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Generalized Chen's Conjecture for Biharmonic Maps on Foliations

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Abstract. In this paper, we prove the generalized Chen's conjecture for $(\mathcal{F},\mathcal{F}')$-biharmonic maps, such maps are critical points of the transversal bienergy functional.

1. Introduction

On a Riemannian geometry, harmonic maps play a central role to study the geometric properties. They are critical points of the energy functional $E(\phi)$ for smooth maps $\phi : (M, g) \to (M', g')$, where

$$E(\phi) = \frac{1}{2}\int_M |d\phi|^2 \mu_M,$$

where μ_M is the volume element. It is well known that harmonic map is a solution of the Euler-Largrange equation $\tau(\phi) = 0$, where $\tau(\phi) = \mathrm{tr}_g(\nabla d\phi)$ is the tension field.

In 1983, J. Eells and L. Lemaire extended the notion of harmonic map to bi-

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harmonic map, which is a critical points of the bienergy functional $E_2(\phi)$, where

$$E_2(\phi) = \frac{1}{2}\int_M |\tau(\phi)|^2 \mu_M.$$

It is well-known [9] that harmonic maps are always biharmonic. But the converse is not true. At first, B.Y. Chen [3] raised so called Chen's conjecture and later, R. Caddeo et al. [2] raised the generalized Chen's conjecture. That is,

Generalized Chen's conjecture: *Every biharmonic submanifold of a Riemannian manifold of non-positive curvature must be harmonic.*

About the generalized Chen's conjecture, Nakauchi et al. [19] showed the following.

Theorem 1.1. [19] *Let (M, g) be a complete Riemannian manifold and (M', g') be of non-positive sectional curvature. Then*

(1) *every biharmonic map $\phi : M \to M'$ with finite energy and finite bienergy must be harmonic.*

(2) *In the case $Vol(M) = \infty$, every biharmonic map with finite bienergy is harmonic.*

Now, we study the generalized Chen's conjecture for biharmonic maps on foliated Riemannian manifolds and extend Theorem 1.1 to foliations. Let $(M, g, \mathcal{F})$ and $(M', g', \mathcal{F}')$ be the foliated Riemannian manifolds. Let $\phi : M \to M'$ be a smooth foliated map, that is, map preserving the leaves. Then ϕ is said to be $(\mathcal{F}, \mathcal{F}')$-*harmonic map* [6] if ϕ is a critical point of the transversal energy $E_B(\phi)$, which is given by

$$E_B(\phi) = \frac{1}{2}\int_M |d_T\phi|^2 \mu_M,$$

where $d_T\phi = d\phi|_Q$ is the differential map of ϕ restricted to the normal bundle Q of $\mathcal{F}$. From the first variational formula for the transversal energy functional [12], it is trivial that $(\mathcal{F}, \mathcal{F}')$-harmonic map is a solution of $\tilde{\tau}_b(\phi) := \tau_b(\phi) + d_T\phi(\kappa_B) = 0$, where $\tau_b(\phi) = tr_Q(\nabla_{tr} d_T\phi)$ is the transversal tension field and κ_B is the basic part of the mean curvature form κ of $\mathcal{F}$.

The map ϕ is said to be $(\mathcal{F}, \mathcal{F}')$-*biharmonic map* if ϕ is a critical point of the transversal bienergy functional $\tilde{E}_{B,2}(\phi)$, where

$$\tilde{E}_{B,2}(\phi) = \frac{1}{2}\int_M |\tilde{\tau}_b(\phi)|^2 \mu_M.$$

By the first variation formula for the transversal bienergy functional $\tilde{E}_{B,2}(\phi)$ (Theorem 3.7), we know that $(\mathcal{F}, \mathcal{F}')$-harmonic map is always $(\mathcal{F}, \mathcal{F}')$-biharmonic. But the converse is not true. So we prove the generalized Chen's conjecture for $(\mathcal{F}, \mathcal{F}')$-biharmonic map. That is, we prove the following theorem

Theorem 1.2. (cf. Theorem 3.10) *Let $(M, g, \mathcal{F})$ be a foliated Riemannian manifold and let $(M', g', \mathcal{F}')$ be of non-positive transversal sectional curvature $K^{Q'}$, that is, $K^{Q'} \leq 0$. Let $\phi : M \to M'$ be a $(\mathcal{F}, \mathcal{F}')$-biharmonic map. Then*

(1) *if M is closed, then ϕ is automatically $(\mathcal{F}, \mathcal{F}')$-harmonic;*

(2) *if M is complete with $Vol(M) = \infty$ and $\tilde{E}_{B,2}(\phi) < \infty$, then ϕ is $(\mathcal{F}, \mathcal{F}')$-harmonic.*

(3) *If M is complete with $E_B(\phi) < \infty$ and $\tilde{E}_{B,2}(\phi) < \infty$, then ϕ is $(\mathcal{F}, \mathcal{F}')$ -harmonic.*

Remark 1.3. On foliations, there is another kinds of harmonic map, called *transversally harmonic map*, which is a solution of the Eular-Lagrange equation $\tau_b(\phi) = 0$ [15]. Also, the *transversally biharmonic map* is defined [4], which is not a critical point of the bienergy $\tilde{E}_{B,2}(\phi)$. Two definitions for harmonic maps are equivalent when the foliation is minimal. The generalized Chen's conjectures for transversally biharmonic map have been proved in [11, 13].

2. Preliminaries

Let $(M, g, \mathcal{F})$ be a foliated Riemannian manifold of dimension n with a foliation $\mathcal{F}$ of codimension $q(= n-p)$ and a bundle-like metric g with respect to $\mathcal{F}$ [18, 23]. Let $Q = TM/T\mathcal{F}$ be the normal bundle of $\mathcal{F}$, where $T\mathcal{F}$ is the tangent bundle of $\mathcal{F}$. Let g_Q be the induced metric by g on Q, that is, $g_Q = \sigma^*(g|_{T\mathcal{F}^\perp})$, where $\sigma : Q \to T\mathcal{F}^\perp$ is the cnonical bundle isomorphism. Then g_Q is the holonomy invariant metric on Q, meaning that $L_X g_Q = 0$ for $X \in T\mathcal{F}$, where L_X is the transverse Lie derivative with respect to X. Let ∇^Q be the transverse Levi-Civita connection on the normal bundle Q [23, 24] and R^Q be the transversal curvature tensor of $\nabla^Q \equiv \nabla$, which is defined by $R^Q(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ for any $X, Y \in \Gamma TM$. Let K^Q and Ric^Q be the transversal sectional curvature and transversal Ricci operator with respect to ∇, respectively. Let $\Omega_B^r(\mathcal{F})$ be the space of all *basic r-forms*, i.e., $\omega \in \Omega_B^r(\mathcal{F})$ if and only if $i(X)\omega = 0$ and $L_X\omega = 0$ for any $X \in \Gamma T\mathcal{F}$, where $i(X)$ is the interior product. Then $\Omega^*(M) = \Omega_B^*(\mathcal{F}) \oplus \Omega_B^*(\mathcal{F})^\perp$ [1]. It is well known that κ_B is closed, i.e., $d\kappa_B = 0$, where κ_B is the basic part of the mean curvature form κ [1, 20] . Let $\bar{*} : \Omega_B^r(\mathcal{F}) \to \Omega_B^{q-r}(\mathcal{F})$ be the star operator given by

