Korean J. Math. **31** (2023), No. 4, pp. 405–417
https://dx.doi.org/10.11568/kjm.2023.31.4.405

RINGS IN WHICH EVERY SEMICENTRAL IDEMPOTENT IS CENTRAL

Muhammad Saad

Abstract. The RIP of rings was introduced by Kwak and Lee as a generalization of the one-sided idempotent-reflexivity property. In this study, we focus on rings in which all one-sided semicentral idempotents are central, and we refer to them as quasi-Abelian rings, extending the concept introduced by RIP. We establish that quasi-Abelianity extends to various types of rings, including polynomial rings, power series rings, Laurent series rings, matrices, and certain subrings of triangular matrix rings. Furthermore, we provide comprehensive proofs for several results that hold for RIP and are also satisfied by the quasi-Abelian property. Additionally, we investigate the structural properties of minimal non-Abelian quasi-Abelian rings.

1. Introduction

Throughout this note, all our rings are assumed to have identity unless indicated otherwise. An idempotent e of a ring R is called *left* (resp. *right*) *semicentral* if $ae = eae$ (resp., $ea = eae$) for all $a \in R$. If an idempotent e is both left and right semicentral, then e is central. For a ring R, we use $\mathcal{I}(R)$, $\mathcal{S}_l(R)$, $\mathcal{S}_r(R)$, and $\mathcal{B}(R)$ to denote the set of all idempotents, left semicentral idempotents, right semicentral idempotents, and central idempotents of R, respectively.

A ring R is called *Abelian* (resp. *semi-Abelian*) if every idempotent of R is central (resp. left or right semicentral). Moreover, R is called *central reduced* (resp. *semicentral reduced*) if it has no nontrivial central (resp. one-sided semicentral) idempotents (see [2, 6, 7]).

Given a ring R, the polynomial ring with an indeterminate x over R, the power series ring with an indeterminate x over R, the Laurent polynomial ring with an indeterminate x over R, the Laurent power series ring with an indeterminate x over R, the n-by-n full matrix ring over R, the n-by-n upper triangular matrix ring over R are denoted respectively by $R[x]$, $R[[x]]$, $R[x, x^{-1}]$, $R[[x, x^{-1}]]$, $\mathbb{M}_n(R)$, and $\mathbb{T}_n(R)$.

Recall that a ring is *reduced* if it has no nonzero nilpotent elements and is called *directly finite* if $ba = 1$ whenever $ab = 1$ for every $a, b \in R$. In [3], a ring R is called 2-*primal* if the prime radical of R coincides with the set of all nilpotent elements of R. According to [19], a ring R is called *NI* if the upper nilradical of R coincides with

Received September 14, 2023. Revised October 26, 2023. Accepted October 26, 2023.
2010 Mathematics Subject Classification: 16S99, 16U80.
Key words and phrases: quasi-Abelian, semicentral idempotent, idempotent-reflexive, RIP.

the set of all nilpotent elements of R. Furthermore, it is worth noting that reduced rings are Abelian, Abelian ring are directly finite, and 2-primal rings are NI.

In [12, 14, 15], a ring R is called *reflexive* if $aRb = 0$ implies $bRa = 0$ for any $a, b \in R$, and is called *right* (resp. *left*) *idempotent-reflexive* if $aRe = 0$ (resp. eRa) implies $eRa = 0$ (resp. aRe) for every $a \in R$ and $e \in \mathcal{I}(R)$. If a ring is both left and right idempotent-reflexive, then the ring is called *idempotent-reflexive*.

According to [16], a ring R is said to have the property of *RIP* if $eRf = 0$ implies $fRe = 0$ for any $e, f \in \mathcal{I}(R)$. The RIP of rings is a generalization of the one-sided idempotent reflexivity property. It is evident that every one-sided semicentral idempotent in a ring with the RIP property is also central (as shown in [16, Proposition 2.10 (1)]). However, the converse is not necessarily true, as demonstrated in the following example.

Example 1.1. Consider an Abelian ring D, and let $S = D \oplus D \oplus D$ equipped with componentwise addition and multiplication. We define the automorphism $\sigma : S \to S$ as $\sigma(x, y, z) = (y, z, x)$ for every $(x, y, z) \in S$. Now, consider the Nagata extension of S by S and σ, denoted as R. In the ring R, we have the idempotents $e = ((0, 1, 0), (0, 0, 0))$ and $f = ((0, 0, 1), (0, 1, 1))$. Although $eRf = 0$, it is noteworthy that $fRe = ((0, 0, 0), (0, D, 0)) \neq 0$. Consequently, R does not possess the RIP property. On the other hand, we observe that $\mathcal{S}_l(R) \cup \mathcal{S}_r(R) = \{((e, e, e), (0, 0, 0)) \mid e \in \mathcal{I}(R)\}$, which constitutes a central subset of the ring R.

Motivated by the previous example, we now delve into introducing a concept in this paper. This concept pertains to rings where the centrality of every semicentral idempotent is extended, thus giving a generalization of RIP rings.

2. Quasi-Abelian rings

We initiate this section by introducing the following definition.

Definition 2.1. A ring R is said to be *quasi-Abelian* if every one-sided semicentral idempotent of R is central; that is $\mathcal{S}_l(R) \cup \mathcal{S}_r(R) = \mathcal{B}(R)$.

Evidently, both Abelian and semicentral reduced rings are contained in the class of quasi-Abelian rings. However, it's important to highlight that there exist quasi-Abelian rings that are neither Abelian nor semicentral reduced.

Example 2.2. The ring $R = \mathbb{M}_2(F)$ of 2-by-2 matrices over some filed F is reflexive by [15, Theorem 2.6(2)], and particularly is quasi-Abelian. However, R is not Abelian. Furthermore, the ring $R \oplus R$ is quasi-Abelian but neither Abelian nor semicentral reduced.

Here are two propositions with straightforward proofs that provide sufficient and necessary conditions for a quasi-Abelian ring to be either Abelian or semicentral reduced.

