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GENERALIZED η-RICCI SOLITONS ON QUASI-SASAKIAN 3-MANIFOLDS ASSOCIATED TO THE SCHOUTEN-VAN KAMPEN CONNECTION

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Abstract. In this paper, we study quasi-Sasakian 3-dimensional manifolds admitting generalized η-Ricci solitons associated to the Schouten-van Kampen connection. We give an example of generalized η-Ricci solitons on a quasi-Sasakian 3-dimensional manifold with respect to the Schouten-van Kampen connection to prove our results.

1. Introduction

The quasi-Sasakian manifold was introduced by Blair [10] as a class of almost contact metric manifolds in order to unify Sasakian and cosymplectic manifolds. Tanno [38] also added some remarks on quasi-Sasakian structures. Three dimensional quasi-Sasakian manifolds were studied by many authors [18, 19, 27, 28, 31, 32]. Recently quasi-Sasakian structures have become a topic of growing interest due to its significant applications to physics, in particular to string theory, super gravity, and magnetic theory [1, 2, 20]. On three-dimensional quasi-Sasakian manifold the structure function was defined by Olszak and with the help of this structure function he obtained a necessary and sufficient condition for such manifolds to be conformally flat [28].

In 1982, Hamilton [21] introduced the notion of Ricci flow on a Riemannian manifold as follows:

$$\frac{\partial}{\partial t}g = -2S,$$

where S is the Ricci tensor of a manifold. The Ricci solitons are special solutions of the Ricci flow equation and generalizations of Einstein metrics. A Ricci soliton [11] is a triplet (g, V, λ) on a pseudo-Riemannian manifold M such that

$$\mathcal{L}_V g + 2S + 2\lambda g = 0, \tag{1}$$

where $\mathcal{L}_V$ is the Lie derivative along the potential vector field V, S is the Ricci tensor, and λ is a real constant. Metrics satisfying (1) are interesting and useful

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in physics and are often referred as quasi-Einstein [13, 14]. The Ricci soliton is said to be shrinking, steady, and expanding according as $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$, respectively. If the vector field V is the gradient of a potential function ψ, then g is called a gradient Ricci soliton. In 2016, Nurowski and Randall [26] introduced the notion of generalized Ricci soliton as follows:

$$\mathcal{L}_V g + 2\mu V^\flat \otimes V^\flat - 2\alpha S - 2\lambda g = 0,$$

where $V^\flat$ is the canonical 1-form associated to V. Also, as a generalization of Ricci soliton, the notion of η-Ricci soliton was introduced by Cho and Kimura [17], which is a 4-tuple (g, V, λ, ρ), such that V is a vector field on M, λ and ρ are constants, and g is a pseudo-Riemannian metric satisfying the equation

$$\mathcal{L}_V g + 2S + 2\lambda g + 2\rho\eta \otimes \eta = 0.$$

Many authors studied the η-Ricci solitons [5, 6, 7, 22, 24, 30, 39]. In particular, if $\rho = 0$, then the η-Ricci soliton equation reduces to the Ricci soliton equation. Motivated by the above studies, Siddiqi [34] introduced the notion of a generalized η-Ricci soliton as follows:

$$\mathcal{L}_V g + 2\mu V^\flat \otimes V^\flat + 2S + 2\lambda g + 2\rho\eta \otimes \eta = 0.$$

Motivated by [3, 12, 25] and the above works, we study generalized η-Ricci solitons on quasi-Sasakian 3-dimensional manifolds associated to the Schouten-van Kampen connection. We give an example of generalized η-Ricci soliton on a quasi-Sasakian 3-dimensional manifold associated to the Schouten-van Kampen connection.

The paper is organized as follows. In Section 2, we recall some necessary and fundamental concepts and formulas on quasi-Sasakian 3-dimensional manifolds which are used throughout the paper. In Section 3, we give the main results and their proofs. In Section 4, we give an example of a quasi-Sasakian 3-dimensional manifold which admits a generalized η-Ricci soliton with respect to the Schouten-van Kampen connection.

2. Preliminaries

A $(2n+1)$-dimensional Riemannian manifold (M, g) is said to be an almost contact metric manifold [8, 9] with an almost contact structure (φ, ξ, η, g) if there exist a $(1,1)$-tensor field φ, a vector field ξ, and a 1-form η such that

$$\varphi^2(X) = -X + \eta(X)\xi, \eta(\xi) = 1, \tag{2}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{3}$$

for any vector fields X, Y. In this case, we have $\varphi\xi = 0$, $\eta \circ \varphi = 0$, and $\eta(X) = g(X, \xi)$. An almost contact metric manifold M is a quasi-Sasakian 3-dimensional manifold if and only if