$$\bar{*}\omega = (-1)^{(n-q)(q-r)} * (\omega \wedge \chi_{\mathcal{F}}), \quad \omega \in \Omega_B^r(\mathcal{F}),$$

where $\chi_{\mathcal{F}}$ is the characteristic form of $\mathcal{F}$ and $*$ is the Hodge star operator associated to g. Let $\langle \cdot, \cdot \rangle$ be the pointwise inner product on $\Omega_B^r(\mathcal{F})$, which is given by

$$\langle \omega_1, \omega_2 \rangle \nu = \omega_1 \wedge \bar{*}\omega_2,$$

where ν is the transversal volume form such that $*\nu = \chi_{\mathcal{F}}$. Let $\delta_B : \Omega_B^r(\mathcal{F}) \to \Omega_B^{r-1}(\mathcal{F})$ be the operator defined by

$$\delta_B\omega = (-1)^{q(r+1)+1}\bar{*}(d_B - \kappa_B\wedge)\bar{*}\omega,$$

where $d_B = d|_{\Omega_B^*(\mathcal{F})}$. It is well known [22] that δ_B is the formal adjoint of d_B with respect to the global inner product. That is,

$$\int_M \langle d\omega_1, \omega_2\rangle \mu_M = \int_M \langle \omega_1, \delta_B\omega_2\rangle \mu_M$$

for any compactly supported basic forms ω_1 and ω_2, where $\mu_M = \nu \wedge \chi_{\mathcal{F}}$ is the volume element.

There exists a bundle-like metric such that the mean curvature form satisfies $\delta_B\kappa_B = 0$ on compact manifolds [5, 16, 17]. The basic Laplacian Δ_B acting on $\Omega_B^*(\mathcal{F})$ is given by

$$\Delta_B = d_B\delta_B + \delta_B d_B.$$

Now we define the bundle map $A_Y : \Gamma Q \to \Gamma Q$ for any $Y \in TM$ by

$$A_Y s = L_Y s - \nabla_Y s, \tag{2.1}$$

where $L_Y s = \pi[Y, Y_s]$ for $\pi(Y_s) = s$. It is well-known [14] that for any infitesimal automorphism Y (that is, $[Y, Z] \in \Gamma T\mathcal{F}$ for all $Z \in \Gamma T\mathcal{F}$ [14])

$$A_Y s = -\nabla_{Y_s}\pi(Y),$$

where $\pi : TM \to Q$ is the natural projection and Y_s is the vector field such that $\pi(Y_s) = s$. So A_Y depends only on $\bar{Y} = \pi(Y)$ and is a linear operator. Moreover, A_Y extends in an obvious way to tensors of any type on Q [14]. Then we have the generalized Weitzenböck formula on $\Omega_B^*(\mathcal{F})$ [10]: for any $\omega \in \Omega_B^r(\mathcal{F})$,

$$\Delta_B\omega = \nabla_{\rm tr}^*\nabla_{\rm tr}\omega + F(\omega) + A_{\kappa_B^\sharp}\omega, \tag{2.2}$$

where $F(\omega) = \sum_{a,b}\theta^a \wedge i(E_b)R^Q(E_b, E_a)\omega$ and

$$\nabla_{\rm tr}^*\nabla_{\rm tr}\omega = -\sum_a \nabla^2_{E_a,E_a}\omega + \nabla_{\kappa_B^\sharp}\omega. \tag{2.3}$$

The operator $\nabla_{\rm tr}^*\nabla_{\rm tr}$ is positive definite and formally self adjoint on the space of basic forms [10]. If ω is a basic 1-form, then $F(\omega)^\sharp = {\rm Ric}^Q(\omega^\sharp)$. Now, we recall the transversal divergence theorem on a foliated Riemannian manifold for later use.

Theorem 2.1. [26] *Let $(M, g, \mathcal{F})$ be a closed, oriented Riemannian manifold with a transversally oriented foliation $\mathcal{F}$ and a bundle-like metric g with respect to $\mathcal{F}$. Then for a transversal infinitesimal automorphism X,*

$$\int_M {\rm div}_\nabla(\pi(X))\mu_M = \int_M g_Q(\pi(X), \kappa_B^\sharp)\mu_M,$$

where ${\rm div}_\nabla\, s$ denotes the transversal divergence of s with respect to the connection ∇.

3. $(\mathcal{F},\mathcal{F}')$-Harmonic and Biharmonic Maps on Foliations

Let $\phi:(M,g,\mathcal{F})\to(M',g',\mathcal{F}')$ be a smooth foliated map, i.e., $d\phi(T\mathcal{F})\subset T\mathcal{F}'$, and $\Omega_B^r(E)=\Omega_B^r(\mathcal{F})\otimes E$ be the space of E-valued basic r-forms, where $E=\phi^{-1}Q'$ is the pull-back bundle on M. We define $d_T\phi:Q\to Q'$ by

$$d_T\phi:=\pi'\circ d\phi\circ\sigma.$$

Trivially, $d_T\phi\in\Omega_B^1(E)$. Let ∇^ϕ and $\tilde\nabla$ be the connections on E and $Q^*\otimes E$, respectively. Then a foliated map $\phi:(M,g,\mathcal{F})\to(M',g',\mathcal{F}')$ is called *transversally totally geodesic* if it satisfies

$$\tilde\nabla_{\rm tr}d_T\phi=0, \tag{3.1}$$

where $(\tilde\nabla_{\rm tr}d_T\phi)(X,Y)=(\tilde\nabla_X d_T\phi)(Y)$ for any $X,Y\in\Gamma Q$. Note that if $\phi:(M,g,\mathcal{F})\to(M',g',\mathcal{F}')$ is transversally totally geodesic with $d\phi(Q)\subset Q'$, then, for any transversal geodesic γ in M, $\phi\circ\gamma$ is also transversal geodesic. From now on, we use ∇ instead of all induced connections if we have no confusion. We define $d_\nabla:\Omega_B^r(E)\to\Omega_B^{r+1}(E)$ by