Proposition 2.3. *A ring R is Abelian if and only if it is quasi-Abelian and semi-Abelian.*

Proposition 2.4. *A ring R is semicentral reduced if and only if R is quasi-Abelian and central reduced.*

Nevertheless, every RIP ring is quasi-Abelian. However, this implication does not hold in the case of rings without identity, as exemplified by the non-unital rings $\begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} \mathbb{Z}_2 & 0 \\ \mathbb{Z}_2 & 0 \end{bmatrix}$ (see [16, Example 4.4]).

The following proposition provides equivalent conditions for the quasi-Abelianity property of rings.

PROPOSITION 2.5. *For a ring R, the following conditions are equivalent:*

(i) *R is quasi-Abelian.*
(ii) *For any $a, b \in R$ such that $ab \in \mathcal{S}_l(R)$, we have $a\mathcal{S}_l(R)b \subseteq \mathcal{S}_l(R)$.*
(iii) *For any $a, b \in R$ such that $ab \in \mathcal{S}_r(R)$, we have $a\mathcal{S}_r(R)b \subseteq \mathcal{S}_r(R)$.*
(iv) *For any $a, b \in R$ such that $ab \in \mathcal{S}_l(R)$, we have $a\mathcal{S}_l(R)b \subseteq \mathcal{I}(R)$.*
(v) *For any $a, b \in R$ such that $ab \in \mathcal{S}_r(R)$, we have $a\mathcal{S}_r(R)b \subseteq \mathcal{I}(R)$.*

Proof. (i)⇒(ii) and (ii)⇒(iv) are straightforward.

(iv)⇒(i): Let f be a semicentral idempotent of R. If f is a left semicentral idempotent, then for every idempotent e of R, we have $ef(1-e), (1-e)fe \in \mathcal{I}(R)$, given that $e(1-e) = (1-e)e = 0 \in \mathcal{S}_l(R)$. Consequently, $ef(1-e) = (1-e)fe = 0$, and $ef = fe$. Thus, it follows that f is central, as indicated in [10, Corollary 2]. On the other hand, if f is a right semicentral idempotent, then $e(1-f)(1-e), (1-e)(1-f)e \in \mathcal{I}(R)$ and $ef = fe$. Again, we can demonstrate that f is central.

To establish the equivalence between (i), (iii), (v), and (vii), a similar argument can be applied as used in the proof of (i)⇔(ii)⇔(iv)⇔(vi). □

In [16], Kwak and Lee raised the following question: "If a ring R has the RIP property, does $\mathbb{M}_n(R)$ have the RIP property for some $n \geq 2$?" We will address this question in the next proposition, but instead of RIP, we will consider the property of quasi-Abelianity.

THEOREM 2.6. *For a ring R, the following conditions are equivalent.*

(i) *R is quasi-Abelian.*
(ii) *$\mathbb{M}_n(R)$ is quasi-Abelian, for every $n \geq 2$.*
(iii) *$\mathbb{M}_n(R)$ is quasi-Abelian, for some $n \geq 2$.*

Proof. (i) ⇒ (ii): Let $e = \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix}$ be a left semicentral idempotent of $\mathbb{M}_2(R)$. For an arbitrary element r of R, consider the elements $\alpha = \begin{bmatrix} r & 0 \\ 0 & 0 \end{bmatrix}$ and $\beta = \begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix}$ in $\mathbb{M}_2(R)$. Since $(1-e)\alpha e = (1-e)\beta e = 0$, we have $(1-e_{11})Re_{ij} = 0$ and $e_{21}Re_{ij}$ for all i and j. So $e_{11} \in \mathcal{S}_l(R)$, and consequently, e_{11} is central due to the quasi-Abelianity of R. Also, $e_{21} = (1-e_{11})e_{21} + e_{21}e_{11} = 0$. Similarly, one can show that e_{22} is a central idempotent and $e_{12} = 0$. Now, $(1-e)\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} e = 0$ gives us $(1-e_{22})e_{11} = 0$, and thus, $e_{22} = e_{11}$. Therefore, e is a central idempotent.

Now, we will use induction on n to show that every left semicentral idempotent of $\mathbb{M}_n(R)$ is a scalar matrix with central entries. We assume that every semicentral idempotent of $\mathbb{M}_n(R)$ is a central scalar matrix. Let $\varepsilon = \begin{bmatrix} a & u' \\ v & f \end{bmatrix}$ be a left semicentral idempotent of $\mathbb{M}_{n+1}(R)$, where $a \in \mathbb{M}_n(R)$, $f \in R$, and $u, v \in R^n$. We should note that R^n is the $(\mathbb{M}_n(R), R)$-bimodule consisting of n-tuples of components from R with unity denoted as $\mathbf{1}_n$ and the notation v' indicates the transpose of vector v. Consider the elements $\begin{bmatrix} m & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & r \end{bmatrix}$ for arbitrary $m \in \mathbb{M}_n(R)$ and $r \in R$. Applying the left semicentrality of ε to these elements, we find that $a \in \mathcal{S}_l(\mathbb{M}_n(R))$ and $f \in \mathcal{S}_l(R)$. From the assumptions, a is a central idempotent scalar matrix of $\mathbb{M}_n(R)$ and $f \in \mathcal{B}(R)$. From $(1-\varepsilon)\begin{bmatrix} 0 & \mathbf{1}'_n \\ 0 & 0 \end{bmatrix}\varepsilon = (1-\varepsilon)\begin{bmatrix} 0 & 0 \\ \mathbf{1}_n & 0 \end{bmatrix}\varepsilon = 0$, we get $a = fI_n$, where I_n represents the identity matrix of $\mathbb{M}_n(R)$. Applying the left semicentrality of ε on the elements $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}$, we get $u = v = 0$. Therefore ε is a scalar central idempotent matrix, and $\mathbb{M}_{n+1}(R)$ is quasi-Abelian.

(ii) $\Rightarrow$ (iii) is a direct implication.

(iii) $\Rightarrow$ (i): If $e \in \mathcal{S}_l(R)$, then $eI_n \in \mathcal{S}_l(\mathbb{M}_n(R))$, and as a result, eI_n is central. Therefore, $e \in \mathcal{B}(R)$, and this implies that R is quasi-Abelian. □

Indeed, the class of quasi-Abelian rings is not closed under subrings, as exemplified by the ring $\mathbb{M}_2(F)$, which is quasi-Abelian, whereas its subring $\mathbb{T}_2(F)$ is not quasi-Abelian. This is evidenced by the element $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, which is a left semicentral idempotent but not central in $\mathbb{T}_2(F)$.

