the Γ-limit of G_n and for every n, x_n is a minimizer of G_n with $x_n \to x$ in X, then x is a minimizer of G. Also, when the limit functional verifies some coercivity properties, the limit minimization problem has always a solution due to the lower semicontinuity of the Γ-limit with respect to the considered topology (see [18, 19]).

2.2. **Lebesgue-Sobolev spaces with variable exponent.** We recall the following properties of Lebesgue and Sobolev spaces with variable exponent (see [20]). Let $\mathbf{p} : \Omega \longrightarrow [1, \infty]$ be a measurable function with $\mathbf{p}^+ :=$ ess sup $\mathbf{p}(x)$ and $\mathbf{p}^- :=$ ess inf $\mathbf{p}(x)$. We define the Lebesgue space with variable exponent $L^{\mathbf{p}(\cdot)}(\Omega; \mathbb{R}^m)$ by

$$L^{\mathbf{p}(\cdot)}(\Omega; \mathbb{R}^m) = \{u : \Omega \longrightarrow \mathbb{R}^m, \text{ measurable, such that } \int_\Omega |u(x)|^{\mathbf{p}(x)} dx < \infty\}.$$

Endowed with the Luxembourg norm introduced by I. Sharapudinov in [21]

$$||u||_{\mathbf{p}(.)} = \inf\{\lambda > 0; \int_\Omega |\frac{u(x)}{\lambda}|^{\mathbf{p}(x)} dx \leq 1\},$$

it is a Banach space and the $||.||_{\mathbf{p}(.)}$ norm is lower semicontinuous with respect to almost everywhere convergence.

If we suppose $\mathbf{p}^+ < \infty$, then it is also a separable space and $C_0^\infty(\Omega; \mathbb{R}^m)$ is dense in $L^{\mathbf{p}(\cdot)}(\Omega; \mathbb{R}^m)$.

If in addition we suppose that $\mathbf{p}^- > 1$, then it is a uniformly convex and reflexive space.

Similarly, we define the Sobolev space $W^{1,\mathbf{p}(\cdot)}(\Omega; \mathbb{R}^m)$ by

$$W^{1,\mathbf{p}(\cdot)}(\Omega; \mathbb{R}^m) = \{u \in L^{\mathbf{p}(\cdot)}(\Omega; \mathbb{R}^m) \text{ such that } \nabla u \in L^{\mathbf{p}(\cdot)}(\Omega; \mathbb{M}^{m\times N})\},$$

where ∇u denotes the distributional gradient of u. It can be endowed with the norm

$$||u||_{1,\mathbf{p}(.)} = ||u||_{\mathbf{p}(.)} + ||\nabla u||_{\mathbf{p}(.)},$$

that makes it a Banach space, which is separable when $\mathbf{p}$ is bounded and uniformly convex, thus reflexive, when $1 < \mathbf{p}^-$. Similarly to the result stating that if $|\Omega| < \infty$ we have

$$\lim_{p\to\infty} ||u||_p = ||u||_\infty,$$

we have also that, if $|\Omega| < \infty$, $u \in L^\infty(\Omega; \mathbb{R}^m)$ and $\mathbf{p}_n$ is a sequence of Lipschitz continuous functions verifying

$$\mathbf{p}_n^- \to \infty \text{ and there exist } \beta > 0 \text{ such that } \mathbf{p}_n^+ < \beta \mathbf{p}_n^-,$$

then,

$$\lim_{n\to\infty} ||u||_{\mathbf{p}_n(.)} = ||u||_\infty.$$

Let $\rho_{\mathbf{p}(.)} : L^{\mathbf{p}(\cdot)}((\Omega;\mathbb{R}^m)) \to \mathbb{R}$ be the modular of $L^{\mathbf{p}(\cdot)}((\Omega;\mathbb{R}^m))$ defined by

$$\rho_{\mathbf{p}(.)}(u) = \int_\Omega |u(x)|^{\mathbf{p}(x)}dx.$$

This modular is (sequentially) lower semicontinuous with respect to weak convergence in $L^{\mathbf{p}(\cdot)}(\Omega;\mathbb{R}^m)$ and almost everywhere convergence. It verifies the unit ball property with the $||.||_{\mathbf{p}(.)}$ norm, more precisely, for every $u \in L^{\mathbf{p}(\cdot)}(\Omega;\mathbb{R}^m)$, we have

$$\text{if } ||u||_{\mathbf{p}(.)} \leq 1 \text{ then } \rho_{\mathbf{p}(.)}(u) \leq ||u||_{\mathbf{p}(.)}$$

and

$$\text{if } ||u||_{\mathbf{p}(.)} > 1 \text{ then } \rho_{\mathbf{p}(.)}(u) \geq ||u||_{\mathbf{p}(.)}.$$

Moreover, we have that for every $u \in L^{\mathbf{p}(\cdot)}((\Omega;\mathbb{R}^m))$

$$||u||_{\mathbf{p}(.)} \leq \max\{(\rho_{\mathbf{p}(.)}(u))^{\frac{1}{\mathbf{p}^-}}, (\rho_{\mathbf{p}(.)}(u))^{\frac{1}{\mathbf{p}^+}}\}. \tag{2.1}$$

Finally, when $\mathbf{p}$ is bounded and there exist $\beta > 0$ such that $\mathbf{p}^+ \leq \beta\mathbf{p}^-$, we have (see Lemma 3.2 in [17]), that for every $1 \leq q \leq \mathbf{p}^-$ and $u \in L^{\mathbf{p}(\cdot)}((\Omega;\mathbb{R}^m))$,

$$||u||_q \leq \max\{|\Omega|^{\frac{1}{q}-\frac{1}{\mathbf{p}^-}}, |\Omega|^{\beta(\frac{1}{q}-\frac{1}{\mathbf{p}^+})}\}[1 + \frac{q(\beta-1)}{\mathbf{p}^+}]^{\frac{1}{q}}||u||_{\mathbf{p}(.)}. \tag{2.2}$$

We have also the following Lemma that will be useful for the computation of the upper bound in Theorem 1.1.