$$d_\nabla(\omega\otimes s)=d_B\omega\otimes s+(-1)^r\omega\wedge\nabla s \tag{3.2}$$

for any $s\in\Gamma E$ and $\omega\in\Omega_B^r(\mathcal{F})$. Let δ_∇ be a formal adjoint of d_∇ with respect to the inner product. Note that

$$d_\nabla(d_T\phi)=0,\quad \delta_\nabla d_T\phi=-\tau_b(\phi)+d_T\phi(\kappa_B^\sharp), \tag{3.3}$$

where $\tau_b(\phi)$ is the *transversal tension field* of ϕ defined by

$$\tau_b(\phi):={\rm tr}_Q(\nabla_{\rm tr}d_T\phi). \tag{3.4}$$

The Laplacian Δ on $\Omega_B^*(E)$ is defined by

$$\Delta=d_\nabla\delta_\nabla+\delta_\nabla d_\nabla.$$

Moreover, the operator A_X is extended to $\Omega_B^r(E)$ as follows:

$$A_X\Psi=L_X\Psi-\nabla_X\Psi,$$

where $L_X=d_\nabla i(X)+i(X)d_\nabla$ for any $X\in\Gamma TM$ and $i(X)(\omega\otimes s)=i(X)\omega\otimes s$. Hence $\Psi\in\Omega_B^*(E)$ if and only if $i(X)\Psi=0$ and $L_X\Psi=0$ for all $X\in\Gamma T\mathcal{F}$. Then the generalized Weitzenböck type formula (2.2) is extended to $\Omega_B^*(E)$ as follows [12]: for any $\Psi\in\Omega_B^r(E)$,

$$\Delta\Psi=\nabla_{\rm tr}^*\nabla_{\rm tr}\Psi+A_{\kappa_B^\sharp}\Psi+F(\Psi), \tag{3.5}$$

where $\nabla_{\rm tr}^*\nabla_{\rm tr}$ is the operator induced from (2.3) and $F(\Psi) = \sum_{a,b=1}^q \theta^a \wedge i(E_b)R(E_b,E_a)\Psi$. Moreover, we have that for any $\Psi \in \Omega_B^r(E)$,

$$\frac{1}{2}\Delta_B|\Psi|^2 = \langle \Delta\Psi, \Psi\rangle - |\nabla_{\rm tr}\Psi|^2 - \langle A_{\kappa_B^\sharp}\Psi, \Psi\rangle - \langle F(\Psi), \Psi\rangle. \tag{3.6}$$

3.1. $(\mathcal{F}, \mathcal{F}')$-harmonic maps

About this section, see [6]. Let Ω be a compact domain of M. Then the *transversal energy functional* of ϕ on Ω is defined by

$$E_B(\phi;\Omega) = \frac{1}{2}\int_\Omega |d_T\phi|^2\mu_M. \tag{3.7}$$

Then Dragomir and Tommasoli [6] defined $(\mathcal{F}, \mathcal{F}')$-*harmonic* if ϕ is a critical point of the transversal energy functional $E_B(\phi)$. Also, we obtain the first variational formula [6, 12]

$$\frac{d}{dt}E_B(\phi_t;\Omega)\Big|_{t=0} = -\int_\Omega \langle \tilde\tau_b(\phi), V\rangle \mu_M, \tag{3.8}$$

where $V = \frac{d\phi_t}{dt}|_{t=0}$ is the normal variation vector field of a foliated variation $\{\phi_t\}$ of ϕ and

$$\tilde\tau_b(\phi) := \tau_b(\phi) - d_T\phi(\kappa_B^\sharp). \tag{3.9}$$

From (3.8), we have the following [6].

Proposition 3.1. *A foliated map ϕ is $(\mathcal{F}, \mathcal{F}')$-harmonic map if and only if $\tilde\tau_b(\phi) = 0$.*

Remark 3.2. (1) If $\phi : M \to \mathbb{R}$ is a basic function, then $\tilde\tau_b(\phi) = -\Delta_B\phi$. So $(\mathcal{F}, \mathcal{F}')$-harmonic map is a generalization of a basic harmonic function.

(2) On foliated manifold, there is another kinds of harmonic map, *transvesally harmonic map*, which is a solution of the Euler-Lagrange equation $\tau_b(\phi) = 0$ by Konderak and Wolak [15]. But the transversally harmonic map is not a critical point of the energy functional $E_B(\phi)$. Two definitions are equivalent when the foliation is minimal.

Now, we define the *transversal Jacobi operator* $J_\phi^T : \Gamma\phi^{-1}Q' \to \Gamma\phi^{-1}Q'$ by

$$J_\phi^T(V) = \nabla_{tr}^*\nabla_{tr}V - {\rm tr}_Q R^{Q'}(V, d_T\phi)d_T\phi. \tag{3.10}$$

Then J_ϕ^T is a formally self-adjoint operator. That is, for any $V, W \in \Gamma\phi^{-1}Q'$,

$$\int_M \langle J_\phi^T(V), W\rangle\mu_M = \int_M \langle V, J_\phi^T(W)\rangle\mu_M. \tag{3.11}$$

Also, we have the second variation formula for the transversal energy functional $E_B(\phi)$.

Theorem 3.3. ([6], The second variation formula) *Let $\phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$ be a $(\mathcal{F}, \mathcal{F}')$-harmonic map and let $\{\phi_{s,t}\}$ be the foliated variation of ϕ supported in a compact domain Ω. Then*

$$\frac{\partial^2}{\partial s \partial t} E_B(\phi_{s,t}; \Omega)\Big|_{(s,t)=(0,0)} = \int_\Omega \langle J_\phi^T(V), W\rangle \mu_M, \tag{3.12}$$

where V and W are the variation vector fields of $\phi_{s,t}$.