In addition, the class of quasi-Abelian rings is not closed under homomorphic images. However, it is worth noting that there exist nontrivial homomorphic images of quasi-Abelian rings that are also quasi-Abelian.

Example 2.7. (i) Let F be a field and $R = F\langle a, b\rangle$ be the free algebra with noncommuting indeterminates a and b over F. Then, R is reduced and so quasi-Abelian. Let I be the ideal of R generated by $arb, a - a^2, ra - ara$, for every $r \in R$. Then $a \in \mathcal{S}_l(R/I)$, but $ar - ara \notin I$, for some $r \in R$ and $a \notin \mathcal{S}_r(R/I)$. Thus R/I is not quasi-Abelian.

(ii) The ring $R = \mathbb{T}_n(F)$, for some filed F, is not quasi-Abelian for any $n \geq 2$. Although, the ring $e_{11}Re_{11} \cong F$ is quasi-Abelian.

(iii) Let $R = \mathbb{H}(\mathbb{Z})$ be the ring of quaternions with integer coefficients. R is a domain, and therefore it is quasi-Abelian. Furthermore, as stated in [9, Exercise 2A], $R/pR \cong \mathbb{M}_n(\mathbb{Z}_p)$, which is a prime ring and hence semicentral reduced. Consequently, R/pR is also quasi-Abelian.

Here, we provide a sufficient condition for an ideal I to make a ring R quasi-Abelian, given that R/I is also quasi-Abelian.

Proposition 2.8. *Consider a ring R/I which is quasi-Abelian for some ideal I of the ring R. If I is a reduced ring (possibly without an identity), then it follows that R is also quasi-Abelian.*

Proof. Let $e \in \mathcal{S}_l(R)$. Given that R/I is quasi-Abelian and $0 = eR(1-e) \subseteq I$, we can deduce that $(1-e)Re \subseteq I$. Consequently, if $((1-e)Re)^2 = 0$, it implies that $(1-e)Re = 0$, since I is a reduced ring. As a result, R is also quasi-Abelian. □

The reducedness of I in the previous proposition cannot be dropped, as demonstrated in the following example.

EXAMPLE 2.9. The ring $R = \mathbb{T}_2(F)$, where F is a field, is not quasi-Abelian. The only non-zero proper ideals of R are

$$I_1 = \begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix}, I_2 = \begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad I_2 = \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix}.$$

But $R/I_1 \cong R/I_2 \cong F$ and $R/I_3 = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + I_3 \mid a, b \in F \right\}$ is a reduced ring, and hence each R/I_i is quasi-Abelian although all I_i are not reduced.

Furthermore, a nontrivial corner of a quasi-Abelian ring may necessarily exhibits the quasi-Abelian property, as demonstrated in the following proposition.

EXAMPLE 2.10. Consider the ring $R = \mathbb{M}_3(\mathbb{R})$, consisting of 2-by-2 matrices with real entries. Let e be the idempotent defined as:

$$e = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

The corner eRe is defined as the set of matrices:

$$eRe = \left\{ \begin{bmatrix} a & a & b \\ -c & -c & -d \\ c & c & c \end{bmatrix} \mid a, b, c, d \in \mathbb{R} \right\}$$

While R is quasi-Abelian, as indicated by Theorem 2.6, eRe does not share this property. This is due to the idempotent $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ in eRe, which is left semicentral but not central.

Notice that even if every corner eRe of a ring R is quasi-Abelian for all non-identity idempotents e, R is not necessarily quasi-Abelian. This can be observed in the non-quasi-Abelian ring $R = \mathbb{T}_2(\mathbb{Z})$ whose every idempotent e satisfies $eRe \cong \mathbb{Z}$.

PROPOSITION 2.11. (i) *The class of quasi-Abelian rings is closed under direct sums and direct products.*

(ii) *For a central idempotent e of a ring R, eR and $(1-e)R$ are both quasi-Abelian if and only if R is quasi-Abelian.*

Proof. The proof is routine. □

Here are some examples that demonstrate the independence of the classes of quasi-Abelian rings, NI rings, and directly finite rings.

EXAMPLE 2.12. (i) Indeed, for a field F, consider the ring $R = \mathbb{T}_2(F)$. It is a 2-primal ring and hence an NI ring. Also, R is directly finite. However, the idempotent $e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ of R is left semicentral but not central. This fact establishes that R is not a quasi-Abelian ring.

(ii) Indeed, for a field K and an integer $n \geq 2$, let's consider the ring $R = K\langle a, b \mid a^n = 0\rangle$, which is the free algebra with two noncommuting indeterminates a and b over K subject to the relation $a^n = 0$. According to [16, Example 2.3 (2)], R is RIP (hence, quasi-Abelian) but not NI.

(iii) Indeed, according to [22, Theorem 1.0], the ring $R = \mathbb{M}_2(D)$ is not directly finite for some domain D. However, R is quasi-Abelian.

3. Quasi-Abelian property for subrings of upper triangular matrix rings

For any ring R and an integer $n \geq 2$, the ring $\mathbb{T}_n(R)$ is not quasi-Abelian, where the element e_{11}, with 1 in the $(1,1)$-position and 0 elsewhere, is left semicentral idempotent but not central. In this section, we discuss the extension of quasi-Abelianity of a ring R to some subrings of its ring of upper triangular matrices.

Initially, we provide a description of the set of left semicentral idempotents for these extensions. It is important to note that all results presented in this section exhibit left-right symmetry. Therefore, any condition for a ring R that depends on $\mathcal{S}_l(R)$ can be equivalently obtained for $\mathcal{S}_r(R)$ analogously.

For a ring R, the trivial extension of R by R is denoted as $T(R,R)$ and is defined as the ring $R \oplus R$ with the usual addition and multiplication operations given by $(r_1, s_1)(r_2, s_2) = (r_1 r_2, r_1 s_2 + s_1 r_2)$. It is worth noting that $T(R,R)$ can be represented as:

$$T(R,R) \cong \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \mid a, b \in R \right\},$$

where a and b are elements of R.