Lemma 2.1. *Suppose that* $\boldsymbol{p}$ *is bounded and let* $v \in L^\infty(\Omega)$*, then we have*

$$||v||_{\boldsymbol{p}(.)} \leq 2^{\frac{1}{\boldsymbol{p}^-}} \max\{||v||_{\boldsymbol{p}^+}, ||v||_{\boldsymbol{p}^-}, ||v||_{\boldsymbol{p}^+}^{\frac{\boldsymbol{p}^+}{\boldsymbol{p}^-}}, ||v||_{\boldsymbol{p}^-}^{\frac{\boldsymbol{p}^-}{\boldsymbol{p}^+}}\}.$$

Proof. Using (2.1), we have that

$$||v||_{\mathbf{p}(.)} \leq \max\{(\rho_{\mathbf{p}(.)}(v))^{\frac{1}{\mathbf{p}^-}}, (\rho_{\mathbf{p}(.)}(v))^{\frac{1}{\mathbf{p}^+}}\}. \tag{2.3}$$

Let $A = \{x \in \Omega,\ |v(x)| \geq 1\}$. Then we have

$$\begin{aligned}\rho_{\mathbf{p}(.)}(v) &\leq \int_A |v(x)|^{\mathbf{p}^+}dx + \int_{\Omega\setminus A} |v(x)|^{\mathbf{p}^-}dx \\ &\leq \int_\Omega |v(x)|^{\mathbf{p}^+}dx + \int_\Omega |v(x)|^{\mathbf{p}^-}dx.\end{aligned}$$

Thus, we have

$$\begin{aligned}(\rho_{\mathbf{p}(.)}(v))^{\frac{1}{\mathbf{p}^+}} &\leq 2^{\frac{1}{\mathbf{p}^+}} \max\{(\int_\Omega |v(x)|^{\mathbf{p}^+}dx)^{\frac{1}{\mathbf{p}^+}}, (\int_\Omega |v(x)|^{\mathbf{p}^-}dx)^{\frac{1}{\mathbf{p}^+}}\} \\ &\leq 2^{\frac{1}{\mathbf{p}^+}} \max\{||v||_{\mathbf{p}^+}, ||v||_{\mathbf{p}^-}^{\frac{\mathbf{p}^-}{\mathbf{p}^+}}\} \end{aligned} \tag{2.4}$$

and

$$(\rho_{\mathbf{p}(.)}(v))^{\frac{1}{\mathbf{p}^-}} \leq 2^{\frac{1}{\mathbf{p}^-}} \max\{(\int_\Omega |v(x)|^{\mathbf{p}^+} dx)^{\frac{1}{\mathbf{p}^-}}, (\int_\Omega |v(x)|^{\mathbf{p}^-} dx)^{\frac{1}{\mathbf{p}^-}}\}$$
$$\leq 2^{\frac{1}{\mathbf{p}^-}} \max\{||v||_{\mathbf{p}^-}, ||v||_{\mathbf{p}^+}^{\frac{\mathbf{p}^+}{\mathbf{p}^-}}\}. \quad (2.5)$$

Using (2.4), (2.5), (2.3) and noticing that $2^{\frac{1}{\mathbf{p}^-}} \geq 2^{\frac{1}{\mathbf{p}^+}}$, we obtain the result.

□

2.3. **Cross quasiconvexity.** The limit model obtained by Γ-convergence techniques, involves an energy functional that is lower semicontinuous with respect to the considered topology. Thus, we define the cross-quasiconvex envelope (see [9, 10, 22, 23]), which is a special case of the notion of $\mathcal{A}$-quasiconvex envelope defined in [24], for $V : [0,1] \times \mathbb{M}^{m\times N} \to \mathbb{R}$, with

$$V(\kappa, A) = \kappa W_1(A) + (1-\kappa)W_2(A),$$

by

$$V_p^\star(\kappa, A) := \inf_{\theta,\varphi} \Big\{ \Big(\int_Q \big(V(\theta(x), A + \nabla\varphi(x))\big)^p dx \Big)^{\frac{1}{p}}, \varphi \in W_\#^{1,p}(Q;\mathbb{R}^m),$$
$$\theta \in L^\infty(\Omega;\{0,1\}),\ \int_\Omega \theta(x)dx = \kappa \Big\},$$

where

$$W_\#^{1,p}(Q;\mathbb{R}^m) = \{\varphi \in W_{loc}^{1,p}(\mathbb{R}^N;\mathbb{R}^m) : \varphi \text{ is } Q \text{ periodic}\},$$

with Q being the unit cube in $\mathbb{R}^N$. We have the following result proved in [1], that will be useful for the computation of the lower bound of the Γ-limit.

Lemma 2.2. *Let $1 < p < p' < \infty$. Then, for every $(\kappa, A) \in [0,1] \times \mathbb{M}^{m\times N}$, we have*

$$V_p^\star(\kappa, A) \leq V_{p'}^\star(\kappa, A).$$

This Lemma enables us to define $V^\star : [0,1] \times \mathbb{M}^{m\times N} \to \mathbb{R}$ by letting

$$V^\star(\kappa, A) := \lim_{p\to+\infty} V_p^\star(\kappa, A) = \sup_{p>1} V_p^\star(\kappa, A).$$

The following Lemmas are a consequence of the dimension reduction studied in [9, 10]. Their proofs follow the same steps as in [9, 10] with simpler arguments since we have no dimension reduction process within it, we therefore omit them. See also [23].

Lemma 2.3. *Let $1 < p < \infty$. Suppose $u_n \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^m)$ and $\chi_n \overset{*}{\rightharpoonup} \chi$ in $L^\infty(\Omega;[0,1])$, then*

$$\liminf_{n\to\infty} ||V_p^\star(\chi_n, \nabla u_n)||_p \geq ||V_p^\star(\chi, \nabla u)||_p.$$

Lemma 2.4. *Let $1 < p < \infty$. For every $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ and $\chi \in L^\infty(\Omega;[0,1])$, there exist $u_n \in W^{1,p}(\Omega;\mathbb{R}^m)$ and $\chi_n \in L^\infty(\Omega;\{0,1\})$ such that $u_n \overset{*}{\rightharpoonup} u$ in $W^{1,p}(\Omega;\mathbb{R}^m)$ and $\chi_n \overset{*}{\rightharpoonup} \chi$ in $L^\infty(\Omega;[0,1])$, with*

$$\limsup_{n\to\infty} ||V_p^\star(\chi_n,\nabla u_n)||_p \leq ||V_p^\star(\chi,\nabla u)||_p.$$

3. Relaxation result

The following relaxation result is a consequence of the Γ-limit result obtained in [1].