Proof. Let $V = \frac{\partial \phi_{s,t}}{\partial s}\Big|_{(s,t)=(0,0)}$ and $W = \frac{\partial \phi_{s,t}}{\partial t}\Big|_{(s,t)=(0,0)}$ be the variation vector fields of $\phi_{s,t}$. Let $\Phi : M \times (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \to M'$ be a smooth map, which is defined by $\Phi(x, s, t) = \phi_{s,t}(x)$. Let ∇^Φ be the pull-back connection on $\Phi^{-1}Q'$. It is trivial that $[X, \frac{\partial}{\partial t}] = [X, \frac{\partial}{\partial s}] = 0$ for any vector field $X \in TM$. From (3.8), we have

$$\frac{\partial^2}{\partial s \partial t} E_B(\phi_{s,t}; \Omega) = -\int_\Omega \langle \frac{\partial^2 \phi_{s,t}}{\partial s \partial t}, \tilde\tau_b(\phi_{s,t})\rangle \mu_M - \int_\Omega \langle \frac{\partial \phi_{s,t}}{\partial t}, \nabla^\Phi_{\frac{\partial}{\partial s}} \tilde\tau_b(\phi_{s,t})\rangle \mu_M.$$

At $(s,t) = (0,0)$, the first term vanishes because of $\tilde\tau_b(\phi) = 0$. So

$$\frac{\partial^2}{\partial s \partial t} E_B(\phi_{s,t}; \Omega)\Big|_{(s,t)=(0,0)} = -\int_\Omega \langle W, \nabla^\Phi_{\frac{\partial}{\partial s}} \tilde\tau_b(\phi_{s,t})\Big|_{(s,t)=(0,0)}\rangle \mu_M. \tag{3.13}$$

At $x \in M$, by a straight calculation, we have

(3.14)
$$\nabla^\Phi_{\frac{\partial}{\partial s}} \tilde\tau_b(\phi_{s,t}) = \sum_a \nabla^\Phi_{E_a} \nabla^\Phi_{E_a} d\Phi(\frac{\partial}{\partial s}) - \nabla^\Phi_{\kappa_B^\sharp} d\Phi(\frac{\partial}{\partial s}) + \sum_a R^\Phi(\frac{d}{dt}, E_a) d\Phi(E_a).$$

Hence at $(s,t) = (0,0)$, we have

$$\nabla^\Phi_{\frac{\partial}{\partial s}} \tilde\tau_b(\phi_{s,t})\Big|_{(s,t)=(0,0)} = -\nabla^*_{tr} \nabla_{tr} V + \mathrm{tr}_Q R^{Q'}(V, d_T\phi) d_T\phi.$$

That is, we have

$$\nabla^\Phi_{\frac{\partial}{\partial s}} \tilde\tau_b(\phi_{s,t})\Big|_{(s,t)=(0,0)} = -J_\phi^T(V). \tag{3.15}$$

Hence the proof of (3.12) follows from (3.13) and (3.15). □

Now, we define the *basic Hessian* $Hess_\phi^T$ of ϕ by

$$Hess_\phi^T(V, W) = \int_M \langle J_\phi^T(V), W\rangle \mu_M. \tag{3.16}$$

Then $Hess_\phi^T(V, W) = Hess_\phi^T(W, V)$ for any $V, W \in \phi^{-1}Q'$. If $Hess_\phi^T$ is positive semi-definite, that is, $Hess_\phi^T(V, V) \geq 0$ for any normal vecor field V along ϕ, then ϕ is said to be *weakly stable*. Hence we have the following corollary.

Corollary 3.4. ([6], Stability) *Let M be a closed Riemannian manifold and M' be of non-positive transversal sectional curvature. Then any $(\mathcal{F},\mathcal{F}')$-harmonic map $\phi:(M,\mathcal{F})\to(M',\mathcal{F}')$ is weakly stable.*

Remark 3.5. For the stability of transversally harmonic map (that is, $\tau_b(\phi)=0$), see [11, Corollary 4.6]. In fact, under the same assumption, a transversally harmonic map is transversally f-stable, that is, $\int_M\langle (J^T_\phi-\nabla_{\kappa_B^\sharp})V,V\rangle e^{-f}\mu_M\geq 0$, where f is a basic function such that $\kappa_B=-df$.

3.2. $(\mathcal{F},\mathcal{F}')$-biharmonic maps

We define the *transversal bienergy functional* $\tilde{E}_{B,2}(\phi)$ on a compact domain Ω by

$$\tilde{E}_{B,2}(\phi;\Omega):=\frac{1}{2}\int_\Omega|\tilde{\tau}_b(\phi)|^2\mu_M. \tag{3.17}$$

Definition 3.6. A foliated map $\phi:(M,g,\mathcal{F})\to(M',g',\mathcal{F}')$ is said to be $(\mathcal{F},\mathcal{F}')$-*biharmonic map* if ϕ is a critical point of the transversal bienergy functional $\tilde{E}_{B,2}(\phi)$.

Theorem 3.7. (The first variation formula) *For a foliated map ϕ,*

$$\frac{d}{dt}\tilde{E}_{B,2}(\phi_t;\Omega)\Big|_{t=0}=-\int_\Omega\langle J^T_\phi(\tilde{\tau}_b(\phi)),V\rangle\mu_M, \tag{3.18}$$

where $V=\frac{d\phi_t}{dt}\Big|_{t=0}$ is the variation vector field of a foliated variation ϕ_t of ϕ.

Proof. Let $\Phi:M\times(-\epsilon,\epsilon)\to M'$ be a smooth map, which is defined by $\Phi(x,t)=\phi_t(x)$. Let ∇^Φ be the pull-back connection on $\Phi^{-1}Q'$. It is trivial that $[X,\frac{\partial}{\partial t}]=0$ for any vector field $X\in TM$. From (3.17), we have

$$\frac{d}{dt}\tilde{E}_{B,2}(\phi_t;\Omega)\Big|_{t=0}=\int_\Omega\langle\nabla^\Phi_{\frac{d}{dt}}\tilde{\tau}_b(\phi_t)|_{t=0},\tilde{\tau}_b(\phi)\rangle\mu_M. \tag{3.19}$$

From (3.11), (3.15) and (3.19), we finish the proof. □

From the first variation formula for the transversal bienergy functional, we know the following fact.

Proposition 3.8. *A $(\mathcal{F},\mathcal{F}')$-biharmonic map ϕ is a solution of the following equation*

$$(\tilde{\tau}_2)_b(\phi):=J^T_\phi(\tilde{\tau}_b(\phi))=0. \tag{3.20}$$

Here $(\tilde{\tau}_2)_b(\phi)$ is called the $(\mathcal{F},\mathcal{F}')$-bitension field of ϕ.