$$\nabla_X \xi = -\sigma\varphi X, \tag{4}$$

for any vector field X, where σ is a certain function on M, such that $\xi\sigma = 0$ and ∇ is the Levi-Civita connection of g [27]. Clearly, a quasi-Sasakian manifold is cosymplectic if and only if $\sigma = 0$ [23]. By virtue of (4), we have

$$(\nabla_X\varphi)Y = \sigma(g(X,Y)\xi - \eta(Y)X),$$
$$(\nabla_X\eta)Y = -\sigma g(\varphi X, Y),$$

for any vector fields X, Y [27]. Using (4) and (5), we find

$$R(X,Y)\xi = -X[\sigma]\varphi Y + Y[\sigma]\varphi X + \sigma^2\{\eta(Y)X - \eta(X)Y\},$$
$$R(X,\xi)\xi = \sigma^2\{X - \eta(X)\xi\},$$
$$R(X,\xi)Y = -X[\sigma]\varphi Y - \sigma^2\{g(X,Y)\xi - \eta(Y)X\} \tag{5}$$

for any vector fields X, Y, where R is the Riemannian curvature tensor. The Ricci tensor S of a quasi-Sasakian 3-dimensional manifold M is determined by (6)

$$S(X,Y) = (\frac{r}{2} - \sigma^2)g(X,Y) + (3\sigma^2 - \frac{r}{2})\eta(X)\eta(Y) - \eta(X)d\sigma(\varphi Y) - \eta(Y)d\sigma(\varphi X)$$

for any vector fields X, Y, where r is the scalar curvature of M. From (6), we also get

$$S(X,\xi) = 2\sigma^2\eta(X) - d\sigma(\varphi X) \tag{7}$$

for any vector field X.

Suppose that M is an almost contact metric manifold and TM is the tangent bundle of M. We have two naturally defined distributions on tangent bundle TM as follows:

$$H = \ker\eta, \qquad \hat{H} = \mathrm{span}\{\xi\}.$$

Thus we get $TM = H \oplus \hat{H}$. Therefore, by this composition we can define the Schouten-van Kampen connection $\bar{\nabla}$ [4, 35] on M with respect to Levi-Civita connection ∇ as follows:

$$\bar{\nabla}_X Y = \nabla_X Y - \eta(Y)\nabla_X\xi + ((\nabla_X\eta)(Y))\xi \tag{8}$$

for any vector fields X, Y. From [29, 35, 36, 37], we have

$$\bar{\nabla}\xi = 0, \qquad \bar{\nabla}g = 0, \qquad \bar{\nabla}\eta = 0,$$

and the torsion $\bar{T}$ of $\bar{\nabla}$ is given by

$$\bar{T}(X,Y) = \eta(X)\nabla_X\xi - \eta(X)\nabla_Y\xi + 2d\eta(X,Y)\xi.$$

Let $\bar{R}$ and $\bar{S}$ be the curvature tensors and the Ricci tensors of the connection $\bar{\nabla}$, respectively. From [29] on a quasi-Sasakian 3-manifold, we have

$$\bar{\nabla}_X Y = \nabla_X Y + \sigma\eta(X)\varphi Y + \sigma g(Y,\varphi X)\xi \tag{9}$$

and

$$\bar{S}(X,Y) = S(X,Y) + (\varphi X)[\sigma]\eta(Y) - 2\sigma^2\eta(X)\eta(Y) \tag{10}$$

where S denotes the Ricci tensor of the connection ∇. Using (10), the Ricci operator $\bar{Q}$ of the connection $\bar{\nabla}$ is determined by

$$\bar{Q}X = QX + (\varphi X)[\sigma]\xi - 2\sigma^2\eta(X)\xi.$$

Let r and $\bar{r}$ be the scalar curvature of the Levi-Civita connection ∇ and the Schouten-van Kampen connection $\bar{\nabla}$, respectively. The equation (10) yields

$$\bar{r} = r - 2\sigma^2.$$

Applying (9) and $\bar{\nabla}g = 0$, we get

$$\overline{\mathcal{L}}_V g = \mathcal{L}_V g$$

for any vector filed V, where $\overline{\mathcal{L}}_V g$ is the Lie derivative along the potential vector field V with respect to the Schouten-van Kampen connection $\bar{\nabla}$ and

$$(\overline{\mathcal{L}}_V g)(Y,Z) := g(\bar{\nabla}_Y V, Z) + g(Y, \bar{\nabla}_Z V)$$

for any vector fields X, Y on M. The generalized η-Ricci soliton associated to the Schouten-van Kampen connection is defined by

$$\alpha\bar{S} + \frac{\beta}{2}\overline{\mathcal{L}}_V g + \mu V^\flat \otimes V^\flat + \rho\eta\otimes\eta + \lambda g = 0, \tag{11}$$

where $\bar{S}$ denotes the Ricci tensor of the connection $\bar{\nabla}$, $V^\flat$ is the canonical 1-form associated to V that is $V^\flat(X) = g(V,X)$ for any vector field X, λ is a smooth function on M, and α, β, μ, ρ are real constants such that $(\alpha, \beta, \mu) \neq (0,0,0)$.