Lemma 3.1. *For a ring R, the set of left semicentral idempotent of the ring $T(R,R)$ is*

$$\mathcal{S}_l(T(R,R)) = \{(f, fr(1-f)) \mid f \in \mathcal{S}_l(R) \text{ and } r \in R\}.$$

Proof. Let $e = \begin{bmatrix} e_1 & e_2 \\ 0 & e_1 \end{bmatrix}$ be a left semicentral idempotent of $T(R,R)$. We can conclude that $e_1 \in \mathcal{S}_l(R)$. Therefore, we have $e_1 x e_2 + e_2 x e_1 - x e_2 = 0$ for every $x \in R$. Multiplying the last equation by e_1 from the left, we obtain $e_1 e_2 R e_1 = 0$, and consequently, $e_2 R e_1 = 0$ since e_1 is a left semicentral idempotent. This implies $(1 - e_1) R e_2 = 0$, and we can deduce that $e_2 \in e_1 R(1 - e_1)$.

Conversely, consider an element $\alpha = \begin{bmatrix} f & fr(1-f) \\ 0 & f \end{bmatrix} \in R$ for some $f \in \mathcal{S}_l(R)$ and $r \in R$. It is obvious that α is idempotent. For every $\beta = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ in $T(R,R)$, we have

$$\begin{aligned} (1-\alpha)\beta\alpha &= \begin{bmatrix} 1-f & -fr(1-f) \\ 0 & 1-f \end{bmatrix} \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \begin{bmatrix} f & fr(1-f) \\ 0 & f \end{bmatrix} \\ &= \begin{bmatrix} (1-f)af & (1-f)bfr(1-f) - fr(1-f)af \\ 0 & (1-f)af \end{bmatrix} = 0, \end{aligned}$$

which implies that α is a left semicentral idempotent. $\square$

Let R be a ring and define subrings of $\mathbb{T}_n(R)$ as follows:

$$D_n(R) = \left\{ \begin{bmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{bmatrix} \mid a, a_{ij} \in R \right\},$$

for all integers $n \geq 2$. The next proposition describes the left semicentral idempotents of $D_n(R)$.

Proposition 3.2. *For a ring R and a positive integer $n \geq 2$, the set of left semicentral idempotents of the ring $D_n(R)$ is*

$$\mathcal{S}_l(D_n(R)) = \left\{ \begin{bmatrix} e & e_{12} & e_{13} & \cdots & e_{1n} \\ 0 & e & e_{23} & \cdots & a_{2n} \\ 0 & 0 & e & \cdots & e_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e \end{bmatrix} \mid e \in \mathcal{S}_l(R) \text{ and } e_{ij} \in eR(1-e) \right\}.$$

Proof. Utilizing induction for the case of n, we can demonstrate the validity of the definition of set for $n = 2$ directly from Lemma 3.1, where $D_2(R) \cong T(R, R)$. Assuming that the provided definition of left semicentral idempotents holds for a certain n, we consider the left semicentral idempotent $\varepsilon = \begin{bmatrix} \alpha & \beta \\ 0 & e \end{bmatrix}$ of $D_{n+1}(R)$, where $\alpha =$ $\begin{bmatrix} e & e_{12} & \cdots & e_{1(n-1)} & e_{1n} \\ 0 & e & \cdots & e_{2(n-1)} & e_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & e & e_{(n-1)n} \\ 0 & 0 & 0 & \cdots & e \end{bmatrix}$ and $\beta = \begin{bmatrix} e_{1(n+1)} \\ e_{2(n+1)} \\ \vdots \\ e_{n(n+1)} \end{bmatrix}$. In fact, from $(1-\varepsilon)D_{n+1}(R)\varepsilon = 0$, we deduce $(1-\alpha)D_n(R)\alpha = 0$, and $\alpha \in \mathcal{S}_l(D_n(R))$. Consequently, $e \in \mathcal{S}_l(R)$, and $e_{ij} \in eR(1-e)$, for every $1 \leq i < j \leq n$, form the assumption. Additionally, the equation $(1-\alpha)a\beta + (1-\alpha)be - \beta a_{11}e = 0$, holds for every $a = [a_{ij}] \in D_n(R)$ and $b \in R^n$. By multiplying the previous equation with $(1-e)I_n$ from the left, where I_n represents the identity matrix of $\mathbb{M}_n(R)$, we obtain $(1-e)a\beta + (1-e)be - (1-e)\beta a_{11}e = 0$. Setting $a = 1$ and $b = 0$, it follows that $(1-e)\beta(1-e) = 0$. Given that $e \in \mathcal{S}_l(R)$ and $(1-e)\beta e = 0$, we conclude that $(1-e)\beta = 0$. Returning to the equation $(1-\alpha)a\beta + (1-\alpha)be - \beta a_{11}e = 0$ and selecting $a = \alpha$ and $b = 0$ yields $\beta e = 0$. Thus, $\beta = e\beta(1-e)$, implying that $e_{i(n+1)} \in eR(1-e)$ for $i = 1, \cdots, n$, thereby completing the proof. □

Let R be a ring and define subrings of $\mathbb{T}_n(R)$ as following:

$$V_n(R) = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ 0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 0 & a_1 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_1 \end{bmatrix} \mid a_i \in R \right\}.$$

The next proposition describes the left semicentral idempotents of $V_n(R)$.

Proposition 3.3. *For a ring R and a positive integer $n \geq 2$, the set of left semicentral idempotents of the ring $V_n(R)$ is*

$$\mathcal{S}_l(D_n(R)) = \left\{ \begin{bmatrix} e_1 & e_2 & e_3 & \cdots & e_n \\ 0 & e_1 & e_2 & \cdots & a_{n-1} \\ 0 & 0 & e_1 & \cdots & e_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e_1 \end{bmatrix} \mid e_1 \in \mathcal{S}_l(R) \text{ and } e_i \in e_1R(1-e_1) \text{ for } i \geq 2 \right\}.$$

Proof. We can employ a similar methodology as demonstrated in the proof of Proposition 3.2, where $V_2(R) \cong T(R,R)$. □

The next theorem establishes the extension of the quasi-Abelian property from a ring R to the subrings $D_n(R)$ and $V_n(R)$ of $\mathbb{T}_n(R)$.