Remark 3.9. (1) From Remark 3.2, if ϕ is a basic function on M, then

$$(\tilde\tau_2)_b(\phi) = J^T_\phi(\tilde\tau_b(\phi)) = -J^T_\phi(\Delta_B\phi) = -\nabla^*_{tr}\nabla_{tr}(\Delta_B\phi) = \Delta^2_B\phi.$$

So $(\mathcal{F},\mathcal{F}')$-biharmonic map is a generalization of basic biharmonic function.

(2) A $(\mathcal{F},\mathcal{F}')$-harmonic map is trivially $(\mathcal{F},\mathcal{F}')$-biharmonic map.

(3) There is another kinds of biharmonic map on foliations, called *transversally biharmonic map*, which is a solution of $(\tau_2)_b(\phi) := J^T_\phi(\tau_b(\phi)) - \nabla_{\kappa_B^\sharp}\tau_b(\phi) = 0$ [11]. Actually, transversally biharmonic map is a critical point of the transversal f-bienergy functional $E_{2,f}(\phi)$, which is defined by

$$E_{2,f}(\phi) = \frac{1}{2}\int_M |\tau_b(\phi)|^2 e^{-f}\mu_M,$$

where f is a solution of $\kappa_B = -df$.

3.3. Generalized Chen's conjecture

Now, we consider the generalized Chen's conjecture for $(\mathcal{F},\mathcal{F}')$-biharmonic maps.

Theorem 3.10. *Let $(M,g,\mathcal{F})$ be a foliated Riemannian manifold and let $(M',g',\mathcal{F}')$ be of non-positive transversal sectional curvature, that is, $K^{Q'}\le 0$. Let $\phi: M\to M'$ be a $(\mathcal{F},\mathcal{F}')$-biharmonic map. Then*

1. *if M is closed, then ϕ is automatically $(\mathcal{F},\mathcal{F}')$-harmonic;*
2. *if M is complete with $Vol(M)=\infty$ and $\tilde E_{B,2}(\phi)<\infty$, then ϕ is $(\mathcal{F},\mathcal{F}')$-harmonic;*
3. *If M is complete with $E_B(\phi)<\infty$ and $\tilde E_{B,2}(\phi)<\infty$, then ϕ is $(\mathcal{F},\mathcal{F}')$ -harmonic.*

Proof. Let $\phi: M\to M'$ be a $(\mathcal{F},\mathcal{F}')$-biharmonic map. Then from (3.20)

$$(\nabla^\phi_{tr})^*(\nabla^\phi_{tr})\tilde\tau_b(\phi) - \sum_a R^{Q'}(\tilde\tau_b(\phi), d_T\phi(E_a))d_T\phi(E_a) = 0, \tag{3.21}$$

where $\{E_a\}$ be a local orthonomal basic frame of Q. From the generalized Weitzenbock formula (3.5) and (3.6), we have

$$\frac{1}{2}\Delta_B|\tilde\tau_b(\phi)|^2 = \langle\nabla^*_{tr}\nabla_{tr}\tilde\tau_b(\phi),\tilde\tau_b(\phi)\rangle - |\nabla_{tr}\tilde\tau_b(\phi)|^2. \tag{3.22}$$

Hence from (3.21), we get

$$\frac{1}{2}\Delta_B|\tilde\tau_b(\phi)|^2 = -|\nabla_{tr}\tilde\tau_b(\phi)|^2 + \sum_a\langle R^{Q'}(\tilde\tau_b(\phi), d_T\phi(E_a))d_T\phi(E_a),\tilde\tau_b(\phi)\rangle.$$

That is,

$$
\begin{aligned}
|\tilde{\tau}_b(\phi)|\Delta_B|\tilde{\tau}_b(\phi)| &= |d_B|\tilde{\tau}_b(\phi)||^2 - |\nabla_{tr}\tilde{\tau}_b(\phi)|^2 \\
&\quad + \sum_a \langle R^{Q'}(\tilde{\tau}_b(\phi), d_T\phi(E_a))d_T\phi(E_a), \tilde{\tau}_b(\phi)\rangle.
\end{aligned} \tag{3.23}
$$

By the Kato's inequality, that is, $|\nabla_{tr}\tilde{\tau}_b(\phi)| \geq |d_B|\tilde{\tau}_b(\phi)||$, and $K^{Q'} \leq 0$, we have

$$
\frac{1}{2}\Delta_B|\tilde{\tau}_b(\phi)| \leq 0. \tag{3.24}
$$

That is, $|\tilde{\tau}_b(\phi)|$ is basic subharmonic.

(1) If M is closed, then $|\tilde{\tau}_b(\phi)|$ is trivially constant. From (3.23), we have that for all a,

$$
\nabla_{E_a}\tilde{\tau}_b(\phi) = 0. \tag{3.25}
$$

Now, we define the normal vector field Y by

$$
Y = \sum_a \langle d_T\phi(E_a), \tilde{\tau}_b(\phi)\rangle E_a.
$$

Then from (3.25), we have

$$
\mathrm{div}_\nabla(Y) = \sum_a \langle \nabla_{E_a} Y, E_a\rangle = \langle \tau_b(\phi), \tilde{\tau}_b(\phi)\rangle. \tag{3.26}
$$

So by integrating (3.26) and by using the transversal divergence theorem (Theorem 2.1), we get

$$
\int_M |\tilde{\tau}_b(\phi)|^2 \mu_M = 0, \tag{3.27}
$$

which implies that $\tilde{\tau}_b(\phi) = 0$, that is, ϕ is the $(\mathcal{F}, \mathcal{F}')$-harmonic map.

(2) Let M be a complete Riemannian manifold. Note that for any basic 1-form ω, it is trivial that $\delta_B\omega = \delta\omega$ and so $\Delta_B f = \Delta f$ for any basic function f. Hence by the Yau's maximum principle [25, Theorem 3], we have following lemma.

Lemma 3.11. *If a nonnegative basic function f is basic-subharmonic, that is, $\Delta_B f \leq 0$, with $\int_M f^p < \infty$ $(p > 1)$, then f is constant.*

Since $\tilde{E}_{B,2}(\phi) < \infty$, by (3.24) and Lemma 3.11, $|\tilde{\tau}_b(\phi)|$ is constant. Moreover, since $Vol(M) = \infty$, $\int_M |\tilde{\tau}_B(\phi)|^2\mu_M < \infty$ implies $\tilde{\tau}_b(\phi) = 0$, that is, ϕ is $(\mathcal{F}, \mathcal{F}')$-harmonic.