Proposition 3.1. *Let $p_0 > 1$. For every $\chi \in L^\infty(\Omega;[0,1])$ and $u \in W^{1,\infty}(\Omega;\mathbb{R}^m)$ let*

$$J(\chi,u) := ||V(\chi,\nabla u)||_\infty \tag{3.1}$$

and

$$G(\chi,u) = ||V^\star(\chi,\nabla u)||_\infty.$$

Then, $G(\chi,u)$ is the lower semi-continuous envelope of $J(\chi,u)$ with respect to the weak-$*$ $L^\infty(\Omega;[0,1]) \times W^{1,p_0}(\Omega;\mathbb{R}^m)$ weak topology.

Proof. The proof is a consequence of the Γ-limit result obtained in [1]. Indeed, it was proved in [1] that the Γ-limit with respect to the weak-$*$ $L^\infty(\Omega;[0,1]) \times W^{1,p_0}(\Omega;\mathbb{R}^m)$ weak topology of the sequence of functionals $(I_p)_{p>p_0}$ defined on $L^\infty(\Omega;[0,1]) \times L^{p_0}(\Omega;\mathbb{R}^m)$ into $\mathbb{R}$ by

$$I_p(\chi,u) = \begin{cases} ||V(\chi,\nabla u)||_p & \text{if} \quad \chi \in L^\infty(\Omega;\{0,1\}),\ u \in W^{1,\infty}(\Omega;\mathbb{R}^m), \\ +\infty & \text{otherwise.} \end{cases}$$

is given by

$$\bar{I}(\chi,u) = \begin{cases} G(\chi,u) & \text{if} \quad u \in W^{1,\infty}(\Omega;\mathbb{R}^m), \\ +\infty & \text{otherwise.} \end{cases}$$

Let $\bar{J}^{p_0}$ be the lower semi-continuous envelope of J with respect to the weak-$*$ $L^\infty(\Omega;[0,1]) \times W^{1,p_0}(\Omega;\mathbb{R}^m)$ weak topology. Following the same steps as in [1] with minor changes (see Appendix), we prove that the same Γ-limit is given by

$$\bar{I}(\chi,u) = \begin{cases} \bar{J}^{p_0}(\chi,u) & \text{if} \quad u \in W^{1,\infty}(\Omega;\mathbb{R}^m), \\ +\infty & \text{otherwise,} \end{cases} \tag{3.2}$$

which gives the result. □

4. Proof of Theorem 1.1

As usual for the computation of the Γ-limit, we split the proof in two steps, the first dealing with the lower bound and the second dealing with the upper bound.

Step 1. **The lower bound**. We suppose that $\min\{\gamma_1,\gamma_2\} < 1$, the proof when $\gamma_i > 1$ is analogous with very minor changes. Let $(\chi,u) \in L^\infty(\Omega;[0,1]) \times W^{1,p_0}(\Omega;\mathbb{R}^m)$ and

$(\chi_n, u_n) \in L^\infty(\Omega; [0,1]) \times W^{1,p_0}(\Omega; \mathbb{R}^m)$ such that $\chi_n \overset{*}{\rightharpoonup} \chi$ in $L^\infty(\Omega; [0,1])$ and $u_n \rightharpoonup u$ in $W^{1,p_0}(\Omega; \mathbb{R}^m)$. We will prove that

$$\liminf I_n(\chi_n, u_n) \geq I(\chi, u).$$

We can suppose that $M = \liminf I_n(\chi_n, u_n) < \infty$. There exist $n_0 \in \mathbb{N}$ such that for every $n > n_0$ we have $I_n(\chi_n, u_n) < M + 1$, which means that for every $n > n_0$, $(\chi_n, u_n) \in L^\infty(\Omega; \{0,1\}) \times W^{1,\mathbf{p}_n(.)}(\Omega; \mathbb{R}^m)$ and

$$I_n(\chi_n, u_n) = ||V(\chi_n, \nabla u_n)||_{\mathbf{p}_n(.)}.$$

Let $p > 1$. Since $\mathbf{p}_n^- \to +\infty$, there exist $n_1 \in \mathbb{N}$ such that for every $n > n_1$ we have $\mathbf{p}_n^- > p$. Using (2.2), we have that for every $n > \max\{n_0, n_1\}$

$$||V(\chi_n, \nabla u_n)||_p \leq \max\{|\Omega|^{\frac{1}{p} - \frac{1}{\mathbf{p}_n^-}}, |\Omega|^{\beta(\frac{1}{p} - \frac{1}{\mathbf{p}_n^+})}\}[1 + \frac{p(\beta - 1)}{\mathbf{p}_n^+}]^{\frac{1}{p}}(M+1) := \delta_{p,n}.$$

Notice that, since $\chi_n(x) \in \{0,1\}$, for a.e. $x \in \Omega$, we have

$$|\chi_n W_1(\nabla u_n) + (1-\chi_n)W_2(\nabla u_n)|^p = \chi_n W_1^p(\nabla u_n) + (1-\chi_n)W_2^p(\nabla u_n).$$