Theorem 3.4. *For a ring R and an integer $n \geq 2$, the following statements are equivalent:*

(i) *R is quasi-Abelian;*
(ii) *$D_n(R)$ is quasi-Abelian;*
(iii) *$V_n(R)$ is quasi-Abelian.*

Proof. (i) ⇒ (ii): Suppose $\varepsilon \in \mathcal{S}_l(D_n(R))$. Then, we can express ε as follows:

$$\varepsilon = \begin{bmatrix} e & e_{12} & e_{13} & \cdots & e_{1n} \\ 0 & e & e_{23} & \cdots & a_{2n} \\ 0 & 0 & e & \cdots & e_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e \end{bmatrix}$$

where $e \in \mathcal{S}_l(R)$ and $e_{ij} \in eR(1-e)$, as show in Proposition 3.2. However, since R is quasi-Abelian, it follows that e is a central element. Consequently, we can deduce that $\varepsilon = eI_n \in \mathcal{B}(D_n(R))$, where I_n represents the identity matrix of $\mathbb{M}_n(R)$. Thus $D_n(R)$ is quasi-Abelian.

(ii) ⇒ (i): Let $e \in \mathcal{S}_l(R)$. This implies that $eI_n \in \mathcal{S}_l(D_n(R))$. Given that $D_n(R)$ is quasi-Abelian, we can infer that $eI_n \in \mathcal{B}(D_n(R))$, which leads to the conclusion that $e \in \mathcal{B}(R)$. Thus R is a quasi-Abelian ring.

(i)⇔(iii) can be established using analogous arguments to those applied in proving the equivalence between (i) and (ii), making use of Proposition 3.3. □

Corollary 3.5. *For a ring R, an integer $m, n \geq 2$, and an indeterminate X, the following statements are equivalent:*

(i) *R is quasi-Abelian;*
(ii) *$T(R,R)$ is quasi-Abelian;*
(iii) *$R[X]/\langle X^m \rangle$ is quasi-Abelian;*

Proof. It is clear since $T(R,R) \cong D_2(R)$ and $R[x]/\langle x^n \rangle \cong V_n(R)$. □

4. Extensions of quasi-Abelian rings

In this section, we establish the quasi-Abelian property of various ring extensions. Recall [23] that if R is a ring and S is a multiplicatively closed subset of R satisfying

$1 \in S$ and $s_1, s_2 \in \Gamma$ implying $s_1 s_2 \in S$, then the *localization* of R at S is a ring denoted by $\Gamma^{-1}R$. This localization comes from a ring homomorphism $\phi : R \to \Gamma^{-1}R$, which satisfies the following properties:

- $\phi(s)$ is invertible for every $s \in \Gamma$.
- Every element in $\Gamma^{-1}R$ can be expressed as $\phi(a)(\phi(s))^{-1}$, where $s \in \Gamma$.
- $\phi(a) = 0$ if and only if $as = 0$ for some $s \in \Gamma$.

Moreover, if Γ consists of central invertible elements, then $\Gamma^{-1}R$ exists, as proven in [23, Chapter 2, Proposition 1.4].

Recalling [16], for a ring R, let Γ be a multiplicatively closed subset of R. It is not generally true that there exist $e \in \mathcal{I}(R)$ and $u \in \Gamma$ such that $\varepsilon = u^{-1}e$, whenever $\varepsilon \in \mathcal{I}(\Gamma^{-1}R)$. However, if $\varepsilon = u^{-1}e$ is a left (resp. right) semicentral idempotent of $\Gamma^{-1}R$, then e must also be a left (resp. right) semicentral idempotent of R, as demonstrated in the following lemma.

LEMMA 4.1. *If R is a ring and Γ a multiplicatively closed subset of R consisting of central regular elements, then $\mathcal{S}_l(\Gamma^{-1}R) \subseteq \Gamma^{-1}\mathcal{S}_l(R)$.*

Proof. Let $\varepsilon = u^{-1}e$, with $e \in R$ and $u \in \Gamma$, be a left semicentral idempotent of $\Gamma^{-1}R$. Hence, for every $r \in R$ and $v \in \Gamma$, we have $(u^{-1}e)(v^{-1}r)(u^{-1}e) = (v^{-1}r)(u^{-1}e)$. Choosing $v = 1$ and $r = u$, we get $u^{-1}e = e$. So, $e(v^{-1}r)e = (v^{-1}r)e$ and $eRe = Re$; that $e \in \mathcal{S}_l(R)$. □

PROPOSITION 4.2. *Let R be a ring and Γ a multiplicatively closed subset of R consisting of central regular elements. Then R is quasi-Abelian if and only if the localization of R at Γ is quasi-Abelian.*

Proof. First, let R be quasi-Abelian and $\varepsilon \in \mathcal{S}_l(\Gamma^{-1}R)$. So, $\varepsilon = u^{-1}e$, for $e \in \mathcal{S}_l(R)$ and $u \in \Gamma$, from Lemma 4.1. So, $0 = (1 - u^{-1}e)(v^{-1}r)u^{-1}e = (uv)^{-1}(1 - u^{-1}e)re$, for every $v \in \Gamma$ and $r \in R$, and so $(1 - u^{-1}e)re = 0$. Choosing $r = 1$, we get $\varepsilon = u^{-1}e = e$. Hence, $(1 - e)Re = 0$ and $e \in \mathcal{S}_l(R)$. So e is central since R is quasi-Abelian. Therefore, ε is central and $\Gamma^{-1}R$ is a quasi-Abelian ring.