(3) Now we define a basic 1-form ω on M by

$$
\omega(X) = \langle d_T\phi(X), \tilde{\tau}_b(\phi)\rangle \tag{3.28}
$$

for any normal vector field X. By using the Schwartz inequality, we get

$$\begin{aligned}\int_M |\omega|\mu_M &= \int_M \Big(\sum_a |\omega(E_a)|^2\Big)^{\frac{1}{2}}\mu_M\\ &= \int_M \Big(\sum_a |\langle d_T\phi(E_a),\tilde\tau_b(\phi)\rangle|^2\Big)^{\frac{1}{2}}\mu_M\\ &\le \int_M |d_T\phi||\tilde\tau_b(\phi)|\mu_M\\ &\le \Big(\int_M |d_T\phi|^2\mu_M\Big)^{\frac{1}{2}}\Big(\int_M |\tilde\tau_b(\phi)|^2\mu_M\Big)^{\frac{1}{2}}\\ &= 2\sqrt{E_B(\phi)E_{B,2}(\phi)} < \infty.\end{aligned}$$

On the other hand, by a straight calculation, we know that

$$\delta_B\omega = -|\tilde\tau_b(\phi)|^2. \tag{3.29}$$

Since $\int_M |\omega|\mu_M < \infty$ and $\int_M(\delta_B)\omega\mu_M = -\tilde E_{B,2}(\infty) < \infty$, by the Gaffney's theorem [8], we know that

$$\int_M |\tilde\tau_b(\phi)|^2\mu_M = -\int_M(\delta_B\omega)\mu_M = -\int_M(\delta\omega)\mu_M = 0. \tag{3.30}$$

Hence $\tilde\tau_b(\phi)=0$, that is, ϕ is $(\mathcal F,\mathcal F')$-harmonic. □

Remark 3.12. Note that for transversally biharmonic map, we need some conditions that the transversal Ricci curvature of M is nonnegative and positive at some point (cf. [11, Theorem 6.5]).

Now, we study the second variation formula for the transversal bienergy functional $\tilde E_{B,2}(\phi)$.

Theorem 3.13. (The second variation formula) *For a foliated map $\phi:(M,g,\mathcal F)\to(M',g',\mathcal F')$, we have*

$$\begin{aligned}\frac{d^2}{dt^2}\tilde E_{B,2}(\phi_t;\Omega)\Big|_{t=0} = &-\int_\Omega\langle\nabla_V V,(\tilde\tau_2)_b(\phi)\rangle\mu_M + \int_\Omega |J_\phi^T(V)|^2\mu_M\\ &-\int_\Omega\langle R^{Q'}(V,\tilde\tau_b(\phi))\tilde\tau_b(\phi),V\rangle\mu_M\\ &-4\int_M\langle R^{Q'}(\nabla_{tr}V,\tilde\tau_b(\phi))d_T\phi,V\rangle\mu_M\\ &+\int_\Omega\langle(\nabla_{\tilde\tau_b(\phi)}R^{Q'})(V,d_T\phi)d_T\phi,V\rangle\mu_M\\ &+2\int_\Omega\langle(\nabla_{tr}R^{Q'})(d_T\phi,V)\tilde\tau_b(\phi),V\rangle\mu_M,\end{aligned}$$

where $V=\frac{d\phi_t}{dt}\Big|_{t=0}$ is the normal variation vector field of $\{\phi_t\}$.

Proof. Let $\Phi : M \times (-\epsilon, \epsilon) \to M'$ be a smooth map, which is defined by $\Phi(x,t) = \phi_t(x)$. Let ∇^Φ be the pull-back connection on $\Phi^{-1}Q'$. It is trivial that $[X, \frac{\partial}{\partial t}] = 0$ for any vector field $X \in TM$. From definition, we have

$$\frac{d^2}{dt^2}\tilde{E}_{B,2}(\phi_t;\Omega) = \int_\Omega \langle \nabla^\Phi_{\frac{d}{dt}} \nabla^\Phi_{\frac{d}{dt}} \tilde{\tau}_b(\phi_t), \tilde{\tau}_b(\phi_t)\rangle \mu_M + \int_\Omega |\nabla^\Phi_{\frac{d}{dt}} \tilde{\tau}_b(\phi_t)|^2 \mu_M.$$

Let $\{E_a\}$ be a local orthonormal basic frame on Q such that $\nabla^\Phi E_a = 0$ at $x \in M$. From (3.14), we have

$$\begin{aligned}\nabla^\Phi_{\frac{d}{dt}} \nabla^\Phi_{\frac{d}{dt}} \tilde{\tau}_b(\phi_t) =& \sum_a \nabla^\Phi_{E_a}\nabla^\Phi_{E_a}\nabla^\Phi_{\frac{d}{dt}} d\Phi(\frac{d}{dt}) - \nabla^\Phi_{\kappa_B^\sharp}\nabla^\Phi_{\frac{d}{dt}} d\Phi(\frac{d}{dt}) + R^\Phi(\kappa_B^\sharp, \frac{d}{dt}) d\Phi(\frac{d}{dt}) \\ &+ \sum_a \nabla^\Phi_{E_a} R^\Phi(\frac{d}{dt}, E_a) d\Phi(\frac{d}{dt}) + \sum_a \nabla^\Phi_{\frac{d}{dt}} R^\Phi(\frac{d}{dt}, E_a) d\Phi(E_a) \\ &+ \sum_a R^\Phi(\frac{d}{dt}, E_a)\nabla^\Phi_{E_a} d\Phi(\frac{d}{dt}).\end{aligned}$$

At $t = 0$, since $d\Phi(\frac{d}{dt})|_{t=0} = \frac{d\phi_t}{dt}|_{t=0} = V$, we have

$$\begin{aligned}\nabla^\Phi_{\frac{d}{dt}} \nabla^\Phi_{\frac{d}{dt}} \tilde{\tau}_b(\phi_t)\Big|_{t=0} =& \sum_a \nabla_{E_a}\nabla_{E_a}\nabla_V V - \nabla_{\kappa_B^\sharp}\nabla_V V + R^{Q'}(d_T\phi(\kappa_B^\sharp), V)V \\ &+ \sum_a \nabla_{E_a} R^{Q'}(V, d_T\phi(E_a))V \\ &+ \sum_a \nabla_V R^{Q'}(V, d_T\phi(E_a)) d_T\phi(E_a) \\ &+ \sum_a R^{Q'}(V, d_T\phi(E_a))\nabla_{E_a} V.\end{aligned}$$