Thus, using the coercivity condition and letting $\alpha = \min\{\alpha_1, \alpha_2\}$, $\gamma = \min\{\gamma_1, \gamma_2\}$, we obtain that

$$\begin{aligned}
\delta_{p,n} &\geq (\int_\Omega \chi_n W_1^p(\nabla u_n) + (1-\chi_n)W_2^p(\nabla u_n)dx)^{\frac{1}{p}} \\
&\geq \alpha(\int_\Omega \chi_n |\nabla u_n|^{\gamma_1 p} + (1-\chi_n)|\nabla u_n|^{\gamma_2 p} dx)^{\frac{1}{p}} \\
&\geq \alpha(\int_{\{|\nabla u_n|>1\}} \chi_n |\nabla u_n|^{\gamma_1 p} + (1-\chi_n)|\nabla u_n|^{\gamma_2 p} dx)^{\frac{1}{p}} \\
&\geq \alpha(\int_{\{|\nabla u_n|>1\}} \chi_n |\nabla u_n|^{\gamma p} + (1-\chi_n)|\nabla u_n|^{\gamma p} dx)^{\frac{1}{p}} \\
&= \alpha(\int_{\{|\nabla u_n|>1\}} |\nabla u_n|^{\gamma p} dx)^{\frac{1}{p}},
\end{aligned}$$

which gives

$$(\int_{\{|\nabla u_n|>1\}} |\nabla u_n|^{\gamma p} dx)^{\frac{1}{\gamma p}} \leq (\frac{\delta_{p,n}}{\alpha})^{\frac{1}{\gamma}}.$$

Next, we have

$$\begin{aligned}
\int_\Omega |\nabla u_n|^{\gamma p} dx &= \int_{\{|\nabla u_n|>1\}} |\nabla u_n|^{\gamma p} dx + \int_{\{|\nabla u_n|\leq 1\}} |\nabla u_n|^{\gamma p} dx \\
&\leq (\frac{\delta_{p,n}}{\alpha})^p + |\Omega|
\end{aligned}$$

and thus

$$||\nabla u_n||_{\gamma p} \leq (\frac{\delta_{p,n}}{\alpha})^{\frac{1}{\gamma}} + |\Omega|^{\frac{1}{\gamma p}}.$$

Since $\lim_{n\to+\infty}\delta_{p,n} = \delta_p := \max\{|\Omega|^{\frac{1}{p}}, |\Omega|^{\frac{\beta}{p}}\}(M+1)$, we have the existence of $n_2 \in \mathbb{N}$ such that for every $n > n_2$, $\delta_{p,n} < \delta_p + 1$. Thus, for every $n > \max\{n_0, n_1, n_2\}$ we have

$$||\nabla u_n||_{\gamma p} \leq (\frac{\delta_p + 1}{\alpha})^{\frac{1}{\gamma}} + |\Omega|^{\frac{1}{\gamma p}}.$$

Next, using Poincaré's Inequality we prove that (u_n) is also uniformly bounded in $L^{\gamma p}(\Omega;\mathbb{R}^m)$ and thus it is uniformly bounded in $W^{1,\gamma p}(\Omega;\mathbb{R}^m)$. Up to a subsequence we have that (u_n) converges weakly to u in $W^{1,\gamma p}(\Omega;\mathbb{R}^m)$. Next, we prove that $u \in W^{1,\infty}(\Omega;\mathbb{R}^m)$. We have, for every $x_0 \in \Omega$ and $t > 0$ such that the open ball $B_t(x_0) \subset \Omega$, that

$$\begin{aligned}\frac{1}{|B_t(x_0)|}\int_{B_{t(x_0)}} |\nabla u| dx &\leq |B_t(x_0)|^{-\frac{1}{\gamma p}} ||\nabla u||_{\gamma p} \\ &\leq |B_t(x_0)|^{-\frac{1}{\gamma p}} \liminf_{n\to\infty} ||\nabla u_n||_{\gamma p} \\ &\leq |B_t(x_0)|^{-\frac{1}{\gamma p}} \Big((\frac{\delta_p + 1}{\alpha})^{\frac{1}{\gamma}} + |\Omega|^{\frac{1}{\gamma p}} \Big).\end{aligned}$$

Letting $p \to \infty$, we obtain that

$$\frac{1}{|B_t(x_0)|}\int_{B_{t(x_0)}} |\nabla u| dx \leq (\frac{M+2}{\alpha})^{\frac{1}{\gamma}} + 1.$$

Then, letting $t \to 0^+$, we obtain that for every Lebesgue point $x_0 \in \Omega$ we have

$$|\nabla u(x_0)| \leq (\frac{M+2}{\alpha})^{\frac{1}{\gamma}} + 1$$

and thus, since Ω is bounded, we obtain that for a.e. $x_0 \in \Omega$

$$|u(x_0)| \leq C(\Omega)\Big((\frac{M+2}{\alpha})^{\frac{1}{\gamma}} + 1 \Big),$$

which gives that $u \in W^{1,\infty}(\Omega;\mathbb{R}^m)$. Finally, we have for every $n > \max\{n_0, n_1\}$

$$\begin{aligned}I_n(\chi_n, u_n) &= ||V(\chi_n, \nabla u_n)||_{\mathbf{p}_n(.)} \\ &\geq \frac{1}{\eta_{p,n}} ||V(\chi_n, \nabla u_n)||_p \\ &\geq \frac{|\Omega|^{\frac{\gamma-1}{\gamma p}}}{\eta_{p,n}} ||V(\chi_n, \nabla u_n)||_{\gamma p} \\ &\geq \frac{|\Omega|^{\frac{\gamma-1}{\gamma p}}}{\eta_{p,n}} ||V^{\star}_{\gamma p}(\chi_n, \nabla u_n)||_{\gamma p},\end{aligned}$$

with $\eta_{p,n} := \frac{\delta_{p,n}}{M+1}$. Since $\chi_n \overset{*}{\rightharpoonup} \chi$ in $L^\infty(\Omega;[0,1])$ and $u_n \rightharpoonup u$ in $W^{1,\gamma p}(\Omega;\mathbb{R}^m)$, the cross quasiconvexity of $V^\star_{\gamma p}$ implies that

$$\begin{aligned}\liminf I_n(\chi_n,u_n) &\geq \frac{|\Omega|^{\frac{\gamma-1}{\gamma p}}}{\eta_p} \liminf ||V^\star_{\gamma p}(\chi_n,\nabla u_n)||_{\gamma p}\\ &\geq \frac{|\Omega|^{\frac{\gamma-1}{\gamma p}}}{\eta_p} ||V^\star_{\gamma p}(\chi,\nabla u)||_{\gamma p}\end{aligned}$$

with $\eta_p = \frac{\delta_p}{M+1}$. Let $q_0 < \gamma p$. We have that

$$\liminf I_n(\chi_n,u_n) \geq \frac{|\Omega|^{\frac{\gamma-1}{\gamma p}}}{\eta_p} |\Omega|^{\frac{q_0-\gamma p}{q_0 \gamma p}} ||V^\star_{\gamma p}(\chi,\nabla u)||_{q_0} = \frac{|\Omega|^{\frac{q_0-p}{q_0 p}}}{\eta_p} ||V^\star_{\gamma p}(\chi,\nabla u)||_{q_0}.$$