Conversely, assume that $\Gamma^{-1}R$ is a quasi-Abelian ring and $(1 - e)Re = 0$ for some $e \in \mathcal{I}(R)$. For any $r \in R$ and $u \in \Gamma$, we have $(1 - e)(v^{-1}r)e = v^{-1}(1 - e)re = 0$. So, $(1 - e)(\Gamma^{-1}R)e = 0$ and therefore $e(\Gamma^{-1}R)(1 - e) = 0$, since $\Gamma^{-1}R$ is quasi-Abelian. Thus $eR(1 - e) = 0$ and R is quasi-Abelian. □

THEOREM 4.3. *For a ring R, the following statements are equivalent:*

(i) *R is quasi-Abelian;*
(ii) *$R[x]$ is quasi-Abelian;*
(iii) *$R[[x]]$ is quasi-Abelian;*
(iv) *$R[x, x^{-1}]$ is quasi-Abelian;*
(v) *$R[[x, x^{-1}]]$ is quasi-Abelian.*

Proof. The equivalence between (ii)$\Leftrightarrow$(iv) and (iii)$\Leftrightarrow$(v) follows directly from Proposition 4.2 by considering $\Gamma = \{1, x, x^2, \ldots\}$, as $R[x, x^{-1}] = \Gamma^{-1}R[x]$ and $R[[x, x^{-1}]] = \Gamma^{-1}R[[x]]$. The implications from (ii) or (iii) to (i) are straightforward since $\mathcal{S}_l(R) \subseteq \mathcal{S}_l(T)$ whenever T is $R[x]$ or $R[[x]]$. For (i)$\Rightarrow$(ii), we consider $e(x) = e_0 + e_1x + \cdots + e_nx^n$ in $\mathcal{S}_l(R[x])$. By [5, Proposition 2.4], we have $e_0 \in \mathcal{S}_l(R)$, $e_0e_i = e_i$, and $e_ie_0 = 0$ for $i = 0, 1, \ldots, n$. Therefore, due to the quasi-Abelianity of R, e_0 is central, and $e_i = 0$ for every $i \geq 1$. Thus, $e(x) = e_0 \in \mathcal{B}(R) \subseteq \mathcal{B}(R[x])$. Hence, $R[x]$ is quasi-Abelian. Similarly, (i)$\Rightarrow$(iii) is proven by employing [5, Proposition 2.5]. □

Let R be an S-algebra over a commutative ring S. Recall that the *Dorroh extension* of R by S is the ring $D = R \times_{\mathrm{Dor}} S$ with operations defined as follows: $(r_1, s_1) + (r_2, s_2) = (r_1 + r_2, s_1 + s_2)$ and $(r_1, s_1)(r_2, s_2) = (r_1 r_2 + s_1 r_2 + s_2 r_1, s_1 s_2)$, for every $r_1, r_2 \in R$ and $s_1, s_2 \in S$. The following proposition extends the notion of quasi-Abelianity from a ring to its Dorroh extension.

Proposition 4.4. *Let R be an algebra over a commutative ring S. Then the Dorroh extension of R by S is quasi-Abelian.*

Proof. Let $e = (e_1, e_2) \in \mathcal{S}_l(R \times_{\mathrm{Dor}} S)$. Then $(1-e)(R \times_{\mathrm{Dor}} S)e = 0$ and $(-e_1, 1-e_2)(r, s)(e_1, e_2) = 0$, for every $s \in S$ and $r \in R$. This yields $-e_1 r e_1 + (1-e_2) r e_1 - s e_1 e_1 - e_2 e_1 r + (1-e_2) s e_1 + (1-e_2) e_2 r - s e_2 e_1 = 0$, and $(1-e_2) s e_2 = 0$. So, e_2 is a central idempotent, and we have the equation $-e_1 r e_1 + r e_1 - e_2 r e_1 - s e_1^2 - e_2 e_1 r + s e_1 - e_2 s e_1 - s e_2 e_1 = 0$. For every $s \in S$ and $r \in R$, we have $(e_1, e_2)(r, s)(-e_1, 1-e_2) = (-e_1 r e_1 - e_2 r e_1 - s e_1 e_1 + (1-e_2) e_1 r - e_2 s e_1 + e_2(1-e_2) r + s(1-e_2) e_1, e_2 s(1-e_2)) = (-e_1 r e_1 - e_2 r e_1 - s e_1^2 + e_1 r - e_2 e_1 r - e_2 s e_1 + s e_1 - s e_2 e_1, 0) = (-e_1 r e_1 - e_2 r e_1 - s e_1^2 + e_1 r - e_2 e_1 r - e_2 s e_1 + s e_1 - s e_2 e_1, 0) = 0$. From these observations, it follows that $e \in \mathcal{S}_r(R \times_{\mathrm{Dor}} S)$ and $R \times_{\mathrm{Dor}} S$ is quasi-Abelian. $\square$

Recall *unitization* of a ring R is the Dorroh extension of R by $\mathbb{Z}$.

Corollary 4.5. *The unitization of a ring R is quasi-Abelian.*

5. Applications

According to [22], a ring R is termed as *torsion-free* if there exists a prime ideal P of R such that $\mathcal{O}(P) = 0$, where $\mathcal{O}(P) = \{a \in R \mid aRb = 0 \text{ for some } b \in R \setminus P\}$. The following proposition outlines a significant property for rings that are both quasi-Abelian and torsion-free.

Proposition 5.1. *If R is a quasi-Abelian and torsion-free ring, then R is semicentral reduced.*

Proof. Assume that $\mathcal{O}(P) = 0$ for some prime ideal P of R, and let $e \in \mathcal{S}_l(R) \cup \mathcal{S}_r(R)$. As R is quasi-Abelian, we have $e \in \mathcal{B}(R)$, and due to quasi-Abelianity, we know that $(1-e)Re = eR(1-e) = 0$. If $e \notin P$, then $1 - e \in \mathcal{O}(P)$, and $e = 1$. If $e \in P$, then $1 - e \notin P$, and hence $e \in \mathcal{O}(P)$, and so $e = 0$. Thus, R is semicentral reduced. $\square$

As a consequence, we get Proposition 8 in [13] as a corollary.