By a straight calculation together with the Bianchi identities, we have

$$\begin{aligned}\sum_a \nabla_{E_a} R^{Q'}(V, d_T\phi(E_a))V =& \sum_a (\nabla_{E_a} R^{Q'})(V, d_T\phi(E_a))V + R^{Q'}(V, \tau_b(\phi))V \\ &+ 2\sum_a R^{Q'}(V, d_T\phi(E_a))\nabla_{E_a} V \\ &- \sum_a R^{Q'}(V, \nabla_{E_a} V) d_T\phi(E_a)\end{aligned}$$

and

$$\begin{aligned}\sum_a \nabla_V R^{Q'}(V, d_T\phi(E_a))d_T\phi(E_a) =& \sum_a (\nabla_V R^{Q'})(V, d_T\phi(E_a))d_T\phi(E_a)\\ &+ \sum_a R^{Q'}(\nabla_V V, d_T\phi(E_a))d_T\phi(E_a)\\ &+ \sum_a R^{Q'}(V, \nabla_{E_a} V)d_T\phi(E_a)\\ &+ \sum_a R^{Q'}(V, d_T\phi(E_a))\nabla_{E_a} V.\end{aligned}$$

By summing the above equations, we have

$$\begin{aligned}\nabla_{\frac{d}{dt}}\nabla_{\frac{d}{dt}}\tilde{\tau}_b(\phi_t)\Big|_{t=0} =& - J_\phi^T(\nabla_V V) + R^{Q'}(V, \tilde{\tau}_b(\phi))V\\ &+ \sum_a (\nabla_V R^{Q'})(V, d_T\phi(E_a))d_T\phi(E_a)\\ &+ \sum_a (\nabla_{E_a} R^{Q'})(V, d_T\phi(E_a))V\\ &+ 4\sum_a R^{Q'}(V, d_T\phi(E_a))\nabla_{E_a} V.\end{aligned}$$

Then by integrating, we get

$$\begin{aligned}\int_\Omega \langle \nabla^\Phi_{\frac{d}{dt}}\nabla^\Phi_{\frac{d}{dt}}\tilde{\tau}_b(\phi_t)\Big|_{t=0}, \tilde{\tau}_b(\phi)\rangle =& - \int_\Omega \langle J_\phi^T(\nabla_V V), \tilde{\tau}_b(\phi)\rangle\\ &+ \int_\Omega \langle R^{Q'}(V, \tilde{\tau}_b(\phi))V, \tilde{\tau}_b(\phi)\rangle\\ &+ \sum_a \int_\Omega \langle (\nabla_V R^{Q'})(V, d_T\phi(E_a))d_T\phi(E_a), \tilde{\tau}_b(\phi)\rangle\\ &+ \sum_a \int_\Omega \langle (\nabla_{E_a} R^{Q'})(V, d_T\phi(E_a))V, \tilde{\tau}_b(\phi)\rangle\\ &+ 4\sum_a \int_\Omega \langle R^{Q'}(V, d_T\phi(E_a))\nabla_{E_a} V, \tilde{\tau}_b(\phi)\rangle.\end{aligned}$$

From the second Bianchi identity, we get

$$\begin{aligned}\langle (\nabla_V R^{Q'})(V, d_T\phi(E_a))d_T\phi(E_a), \tilde{\tau}_b(\phi)\rangle =& \langle (\nabla_{E_a} R^{Q'})(V, d_T\phi(E_a))V, \tilde{\tau}_b(\phi)\rangle\\ &+ \langle (\nabla_{\tilde{\tau}_b(\phi)} R^{Q'})(V, d_T\phi(E_a))d_T\phi(E_a), V\rangle.\end{aligned}$$

From the above equation, we get

$$\begin{aligned}\int_\Omega \langle \nabla^\Phi_{\frac{d}{dt}} \nabla^\Phi_{\frac{d}{dt}} \tilde{\tau}_b(\phi_t)\Big|_{t=0}, \tilde{\tau}_b(\phi)\rangle = &- \int_\Omega \langle J^T_\phi(\nabla_V V), \tilde{\tau}_b(\phi)\rangle + \int_\Omega \langle R^{Q'}(V, \tilde{\tau}_b(\phi))V, \tilde{\tau}_b(\phi)\rangle \\ &+ \sum_a \int_\Omega \langle (\nabla_{\tilde{\tau}_b(\phi)} R^{Q'})(V, d_T\phi(E_a))d_T\phi(E_a), V\rangle \\ &+ 2\sum_a \int_\Omega \langle (\nabla_{E_a} R^{Q'})(V, d_T\phi(E_a))V, \tilde{\tau}_b(\phi)\rangle \\ &+ 4\sum_a \int_\Omega \langle R^{Q'}(V, d_T\phi(E_a))\nabla_{E_a} V, \tilde{\tau}_b(\phi)\rangle.\end{aligned}$$

From the above equation and (3.15), by using the curvature properties and self-adjointness of J^T_ϕ, the proof follows. □

Definition 3.14. A $(\mathcal{F}, \mathcal{F}')$-biharmonic map $\phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$ is said to be *weakly stable* if $\frac{d^2}{dt^2} \tilde{E}_{B,2}(\phi_t)\Big|_{t=0} \geq 0$.

Now, we consider the generalized Chen's conjecture for $(\mathcal{F}, \mathcal{F}')$-biharmonic map when the transversal sectional curvature of M' is positive, that is, $K^{Q'} > 0$. In case of $K^{Q'} \leq 0$, see Theorem 3.10.

Let us recall the transversal stress-energy tensor $S_T(\phi)$ of ϕ [4, 11]:

$$S_T(\phi) = \frac{1}{2}|d_T\phi|^2 g_Q - \phi^* g_{Q'}. \tag{3.31}$$

Note that for any vector field $X \in \Gamma Q$,

$$(\mathrm{div}_\nabla S_T(\phi))(X) = -\langle \tau_b(\phi), d_T\phi(X)\rangle. \tag{3.32}$$

If $\mathrm{div}_\nabla S_T(\phi) = 0$, then we say that ϕ satisfies the *transverse conservation law* [4]. If there exists a basic function λ^2 such that $\phi^* g_{Q'} = \lambda^2 g_Q$, then ϕ is called a *transversally weakly conformal map*. In the case of λ being nonzero constant, ϕ is called a *transversally homothetic* map. Hence we have the following propositions.

