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Some Triviality Characterizations on Gradient Almost Yamabe Solitons

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Abstract. An almost Yamabe soliton is a generalization of the Yamabe soliton. In this article, we deduce some results regarding almost gradient Yamabe solitons. More specifically, we show that a compact almost gradient Yamabe soliton having non-negative Ricci curvature is trivial. Again, we prove that an almost gradient Yamabe soliton with a non-negative potential function and scalar curvature bound admitting an integral condition is trivial. Moreover, we give a characterization of a compact almost gradient Yamabe solitons.

1. Introduction

Barbosa and Ribeiro [1] laid the foundation of almost Yamabe solitons. A complete Riemannian manifold (M^n, g) of dimension n is known as an almost Yamabe soliton if there is a complete vector field X on M as well as $\lambda \in C^\infty(M)$ so that

$$\frac{1}{2}\pounds_X g = (R - \lambda)g,$$

where the notation R denotes the scalar curvature of g. If X becomes the gradient vector field of some smooth function u on M, then the above equation reduces to the following:

$$\nabla^2 u = (R - \lambda)g, \tag{1.1}$$

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where ∇^2 denotes the Hessian operator. In this case, the almost Yamabe soliton is known as the almost gradient Yamabe soliton and is denoted by (M^n, g, u, λ). The functions u and λ are called the potential function and the soliton function, respectively. If λ is a constant, then the almost Yamabe soliton takes the form of the Yamabe soliton, which is a special type soliton to the Yamabe flow, introduced by Hamilton [10] in studying the Yamabe metric on a compact Riemannian manifold. For more work related to almost Yamabe solitons see [5, 6, 7]. All the manifolds in this article are taken without boundary.

Curvature estimation in Yamabe solitons is an active field of research in the area of differential geometry. In the case of compact Yamabe solitons, the scalar curvature can be fully estimated. Daskalopoulos and Sesum [4] first showed that in a compact gradient Yamabe soliton, the scalar curvature is constant. Later, Hsu [11] gave a shorter proof by showing that scalar curvature equals to the soliton constant. Barbosa and Ribeiro [1] proved that it is not possible for a compact Riemannian manifold with negative Ricci curvature to satisfy the nontrivial almost Yamabe soliton condition. For more work on Yamabe solitons see [3, 8, 12, 13]. In this paper, it is shown that a compact almost gradient Yamabe soliton admitting non-negative Ricci curvature is trivial. Our main results are the followings:

Theorem 1.1. *Suppose (M^n, g, u, λ) is an n-dimensional $(n \geq 3)$ compact orientable non-trivial almost gradient Yamabe soliton. If the Ricci curvature is non-negative, then the soliton is trivial.*

Theorem 1.2. *Let (M^n, g, u, λ) be an n-dimensional $(n \geq 3)$ almost gradient Yamabe soliton with $R \geq \lambda$ and $u \geq 0$. Then (M^n, g, u, λ) becomes trivial, if the following condition holds*

$$\lim_{r\to\infty} \frac{1}{r} \int_{B_p(r)} u^2 dv < \infty,$$

where $B_p(r)$ indicates an open ball with radius $r > 0$ and center at p.

Since the subharmonic property of a smooth convex function is indicated by its convexity [9], the following corollary can be easily drawn:

Corollary 1.1. *Let (M^n, g, u, λ) be an n-dimensional $(n \geq 3)$ almost gradient Yamabe soliton with non-negative convex potential function u. Then (M^n, g, u, λ) becomes trivial, if the following condition holds*

$$\lim_{r\to\infty} \frac{1}{r} \int_{B_p(r)} u^2 dv < \infty,$$

where $B_p(r)$ indicates an open ball with radius $r > 0$ and center at p.

Theorem 1.3. *Suppose (M^n, g, u, λ) is a compact n-dimensional $(n \geq 3)$ almost gradient Yamabe soliton having non-negative Ricci curvature. Then, $max_M(\lambda) \geq 0$.*

From the above theorem, the following corollary follows immediately:

Corollary 1.2. *Suppose (M^n, g, u, λ) is a compact n-dimensional $(n \geq 3)$ gradient Yamabe soliton having non-negative Ricci curvature. Then, the soliton is non-expanding, i.e., $\lambda \geq 0$.*

2. Proofs

To prove Theorem 1.1, the following lemmas are required.