Letting $p \to \infty$ we obtain that

$$\liminf I_n(\chi_n,u_n) \geq |\Omega|^{\frac{-1}{q_0}} ||V^\star(\chi,\nabla u)||_{q_0}.$$

Finally, letting $q_0 \to +\infty$ we obtain that

$$\liminf I_n(\chi_n,u_n) \geq ||V^\star(\chi,\nabla u)||_\infty,$$

which concludes the lower bound.

Step 2. **The upper bound**. Let $(\chi,u) \in L^\infty(\Omega;[0,1]) \times L^{p_0}(\Omega;\mathbb{R}^m)$. If $u \notin W^{1,\infty}(\Omega;\mathbb{R}^m)$ then $I(\chi,u) = \infty$ and there is nothing to prove. Thus, we suppose that $u \in W^{1,\infty}(\Omega;\mathbb{R}^m)$. Using the upper bound of the Γ-limit result obtained in [1], we have the existence of $(\chi_n,u_n) \in L^\infty(\Omega;\{0,1\}) \times W^{1,n}(\Omega;\mathbb{R}^m)$ such that $\chi_n \overset{*}{\rightharpoonup} \chi$ in $L^\infty(\Omega;[0,1])$, $u_n \rightharpoonup u$ in $W^{1,p_0}(\Omega;\mathbb{R}^m)$ verifying

$$\limsup_{n\to\infty} ||V(\chi_n,\nabla u_n)||_n \leq ||V^\star(\chi,\nabla u)||_\infty. \tag{4.1}$$

We suppose that for some $N \in \mathbb{N}$, we have $p_n^+ \leq n$ for every $n \geq N$, otherwise we chose a subsequence n' verifying $p_n^+ \leq n'$ that we still label n. Then, using Lemma 2.1, we have for every $n \geq N$,

$$\begin{aligned}I_n(\chi_n,u_n) &= ||V(\chi_n,\nabla u_n)||_{\mathbf{p}_n(.)}\\ &\leq 2^{\frac{1}{\mathbf{p}_n^-}} \max\{||V(\chi_n,\nabla u_n)||_{\mathbf{p}_n^+}, ||V(\chi_n,\nabla u_n)||_{\mathbf{p}_n^-}\\ &\qquad , ||V(\chi_n,\nabla u_n)||_{\mathbf{p}_n^+}^{\frac{\mathbf{p}_n^+}{\mathbf{p}_n^-}}, ||V(\chi_n,\nabla u_n)||_{\mathbf{p}_n^-}^{\frac{\mathbf{p}_n^-}{\mathbf{p}_n^+}}\}.\end{aligned}$$

Using Hölder inequality, we obtain that

$$\begin{aligned}I_n(\chi_n,u_n) \leq 2^{\frac{1}{\mathbf{p}_n^-}} \max\{&|\Omega|^{\frac{n-\mathbf{p}_n^+}{n\mathbf{p}_n^+}} ||V(\chi_n,\nabla u_n)||_n, |\Omega|^{\frac{n-\mathbf{p}_n^-}{n\mathbf{p}_n^-}} ||V(\chi_n,\nabla u_n)||_n\\ &, |\Omega|^{\frac{n-\mathbf{p}_n^+}{n\mathbf{p}_n^+}\frac{\mathbf{p}_n^+}{\mathbf{p}_n^-}} ||V(\chi_n,\nabla u_n)||_n^{\frac{\mathbf{p}_n^+}{\mathbf{p}_n^-}}, |\Omega|^{\frac{n-\mathbf{p}_n^-}{n\mathbf{p}_n^-}\frac{\mathbf{p}_n^-}{\mathbf{p}_n^+}} ||V(\chi_n,\nabla u_n)||_n^{\frac{\mathbf{p}_n^-}{\mathbf{p}_n^+}}\}.\end{aligned}$$

Making $n \to \infty$ and using (4.1), we obtain that

$$\limsup_{n\to\infty} I_n(\chi_n, u_n) \le ||V^\star(\chi, \nabla u)||_\infty$$

which gives the result

5. An Auxiliary Result

In Theorem 1.1, we obtained the integral representation of the Γ-limit of the sequence of energy functionals defined with the variable exponent that is, for every $\chi \in L^\infty(\Omega; [0,1]), u \in W^{1,\infty}(\Omega; \mathbb{R}^m)$,

$$I(\chi, u) = \inf_{\{\chi_n\},\{u_n\}} \{\liminf ||V(\chi_n, \nabla u_n)||_{\mathbf{p}_n(.)},\ \chi_n \in L^\infty(\Omega; \{0,1\}),$$
$$u_n \in W^{1,\mathbf{p}_n(\cdot)}(\Omega; \mathbb{R}^m),\ \chi_n \overset{*}{\rightharpoonup} \chi \text{ in } L^\infty(\Omega; [0,1]),\ u_n \rightharpoonup u \text{ in } W^{1,p_0}(\Omega; \mathbb{R}^m)\}.$$

As in [9], we can deduce the following result, using the same arguments.