Corollary 5.2. *Let R be an idempotent-reflexive p.q.-Baer ring. Then following conditions are equivalent:*

(i) *R is prime.*
(ii) *R is torsion-free.*

Proposition 5.3. *Let R be a quasi-Abelian ring. If R contains an injective maximal right ideal, then R is right self-injective.*

Proof. Let M be an injective maximal right ideal of R. Then, $R = M \oplus N$, where N is a minimal right ideal. Therefore, we can express M as $M = eR$ and $N = (1-e)R$, where e is a nontrivial idempotent of R. If $NM = 0$, it follows that $(1-e)Re = 0$, implying that e is central due to the quasi-Abelian property of R. Consequently, we have $R = Re \oplus R(1-e)$, and the left R-module R/M is projective. Hence, using [21, Lemma 1.1], R/M is R-flat. From [21, Proposition 1.4], it follows that R/M is injective. Consequently, N is an injective R-module. If $NM \neq 0$, we have $NM = N$, implying that there exists $b \in N$ such that $bM \neq 0$. Hence, we define the epimorphism $f : M \to N$ as $\phi(m) = bm$ for every $m \in M$. As the right R-module N is projective and $M/\ker\phi \cong N$, we can express M as $M \cong \ker\phi \oplus M/\ker\phi \cong \ker\phi \oplus N$ as right R-modules. Thus, N is an injective R-module. Consequently, $R = M \oplus N$ is right self-injective. □

In [11], a ring R is said to be a *right HI-ring* if R is a right hereditary ring containing an injective maximal right ideal. The next corollary extends Corollary 8 in [14].

COROLLARY 5.4. *For a ring R, the following statements are equivalent.*

(i) *R is semisimple and Artinian;*
(ii) *R is idempotent-reflexive and right HI-ring;*
(iii) *R is quasi-Abelian and right HI-ring.*

Proof. The proof is direct from the previous proposition and [20, Corollary 7]. □

For a nonempty subset S of a ring R, we denote the *right annihilator* of S in R as $r(S)$, defined as $r(S) = \{x \in R \mid Sx = 0\}$. Similarly, the *left annihilator* is defined and denoted as $\ell(S)$. In [6], a ring R is termed *right* (resp. *left*) *p.q.-Baer* if the right (resp. left) annihilator of a principally right (resp. left) ideal is a right (resp. left) ideal generated by an idempotent. Moreover, from [4], R is defined as a *right* (resp. *left*) $\mathfrak{cP}$*-Baer ring* if the right (resp. left) annihilator of a principally right (resp. left) idempotent-generated ideal is a right (resp. left) idempotent-generated ideal.

The next two propositions utilize the aforementioned Baer-like properties as sufficient conditions for a quasi-Abelian ring to be reflexive or idempotent-reflexive.

PROPOSITION 5.5. *If R is a quasi-Abelian and right (or left) p.q.-Baer ring, then R is reflexive.*

Proof. Let $aRb = 0$, for some $a, b \in R$. Then $b \in r(aR) = eR$ for some $e \in \mathcal{S}_l(R)$. Consequently, $aRe = 0$ and $b = eb$. However, since R is quasi-Abelian, e is central. Hence, $bRa = ebRa = bRae \subseteq bRaRe = 0$, which implies that R is reflexive. □

PROPOSITION 5.6. *If a ring R is quasi-Abelian and right (resp. left) $\mathfrak{cP}$-Baer, then R is left (resp. right) idempotent-reflexive.*

Proof. Indeed, it is sufficient to prove the right $\mathfrak{cP}$-Baer case. Let $eRa = 0$ for some idempotent $e^2 = e$ and $a \in R$. Then, $a \in r(eR) = fR$ for some $f \in \mathcal{S}_l(R)$. Consequently, we have $eRf = 0$ and $a = fa$. However, since R is quasi-Abelian, f is central. Hence, $aRe = faRe = aRef = 0$, and this demonstrates that R is idempotent-reflexive. □

Finally, we proceed to characterize the structure of non-Abelian quasi-Abelian rings of minimal order.

THEOREM 5.7. *If R is a non-Abelian quasi-Abelian ring of minimal order, then R is of order 16 and is isomorphic to $\mathbb{M}_2(\mathbb{Z}_2)$ or the ring* $\begin{bmatrix} \mathbb{Z}_2 & \overline{x}\mathbb{Z}_2[x]/\langle x^2\rangle \\ \overline{x}\mathbb{Z}_2[x]/\langle x^2\rangle & \mathbb{Z}_2 \end{bmatrix}$.

Proof. Consider a non-Abelian quasi-Abelian ring R of minimal order. It is worth noting that R cannot be a local ring, as local rings are known to be Abelian. Using the results in [8], it becomes apparent that any noncommutative ring of minimal order is isomorphic to $\mathbb{T}_2(\mathbb{Z}_2)$. However, $\mathbb{T}_2(\mathbb{Z}_2)$ is not quasi-Abelian. Consequently, the order of R must be greater than or equal to 16. Moreover, referring to [8], if a finite ring possesses a cube-free factorization, it must be commutative. However, since $\mathbb{M}_2(\mathbb{Z}_2)$ is both non-Abelian and quasi-Abelian, we can conclude that the order of R must indeed be 16.

According to the Wedderburn-Artin theorem, we have $R/\mathcal{J}(R) \cong \sum_{i=1}^{m} M_i(D_i)$, where k_i are positive integers and D_i are division rings. Next, let's consider the case where $k_i = 1$ for all i, and observe that $\mathbb{M}_2(\mathbb{Z}_2)$ is a quasi-Abelian ring. Therefore, we encounter three possibilities: $|\mathcal{J}(R)| = 2$, $|\mathcal{J}(R)| = 4$, and $|\mathcal{J}(R)| = 8$.

If $|\mathcal{J}(R)| = 8$, then we would have $R/\mathcal{J}(R) \cong \mathbb{Z}_2$, implying that R is a local ring. However, this contradicts the assumption that R is non-Abelian.

In the case of $|\mathcal{J}(R)| = 4$, we find that $R/\mathcal{J}(R) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. As R is non-Abelian, there exists a nontrivial idempotent e in R such that $er - re \neq 0$ for some $r \in R$. Consider the set $S_1 = \{0, 1, e, 1-e\}$ to obtain the expression $R = \{a + b \mid a \in S_1, b \in \mathcal{J}(R)\}$. Consequently, by [1, Proposition 2.7(1)], R has RIP, and specifically, it is quasi-Abelian.