Proposition 3.15. [7] *Let $\phi : (M, g, \mathcal{F}) \to (M', g', \mathcal{F}')$ be a transversally weakly conformal map with codim$(\mathcal{F}) > 2$. Then ϕ is transversally homothetic if and only if ϕ satisfies the transverse conservation law.*

Theorem 3.16. *Let $(M, g, \mathcal{F})$ be a closed foliated Riemannian manifold and $(M', g', \mathcal{F}')$ be a foliated Riemannian manifold with a positive constant transversal sectional curvature $K^{Q'}$. Let $\phi : M \to M'$ be a $(\mathcal{F}, \mathcal{F}')$-biharmonic map such that ϕ is transversally weakly stable and satisfies the transverse conservation law. If $\mathcal{F}$ is minimal or ϕ is transversally weakly conformal with codim$\mathcal{F} > 2$, then ϕ is $(\mathcal{F}, \mathcal{F}')$-harmonic.*

Proof. Let $\phi : M \to M'$ be a $(\mathcal{F}, \mathcal{F}')$-biharmonic map, that is, $(\tilde{\tau}_2)_b(\phi) = 0$. Let $K^{Q'} = c > 0$, where c is a positive constant. Then for any $X, Y, Z \in \Gamma Q'$

$$R^{Q'}(X,Y)Z = c\{\langle Y, Z\rangle X - \langle X, Z\rangle Y\}. \tag{3.33}$$

So $(\nabla_X R^{Q'})(Y,Z) = 0$. Hence if we take $V = \tilde{\tau}_b(\phi)$ in Theorem 3.13, then from (3.33)

$$\begin{aligned}
\frac{d^2}{dt^2}\tilde{E}_{B,2}(\phi_t)\Big|_{t=0} &= -4\int_M \langle R^{Q'}(\nabla_{tr}\tilde{\tau}_b(\phi), \tilde{\tau}_b)d_T\phi, \tilde{\tau}_b(\phi)\rangle\mu_M \\
&= -4c\int_M \langle \tilde{\tau}_b(\phi), d_T\phi\rangle\langle \nabla_{tr}\tilde{\tau}_b(\phi), \tilde{\tau}_b(\phi)\rangle\mu_M \\
&\quad + 4c\int_M \langle d_T\phi, \nabla_{tr}\tilde{\tau}_b(\phi)\rangle|\tilde{\tau}_b(\phi)|^2\mu_M \\
&= -4c\int_M \langle \tau_b(\phi), \tilde{\tau}_b(\phi)\rangle|\tilde{\tau}_b(\phi)|^2\mu_M \\
&\quad + 4c\sum_a\int_M E_a(\langle d_T\phi(E_a), \tilde{\tau}_b(\phi)\rangle|\tilde{\tau}_b(\phi)|^2)\mu_M \\
&\quad - 12c\sum_a\int \langle d_T\phi(E_a), \tilde{\tau}_b(\phi)\rangle\langle \nabla_{E_a}\tilde{\tau}_b(\phi), \tilde{\tau}_b(\phi)\rangle\mu_M.
\end{aligned}$$

If we choose a normal vector field X as

$$\langle X, Y\rangle = \langle \tilde{\tau}_b(\phi), d_T\phi(Y)\rangle|\tilde{\tau}_b(\phi)|^2$$

for any normal vector field Y, then

$$\mathrm{div}_\nabla X = \sum_a E_a(\langle \tilde{\tau}_b(\phi), d_T\phi(E_a)\rangle|\tilde{\tau}_b(\phi)|^2).$$

Hence by the transversal divergence theorem, we have

$$\begin{aligned}
\int\sum_a E_a(\langle d_T\phi(E_a), \tilde{\tau}_b(\phi)\rangle|\tilde{\tau}_b(\phi)|^2)\mu_M &= \int \mathrm{div}_\nabla(X)\mu_M = \int\langle X, \kappa_B^\sharp\rangle\mu_M \\
&= \int\langle d_T\phi(\kappa_B^\sharp), \tilde{\tau}_b(\phi)\rangle|\tilde{\tau}_b(\phi)|^2\mu_M.
\end{aligned}$$

Combining the above equations, we have

$$\begin{aligned}
\frac{d^2}{dt^2}\tilde{E}_{B,2}(\phi_t)\Big|_{t=0} &= -4c\int_M |\tilde{\tau}_b(\phi)|^4\mu_M \\
&\quad - 12c\sum_a\int_M \langle \tilde{\tau}_b(\phi), d_T\phi(E_a)\rangle\langle \nabla_{E_a}\tilde{\tau}_b(\phi), \tilde{\tau}_b(\phi)\rangle\mu_M.
\end{aligned} \tag{3.34}$$

Since ϕ satisfies the transverse conservation law, that is, $(\mathrm{div}_\nabla S_T(\phi))(X) = 0$ for any X, we have

$$\langle \tau_b(\phi), d_T\phi(E_a)\rangle = (\mathrm{div}_\nabla S_T(\phi))(E_a) = 0.$$

Moreover, since ϕ is transversally weakly conforml, from Proposition 3.15, ϕ is transversally homothetic. Hence

$$\sum_a \langle d_T\phi(\kappa_B^\sharp), d_T\phi(E_a)\rangle\langle \nabla_{E_a}\tilde\tau_b(\phi), \tilde\tau_b(\phi)\rangle = \alpha\langle \nabla_{\kappa_B^\sharp}\tilde\tau_b(\phi), \tilde\tau_b(\phi)\rangle$$

for some constant α. So if we choose the bundle-like metric such that $\delta_B\kappa_B = 0$, then

$$\begin{aligned}
&\int_M \sum_a \langle \tilde\tau_b(\phi), d_T\phi(E_a)\rangle\langle \nabla_{E_a}\tilde\tau_b(\phi), \tilde\tau_b(\phi)\rangle\mu_M\\
&\quad= -\alpha\int_M \langle \nabla_{\kappa_B^\sharp}\tilde\tau_b(\phi), \tilde\tau_b(\phi)\rangle\mu_M\\
&\quad= -\frac{\alpha}{2}\int_M \langle \delta_B\kappa_B, |\tilde\tau_b(\phi)|\rangle\mu_M\\
&\quad= 0.
\end{aligned}$$

Hence from (3.34), we have

$$\frac{d^2}{dt^2}\tilde E_{B,2}(\phi_t)\Big|_{t=0} = -4c\int |\tilde\tau_b(\phi)|^4\mu_M. \tag{3.35}$$

In case $\mathcal{F}$ is minimal, (3.35) also holds. Hence since ϕ is weakly stable and $c > 0$, we have $\tilde\tau_b(\phi) = 0$, that is, ϕ is $(\mathcal{F}, \mathcal{F}')$-harmonic. □

Remark 3.17. The generalized Chen's conjectures for the transversally biharmonic map have been studied in [11, 13] under some additional conditions such that the transversal Ricci curvature of M is nonnegative.

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