Theorem 5.1. *Consider the same hypothesis of Theorem1.1, supposing $\gamma_1 = \gamma_2 = \gamma$ and the additional growth condition : there exist $C_1, C_2 > 0$ such that*

$$W_i(A) \le C_i(1 + |A|^\gamma).$$

Let $0 < \lambda < 1$,

$$G_\lambda(u) = \inf_{\{\chi_n\},\{u_n\}} \{\liminf ||V(\chi_n, \nabla u_n)||_{\boldsymbol{p}_n(.)},\ \chi_n \in L^\infty(\Omega; \{0,1\}),$$
$$u_n \in W^{1,\boldsymbol{p}_n(\cdot)}(\Omega; \mathbb{R}^m),\ u_n \rightharpoonup u \text{ in } W^{1,p_0}(\Omega; \mathbb{R}^m),\ \frac{1}{|\Omega|}\int_\Omega \chi_n(x)dx = \lambda\}$$

and

$$I_\lambda(u) = \inf_\chi \{I(\chi, u);\ \frac{1}{|\Omega|}\int_\Omega \chi(x)dx = \lambda\}.$$

Then

$$G_\lambda = I_\lambda.$$

Proof. Let $u \in W^{1,\infty}(\Omega; \mathbb{R}^m)$. First notice that

$$G_\lambda(u) \ge I_\lambda(u).$$

Indeed, let $\chi_n \in L^\infty(\Omega; \{0,1\})$ such that $\dfrac{1}{|\Omega|}\displaystyle\int_\Omega \chi_n(x)dx = \lambda$, then, for a subsequence we have $\chi_n \overset{*}{\rightharpoonup} \chi$ in $L^\infty(\Omega; [0,1])$ with $\chi \in L^\infty(\Omega; [0,1])$ which gives the first inequality.

For the second inequality, let $\varepsilon > 0$, then there exist $\chi \in L^\infty(\Omega; [0,1])$, with $\frac{1}{|\Omega|}\int_\Omega \chi(x)dx = \lambda$ such that

$$\varepsilon + I_\lambda(u) \ge I(\chi, u).$$

Let $\chi_n \in L^\infty(\Omega;\{0,1\})$, $u_n \in W^{1,p_n(.)}(\Omega;\mathbb{R}^m)$ such that $\chi_n \overset{*}{\rightharpoonup} \chi$ in $L^\infty(\Omega;[0,1])$ and $u_n \rightharpoonup u$ in $W^{1,p_0}(\Omega;\mathbb{R}^m)$ with

$$\lim_{n\to+\infty} ||V(\chi_n,\nabla u_n)||_{\mathbf{p}_n(.)} = I(\chi,u) \le \varepsilon + I_\lambda(u).$$

If $\dfrac{1}{|\Omega|}\displaystyle\int_\Omega \chi_n(x)dx = \lambda$ then we obtain the result letting $\varepsilon \to 0$. Otherwise, we need to construct $\tilde{\chi}_n \in L^\infty(\Omega;\{0,1\})$ with $\dfrac{1}{|\Omega|}\displaystyle\int_\Omega \tilde{\chi}_n(x)dx = \lambda$ and

$$\lim_{n\to+\infty} ||V(\tilde{\chi}_n,\nabla u_n)||_{\mathbf{p}_n(.)} \le \lim_{n\to+\infty} ||V(\chi_n,\nabla u_n)||_{\mathbf{p}_n(.)}.$$

We suppose that $\chi_n = \chi_{E_n}$, with $E_n \subset \Omega$ and $0 < \dfrac{|E_n|}{|\Omega|} < \lambda$. Let

$$K_n = \Big[\frac{|\Omega| - |E_n|}{\lambda|\Omega| - |E_n|}\Big],$$

where $[x]$ denotes the integer part of x. We have

$$0 < \Big[\frac{(1-\lambda)|\Omega|}{\lambda|\Omega| - |E_n|}\Big] \le K_n \le \frac{|\Omega| - |E_n|}{\lambda|\Omega| - |E_n|}$$

and since $\chi_n \overset{*}{\rightharpoonup} \chi$ in $L^\infty(\Omega;[0,1])$ with $\dfrac{1}{|\Omega|}\displaystyle\int_\Omega \chi(x)dx = \lambda$, then $|E_n| \to \lambda|\Omega|$ and thus $K_n \to \infty$. Next, since

$$|\Omega\backslash E_n| > (1-\lambda)|\Omega| > 0,$$

we can have the following disjoint decomposition

$$|\Omega\backslash E_n| = \cup_{i=1}^{K_n} \hat{E}_i \cup B,$$

with $|\hat{E}_i| = \lambda|\Omega| - |E_n|$. Then, the coercivity condition implies the existence of $c > 0$ such that

$$\sum_{i=1}^{K_n} \int_{\hat{E}_i} |\nabla u_n|^{\gamma \mathbf{p}_n(x)} dx \le \int_{\Omega\backslash E_n} |\nabla u_n|^{\gamma \mathbf{p}_n(x)} dx < c.$$

Thus, there exist $1 \le i(n) \le K_n$ such that

$$K_n \int_{\hat{E}_{i(n)}} |\nabla u_n|^{\gamma \mathbf{p}_n(x)} dx \le c,$$

which gives that

$$\lim_{n\to+\infty} \int_{\hat{E}_{i(n)}} |\nabla u_n|^{\gamma \mathbf{p}_n(x)} dx = 0.$$

Using the growth condition, we obtain that

$$\lim_{n\to+\infty} \int_{\hat{E}_{i(n)}} W_1(\nabla u_n)^{\mathbf{p}_n(x)} dx = \lim_{n\to+\infty} \int_{\hat{E}_{i(n)}} W_2(\nabla u_n)^{\mathbf{p}_n(x)} dx = 0. \tag{5.1}$$

Let $\tilde{\chi}_n = \chi_{E_n} + \chi_{\hat{E}_{i(n)}}$. We have $\tilde{\chi}_n \in L^\infty(\Omega;\{0,1\})$ with $\dfrac{1}{|\Omega|}\displaystyle\int_\Omega \tilde{\chi}_n(x)dx = \lambda$. Moreover,

$$||V(\tilde{\chi}_n, \nabla u_n)||_{\mathbf{p}_n(.)} = ||V(\chi_n, \nabla u_n) + \chi_{\hat{E}_{i(n)}}(W_1(\nabla u_n) - W_2(\nabla u_n))||_{\mathbf{p}_n(.)}$$

and thus

$$||V(\tilde{\chi}_n, \nabla u_n)||_{\mathbf{p}_n(.)} \leq ||V(\chi_n, \nabla u_n)||_{\mathbf{p}_n(.)} + ||\chi_{\hat{E}_{i(n)}} W_1(\nabla u_n)||_{\mathbf{p}_n(.)} + ||\chi_{\hat{E}_{i(n)}} W_2(\nabla u_n))||_{\mathbf{p}_n(.)}.$$