Lemma 2.1. ([1]) *Suppose (M^n, g, u, λ) is a almost gradient Yamabe soliton, then*

1. $(R - \lambda)n = \Delta u$,
2. *if M is compact and $\int_M \langle \nabla\lambda, \nabla u\rangle \geq 0$, then the soliton is trivial.*

Lemma 2.2. ([2]) *Consider a compact Riemannian manifold admitting a conformal vector field X, the following identity satisfies:*

$$\int_M X \cdot R dv = 0.$$

Proof. [Proof of Theorem 1.1]
As the Ricci curvature satisfies the non-negativity condition everywhere in M, we can choose a constant $K \geq 0$ such that

$$Ric \geq (n-1)K.$$

Consider the following smooth function defined on M

$$Q = |\nabla u|^2 + \frac{1}{n}u^2.$$

By using Ricci identity and the summation convention on repeated indices, we compute

$$\begin{aligned}
\Delta Q &= \Big(2u_j u_{ji} + \frac{2}{n}uu_i\Big)_i \\
&= 2u_{ji}^2 + 2u_j u_{jii} + \frac{2}{n}u_i^2 + \frac{2}{n}u(\Delta u) \\
&\geq 2u_{ii}^2 + 2R_{ij}u_iu_j + 2u_j(\Delta u)_j + \frac{2}{n}|\nabla u|^2 + \frac{2}{n}u(\Delta u) \\
&\geq \frac{2(u_{ii})^2}{n} + 2R_{ij}u_iu_j + 2u_j(\Delta u)_j + \frac{2}{n}|\nabla u|^2 + \frac{2}{n}u(\Delta u) \\
&\geq 2(n-1)K|\nabla u|^2 + 2nu_j(R-\lambda)_j + \frac{2}{n}|\nabla u|^2 + \frac{2}{n}\Big(\frac{1}{2}\Delta(u^2) - |\nabla u|^2\Big) \\
&\geq 2K|\nabla u|^2 + 2u_j(R-\lambda)_j + \frac{1}{n}\Delta(u^2) \\
&= 2K|\nabla u|^2 + 2n\langle\nabla R, \nabla u\rangle - 2n\langle\nabla\lambda, \nabla u\rangle + \frac{1}{n}\Delta(u^2). \qquad (2.1)
\end{aligned}$$

Again, the manifold is closed. Hence, Stokes theorem implies

$$\int_M \Delta Q dv = 0. \tag{2.2}$$

Now, (2.1) and (2.2) together imply that

$$\begin{aligned} 0 &\geq \int_M \Big\{2K|\nabla u|^2 + 2n\langle \nabla R, \nabla u\rangle - 2n\langle \nabla\lambda, \nabla u\rangle + \frac{1}{n}\Delta(u^2)\Big\}dv \\ &= 2K\int_M |\nabla u|^2 + 2n\int_M \langle \nabla R, \nabla u\rangle - 2n\int_M \langle \nabla\lambda, \nabla u\rangle \\ &= 2K\int_M |\nabla u|^2 - 2n\int_M \langle \nabla\lambda, \nabla u\rangle. \end{aligned}$$

In the above inequality, we have used lemma 2.2 and the Divergence theorem. Hence, we get

$$n\int_M \langle \nabla\lambda, \nabla u\rangle \geq K\int_M |\nabla u|^2. \tag{2.3}$$