For $|\mathcal{J}(R)| = 2$, we find that $R/\mathcal{J}(R) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, and thus $\mathcal{J}(R)$ is nilpotent. By using [18, Proposition 3.6.2], we can establish the existence of orthogonal nontrivial idempotents e_1, e_2, e_3 in R satisfying $e_1+e_2+e_3 = 1$. This collection of nontrivial idempotents forms a subset $S_2 = \{0, 1, e_1, e_2, e_3, 1-e_1, 1-e_2, 1-e_3\}$ of $\mathcal{I}(R)$. Furthermore, for any $r \in R$, we have $e_j r(1-e_j) = e_j r(e_i + e_k) = e_j r e_i + e_j r e_k = e_j b e_i + e_j b e_k = 0$, and $e_j R(1-e_j) = 0$, due to the quasi-Abelianity of R. Hence, $(1-e_j)Re_j = 0$, by the quasi-Abelianity of R. Therefore, $0 = (1-e_j)be_j = (e_i+e_k)be_j = e_i b e_j + e_k b e_j = e_i b e_j$, a contradiction. Consequently, we can conclude that $e_i R e_j = 0$ for all i, j with $i \neq j$. Given that R is non-Abelian, we can conclude that S_2 contains a non-central element, denoted as e_i, and $e_i x \neq x e_i$ for some $x \in R$. But $e_i x - x e_i = (e_i + e_j + e_k)(e_i x - x e_i)(e_i + e_j + e_k) = e_i(e_i x - x e_i)e_i + e_j(e_i x - x e_i)e_j + e_k(e_i x - x e_i)e_k = e_i x e_i - e_i x e_i = 0$, a contradiction. Thus, the case of $|\mathcal{J}(R)| = 8$ is impossible.

The remaining part of the proof can be deduced directly from references [1, Example 2.10] and [17, Theorem 4.2]. □

Observe that $\mathbb{M}_n(\mathbb{Z}_2)$ is semiprime for every $n \geq 1$, and as a result, we obtain the following corollary.

Corollary 5.8. *For a non-Abelian ring R, the following conditions are equivalent.*

(i) *R is a semiprime ring of minimal order if;*
(ii) *R is a quasi-Abelian ring of minimal order;*
(iii) *R is a RIP ring of minimal order;*
(iv) *R is a reflexive ring of minimal order;*
(v) *R is a one-sided idempotent-reflexive ring of minimal order;*
(vi) *R is a left idempotent-reflexive ring of minimal order.*

References

[1] A. M. Abdul-Jabbar, C. A. K. Ahmed, T. K. Kwak, and Y. Lee, *Reflexivity with maximal ideal axes,* Comm. Algebra **45** (10) (2017), 4348–4361.

[2] G. F. Birkenmeier, *Idempotents and completely semiprime ideals,* Comm. Algebra **11** (1983), 567–580.
[3] G. F. Birkenmeier, H. E. Heatherly, and E. K. Lee, *Completely prime ideals and associated radicals,* In Proc. Biennial Ohio State-Denison Conference, pages 102–129. World Scientific, 1992.
[4] G. F. Birkenmeier and B. J. Heider, *Annihilators and extensions of idempotent-generated ideals,* Comm. Algebra **47** (3) (2019), 1348 – 1375.
[5] G. F. Birkenmeier, J. Y. Kim, and J. K. Park, *On polynomial extensions of principally quasi-Baer rings,* Kyungpook Math. J. **40** (2000), 247–253.
[6] G. F. Birkenmeier, J. Y. Kim, and J. K. Park, *Principally quasi-Baer rings,* Comm. Algebra **29** (2001), 639–660.
[7] W. Chen, *On semiabelian π-regular rings,* Int. J. Math. Math. Sci **2007** (2007).
[8] K. Eldridge, *Orders for finite noncommutative rings with unity,* The American Mathematical Monthly **75** (5) (1968), 512–514.
[9] K. R. Goodearl and R. B. Warfield Jr, An introduction to noncommutative Noetherian rings, *Cambridge University Press* (2004).
[10] H. Heatherly and R. P. Tucci, *Central and semicentral idempotents,* Kyungpook Math. J **40** (2) (2000), 255–258.
[11] Z. Jule and D. Xianneng, *Hereditary rings containing an injective maximal left ideal,* Comm. Algebra **21** (12) (1993), 4473–4479.
[12] J. Y. Kim, *Certain rings whose simple singular modules are GP-injective,* Proc. Japan Acad. Ser. A Math. Sci. **81** (7) (2005), 125–128.
[13] J. Y. Kim, *On reflexive principally quasi-Baer rings,* Korean J. Math. **17** (3) (2009), 233–236.
[14] J. Y. Kim and J. U. Baik, *On idempotent reflexive rings,* Kyungpook Math. J. **46** (2006), 597–601.
[15] T. K. Kwak and Y. Lee, *Reflexive property of rings,* Comm. Algebra **40** (4) (2012), 1576–1594.
[16] T. K. Kwak and Y. Lee, *Reflexive property on idempotents,* Bull. Korean Math. Soc. **50** (6) (2013), 1957–1972.
[17] T. K. Kwak and Y. Lee, *Corrigendum to "reflexive property on idempotents" bull. korean math. soc.* 50 (2013), no. 6, 1957–1972], Bull. Korean Math. Soc. **53** (6) (2016), 1913–1915.
[18] J. Lambek, Lectures on rings and modules, volume 283, *American Mathematical Soccity* (2009).
[19] G. Marks, *On 2-primal ore extensions,* Comm. Algebra **29** (5) (2001), 2013–2023.
[20] B. Osofsky, *Rings all of whose finitely generated modules are injective,* J. Math. **14** (1964), 645–650.
[21] V. Ramamurthi, *On the injectivity and flatness of certain cyclic modules,* Proc. Am. Math. Soc. **48** (1) (1975), 21–25.
[22] J. Shepherdson, *Inverses and zero divisors in matrix rings,* Proc. London Math. Soc. **3** (1) (1951), 71–85.
[23] B. Stenström, Rings of quotients, *Springer-Verlag, Berlin Heidelberg New York* (1975).

Muhammad Saad
Department of Mathematics and Computer Science, Faculty of Science,
Alexandria University, Alexandria 21568, Egypt
E-mail: m.saad@alexu.edu.eg