Using (5.1), we obtain that

$$\lim_{n\to+\infty} ||V(\tilde{\chi}_n, \nabla u_n)||_{\mathbf{p}_n(.)} \leq \lim_{n\to+\infty} ||V(\chi_n, \nabla u_n)||_{\mathbf{p}_n(.)},$$

which concludes the proof in the case $\dfrac{|E_n|}{|\Omega|} < \lambda$. A similar construction can be made when $\dfrac{|E_n|}{|\Omega|} > \lambda$ setting $K_n = \Big[\dfrac{|\Omega| - |E_n|}{|E_n| - \lambda|\Omega|}\Big]$. □

6. Appendix

In this section we will prove an analogous result to Theorem 1.1 in [1] where the limit functional contains $\bar{J}^{p_0}$ instead of G, thus obtaining (3.2) in the proof of Theorem 5.1. The proof of the next Theorem follows the same steps as in [1] with minor changes and thus we will focus on these changes.

Theorem 6.1. *Let* $1 < p_0 < \infty$. *Consider the sequence of functionals* $(I_p)_{p>p_0}$, *where* p *denotes a sequence* $p_n \to +\infty$, *defined on* $L^\infty(\Omega;[0,1]) \times L^{p_0}(\Omega;\mathbb{R}^m)$ *by*

$$I_p(\chi, u) = \begin{cases} \Big(\displaystyle\int_\Omega \chi W_1(\nabla u)^p + (1-\chi) W_2(\nabla u)^p dx\Big)^{\frac{1}{p}} & \text{if } \begin{cases} \chi \in L^\infty(\Omega;\{0,1\}), \\ u \in W^{1,p}(\Omega;\mathbb{R}^m), \end{cases} \\ +\infty & \text{otherwise,} \end{cases},$$

where $W_i : \mathbb{M}^{m\times N} \to \mathbb{R}$ *are continuous functions verifying linear growth and coercivity hypotheses: there exist* $\alpha_i, \beta_i > 0$ *such that*

$$\beta_i |A| \leq W_i(A) \leq \alpha_i(1 + |A|).$$

Let I *be defined on* $L^\infty(\Omega;[0,1]) \times L^{p_0}(\Omega;\mathbb{R}^m)$ *by*

$$I(\chi, u) = \begin{cases} \bar{J}^{p_0}(\chi, u) & \text{if } u \in W^{1,\infty}(\Omega;\mathbb{R}^m), \\ +\infty & \text{otherwise,} \end{cases}$$

where J *is defined in (3.1) and* $\bar{J}^{p_0}$ *is its lower semicontinuous envelope with respect to the weak-*$*$ $L^\infty(\Omega;[0,1]) \times W^{1,p_0}(\Omega;\mathbb{R}^m)$ *weak topology. Then, the sequence of functionals* $(I_p)_{p>p_0}$ Γ*-converges to* I *as* p *goes to* $+\infty$ *with respect to the* $L^\infty(\Omega;[0,1])$ *weak-*$*\times$ $W^{1,p_0}(\Omega;\mathbb{R}^m)$ *weak topology.*

Proof. Step 1. **The lower bound**. Let $(\chi, u) \in L^\infty(\Omega; [0,1]) \times L^{p_0}(\Omega; \mathbb{R}^m)$ and $(\chi_p, u_p) \in L^\infty(\Omega; [0,1]) \times L^{p_0}(\Omega; \mathbb{R}^m)$ such that $\chi_p \overset{*}{\rightharpoonup} \chi$ in $L^\infty(\Omega; [0,1])$ and $u_p \rightharpoonup u$ in $W^{1,p_0}(\Omega; \mathbb{R}^m)$. We will prove that

$$\liminf_{p\to\infty} I_p(\chi_p, u_p) \geq I(\chi, u).$$

We can suppose that $M = \liminf_{p\to\infty} I_p(\chi_p, u_p) < \infty$, which implies that $\chi_p \in L^\infty(\Omega; \{0,1\})$ and $u_p \in W^{1,p}(\Omega; \mathbb{R}^m)$. As in [1], we obtain that for some $p_1 \geq p_0$ and for every $r \geq p_1$, $(u_p)_p$ is uniformly bounded in $W^{1,r}(\Omega; \mathbb{R}^m)$ and thus, up to a sub-sequence, it converges weakly in $W^{1,r}(\Omega; \mathbb{R}^m)$ to $u \in W^{1,\infty}(\Omega; \mathbb{R}^m)$. Then, still as in [1], we obtain that for every $r \geq p_1$

$$I_p(\chi_p, u_p) \geq |\Omega|^{\frac{r-p}{pr}} ||V(\chi_p, \nabla u_p)||_r,$$

where V is defined in (1.1). Making $r \to \infty$, we obtain that

$$I_p(\chi_p, u_p) \geq |\Omega|^{\frac{1}{p}} ||V(\chi_p, \nabla u_p)||_\infty,$$

and thus, using the lower semicontinuity of $\bar{J}^{p_0}$, we obtain that

$$\begin{aligned}\liminf_{p\to\infty} I_p(\chi_p, u_p) &\geq \liminf_{p\to\infty} ||V(\chi_p, \nabla u_p)||_\infty \\ &= \liminf_{p\to\infty} J(\chi_p, u_p) \\ &\geq \liminf_{p\to\infty} \bar{J}^{p_0}(\chi_p, u_p) \\ &\geq \bar{J}^{p_0}(\chi, u).\end{aligned}$$

This last inequality insures that we have

$$\Gamma - \liminf I_p(\chi, u) \geq I(\chi, u),$$

for every $(\chi, u) \in L^\infty(\Omega; [0,1]) \times L^{p_0}(\Omega; \mathbb{R}^m)$.