According to our assumption $K \geq 0$, which implies that

$$\int_M \langle \nabla\lambda, \nabla u\rangle \geq 0, \tag{2.4}$$

which shows that, according to the lemma 2.1, the soliton is trivial. □

Proof. [Proof of Theorem 1.2]
Putting the inequality $R \geq \lambda$ in the first equation of lemma 2.1, we get $\Delta u \geq 0$, i.e., u is subharmonic. Consider a cut-off function $\varphi \in C_0^\infty(p, 2r)$ such that

$$\begin{cases} 0 \leq \varphi \leq 1 & \text{in } B_p(2r) \\ \varphi = 1 & \text{in } B_p(r) \\ \varphi = 0 & \text{on } \partial B_p(2r) \\ |\nabla\varphi| \leq C/2r & \text{in } B_p(2r). \end{cases}$$

Using the product rule of Laplacian operator,

$$\Delta(u^2) = 2u\Delta u + 2|\nabla u|^2,$$

we get

$$\begin{aligned} 0 &\leq 2\int_{B_p(2r)} \varphi^2 u\Delta u dv \\ &= -2\int_{B_p(2r)} \varphi^2|\nabla u|^2 dv + \int_{B_p(2r)} \varphi^2\Delta(u^2)dv \\ &= -2\int_{B_p(2r)} \varphi^2|\nabla u|^2 dv - \int_{B_p(2r)} \langle \nabla\varphi^2, \nabla u^2\rangle dv \\ &= -2\int_{B_p(2r)} \varphi^2|\nabla u|^2 dv - \int_{B_p(2r)} \varphi u\langle \nabla\varphi, \nabla u\rangle dv. \end{aligned}$$

The Hölder inequality and the above inequality together imply that

$$\begin{aligned}\int_{B_p(2r)} \varphi^2|\nabla u|^2 dv &\leq -\int_{B_p(2r)} \varphi u\langle\nabla\varphi, \nabla u\rangle dv \\ &\leq \Big(\int_{B_p(2r)} \varphi^2|\nabla u|^2 dv\Big)^{1/2}\Big(\int_{B_p(2r)} u^2|\nabla\varphi|^2 dv\Big)^{1/2}.\end{aligned}$$

It follows that

$$\begin{aligned}\int_{B_p(2r)} |\nabla u|^2 dv &\leq \int_{B_p(2r)} \varphi^2|\nabla u|^2 dv \\ &\leq 2\int_{B_p(2r)} u^2|\nabla\varphi|^2 dv \\ &\leq \frac{C}{r^2}\int_{B_p(2r)} u^2 dv.\end{aligned}$$

Now taking limit in both sides, we get

$$\lim_{r\to\infty}\int_{B_p(2r)} |\nabla u|^2 dv \leq 0,$$

which implies that u is constant and the almost Yamabe soliton becomes trivial. □

Proof. [Proof of Theorem 1.3]
Let $p \in M$. Now choose an orthonormal basis $\{e_1, \cdots, e_n\}$ of T_pM in such a way that

$$Ric_p(e_i) = \mu_i e_i \ \forall i = 1, 2, \cdots, n,$$

where each μ_i's is non-zero real number. Then, for any $v \in \mathbb{S}^{n-1}$, we have

$$v = x_i e_i.$$

Therefore,

$$\sum_{i=1}^{n} x_i^2 = 1, \text{ and } R_p = \sum_{i=1}^{n} \mu_i.$$

Now, using the Stokes theorem, we calculate

$$\begin{aligned}\int_M \int_{v\in\mathbb{S}^{n-1}} Ric_p(v)dsdv &= \int_M \int_{x\in\mathbb{S}^{n-1}} \sum_{i=1}^{n} x_i^2\mu_i^2 dxdv \\ &= \int_M \Big\{ \sum_{i=1}^{n} \mu_i^2 \int_{x\in\mathbb{S}^{n-1}} x_i^2 dx \Big\} dv \\ &= \frac{Vol(\mathbb{S}^{n-1})}{n} \int_M R(p)dv \\ &= \frac{Vol(\mathbb{S}^{n-1})}{n} \int_M \Big\{ \frac{\Delta u}{n} + \lambda \Big\} dv \\ &\leq \frac{Vol(\mathbb{S}^{n-1})}{n} \max{}_M(\lambda) Vol(M).\end{aligned}$$

Since the left hand side is non-negative, it shows that $\max{}_M(\lambda) \geq 0$. □

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