Step 2. **The upper bound**. We need to prove the converse inequality stating that

$$\Gamma - \limsup I_p(\chi, u) \leq I(\chi, u)$$

for every $(\chi, u) \in L^\infty(\Omega; [0,1]) \times L^{p_0}(\Omega; \mathbb{R}^m)$. If $u \notin W^{1,\infty}(\Omega; \mathbb{R}^m)$ then there is nothing to prove. Then, let $(\chi, u) \in L^\infty(\Omega; [0,1]) \times W^{1,\infty}(\Omega; \mathbb{R}^m)$. In [1], we obtained that

$$\Gamma - \limsup I_p(\chi, u) = ||V^\star(\chi, \nabla u)||_\infty.$$

Thus, we have

$$\Gamma - \limsup I_p(\chi, u) \leq ||V(\chi, \nabla u)||_\infty = J(\chi, u).$$

Finally, taking the lower semicontinuous envelop with respect to the $L^\infty(\Omega; [0,1])$ weak-$*$ $\times$ $W^{1,p_0}(\Omega; \mathbb{R}^m)$ weak topology on both sides of the last inequality, we obtain that

$$\Gamma - \limsup I_p(\chi, u) \leq \bar{J}^{p_0}(\chi, u)$$

and thus the result. □

ACKNOWLEDGMENTS

I wish to thank Professor Hervé Le Dret for many useful discussions concerning the subject of this paper. The author's work was supported by IMSIU.

REFERENCES

[1] H. ZORGATI: *A* Γ*-convergence result for optimal design problems*, C. R., Math., Acad. Sci. Paris., **360** (2022), 1145-1151.

[2] E. ACERBI, G. MINGIONE, *Regularity results for a class of functionals with non-standard growth*, Arch. Rational Mech. Anal., **156** (2001), 121–140.

[3] K. RAJAGOPAL, M. RUŽIČKA *Mathematical modeling of electrorheological materials*, Cont. Mech. and Thermodynamics, **13** (2001), 59–78.

[4] M. RUŽIČKA *Electrorheological fluids: modeling and mathematical theory*, Lecture Notes in Mathematics, 1748, Springer-Verlag, Berlin, 2000.

[5] A. WINEMAN, K. RAJAGOPAL *On constitutive equations for electrorheological materials*, Contin. Mech. Thermodyn., **7** (1995), 1–22.

[6] S. ANTONTSEV, J. RODRIGUES, *On stationary thermo-rheological viscous flows*, Ann. Univ. Ferrara Sez. VII Sci. Mat., **52** (2006), 19–36.

[7] Y. CHEN, S. LEVINE, M. RAO, *Variable Exponent, Linear Growth Functionals in Image Restoration*, SIAM J. Appl. Math., **66** (2006), 1383–1406.

[8] F. LI, Z. LI, L. PI, *Variable exponent functionals in image restoration*, Applied Mathematics and Computation, **216** (2010), 870–882.

[9] I. FONSECA, G. FRANCFORT, *3D-2D asymptotic analysis of an optimal design problem for thin films*, J. Reine Angew. Math., **505** (1998), 173-202.

[10] A. BRAIDES, I. FONSECA, G. FRANCFORT, *3D-2D asymptotic analysis for inhomogeneous thin films*, Indiana Univ. Math. J.,**49** (2000), 1367-1404.

[11] A. GARRONI, V. NESI, M. PONSIGLIONE, *Dielectric breakdown: Optimal bounds*, Proc. R. Soc. A,, **457** (2001) 2317–2335.

[12] M. BOCEA, M. MIHILESCU, Γ*-Convergence of power-law functionals with variable exponents*, Nonlinear Analysis: Theory, Methods and Applications, **73** (2010), 110-121.

[13] M. BOCEA, V. NESI, Γ*-convergence of power-law functionals, variational principles in* L^∞*, and applications*, SIAM J. Math. Anal., **39** (2008), 1550-1576.

[14] N. ANSINI, F. PRINARI, *Power-law approximation under differential constraints*, SIAM J. Math. Anal., **46** (2014), 1085-1115.

[15] N. ANSINI, F. PRINARI, *On the lower semicontinuity of supremal functional under differential constraints*, ESAIM, Control Optim. Calc. Var., **21** (2015), 1053-1075.

[16] T. CHAMPION, L. DE PASCALE, F. PRINARI, Γ*-convergence and absolute minimizers for supremal functionals*, ESAIM: COCVC,, **10** (2004) 14-27.

[17] M. ELEUTERI, F. PRINARI, Γ*-convergence for power-law functionals with variable exponents*, Nonlinear Analysis: Real World Applications, **58** (2021) 103221.

[18] G. DAL MASO, *An Introduction to* Γ*-Convergence*, Birkhauser, Boston, 1993.

[19] DE GIORGI E, *Sulla convergenza di alcune successioni di integrali del tipo dell area*, Rend. Mat., (IV), **8** (1975), 277-294.

[20] L. DIENING, P. HARJULEHTO, P. HASTO, M. RUZICKA, *Lebesgue and Sobolev Spaces with Variable Exponents* Lecture Notes in Mathematics, vol. 2017, Springer-Verlag, Berlin, 2011.

[21] I. SHARAPUDINOV, *On the topology of the space* $L^{p(t)}([0,1])$, Math. Notes., **26** (1979), 796–806.

[22] C.B. MORREY JR, *Quasiconvexity and the semicontinuity of multiple integrals*, Pacific J. Math., **2** (1952), 25-53.

[23] H. LE DRET, A. RAOULT, *Variational Convergence for Nonlinear Shell Models with Directors and Related Semicontinuity and Relaxation Results*, Arch. Rational Mech. Anal., **154** (2000), 101–134.

[24] I. FONSECA, S. MULLER, $\mathcal{A}$*-quasiconvexity, lower semicontinuity and Young measures*, SIAM J. Math. Anal., **30** (1999), 1355–1390.