The generalized η-Ricci soliton equation reduces to

(1) the η-Ricci soliton equation when $\alpha = 1$ and $\mu = 0$,
(2) the Ricci soliton equation when $\alpha = 1$, $\mu = 0$, and $\rho = 0$, and
(3) the generalized Ricci soliton equation when $\rho = 0$.

3. Main results and their proofs

A quasi-Sasakian 3-dimensional manifold is said to η-Einstein if its Ricci tensor S is of the form

$$S = ag + b\eta\otimes\eta,$$

where a and b are smooth functions on manifold. Let M be a non-cosymplectic quasi-Sasakian 3-manifold. Now, we consider M which satisfies the generalized η-Ricci soliton (11) associated to the Schouten-van Kampen connection, and the potential vector field V is a pointwise collinear vector field with the structure vector field ξ, that is, $V = f\xi$ for some function f on M. Using (4), we get

$$\begin{aligned}\overline{\mathcal{L}}_{f\xi} g(X,Y) &= g(\bar{\nabla}_X f\xi, Y) + g(X, \bar{\nabla}_Y f\xi)\\ &= X[f]\eta(Y) + Y[f]\eta(X)\end{aligned} \tag{12}$$

for any vector fields X, Y. Also, we have

$$\xi^\flat \otimes \xi^\flat(X,Y) = \eta(X)\eta(Y) \tag{13}$$

for any vector fields X, Y. Applying $V = f\xi$, (10), (12), and (13) in the equation (11), we infer

$$\begin{aligned} &\alpha S(X,Y) + \alpha(\varphi X)[\sigma]\eta(Y) + \frac{\beta}{2}X[f]\eta(Y) \\ &+\frac{\beta}{2}Y[f]\eta(X) + (\mu f^2 + \rho - 2\alpha\sigma^2)\eta(X)\eta(Y) + \lambda g(X,Y) = 0 \end{aligned} \tag{14}$$

for any vector fields X, Y. We plug $Y = \xi$ in the above equation and using (7) to yield

$$\frac{\beta}{2}X[f] + \frac{\beta}{2}\xi[f]\eta(X) + (\mu f^2 + \rho + \lambda)\eta(X) = 0. \tag{15}$$

Taking $X = \xi$ in (15) gives

$$\beta\xi[f] = -(\mu f^2 + \rho + \lambda). \tag{16}$$

Inserting (16) in (15), we conclude

$$\beta X[f] = -(\mu f^2 + \rho + \lambda)\eta(X),$$

which yields

$$\beta df = -(\mu f^2 + \rho + \lambda)\eta. \tag{17}$$

Applying (17) in (14), we obtain

$$\alpha\bar{S}(X,y) = \lambda(-g(X,Y) + \eta(X)\eta(Y)), \tag{18}$$

which implies $\alpha\bar{r} = -2\lambda$.

Therefore, this leads to the following theorem:

Theorem 3.1. *Let $(M, g, \varphi, \xi, \eta)$ be a non-cosymplectic quasi-Sasakian 3-dimensional manifold. If M admits a generalized η-Ricci soliton $(g, V, \alpha, \beta, \mu, \rho, \lambda)$ with respect to the Schouten-van Kampen connection such that $\alpha \neq 0$ and $V = f\xi$ for some smooth function f on M, then M is an η-Einstein manifold with respect to the Schouten-van Kampen connection.*

From (18), we also have the following corollary:

Corollary 3.2. *Let $(M, g, \varphi, \xi, \eta)$ be a non-cosymplectic quasi-Sasakian 3-dimensional manifold. If M admits a generalized η-Ricci soliton $(g, V, \alpha, \beta, \mu, \rho, \lambda)$ with respect to the Schouten-van Kampen connection such that $V = f\xi$ for some smooth function f on M, then $\alpha\bar{r} = -2\lambda$.*

Now, let M be an η-Einstein quasi-Sasakian 3-dimensional manifold with respect to the Schouten-van Kampen connection and $V = \xi$. Then, we get $\bar{S} = ag + b\eta \otimes \eta$ for some functions a and b on M. From (2), (3), and (4), we have

$$\begin{aligned} \overline{\mathcal{L}}_\xi g(X,Y) &= \mathcal{L}_\xi g(X,Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) \\ &= g(-\sigma\varphi X, Y) + g(X, -\sigma\varphi Y) \\ &= -\sigma(g(\phi X, Y) + g(X, \varphi Y)) = 0 \end{aligned}$$

for any vector fields X, Y. Therefore,

$$\begin{aligned}&\alpha\bar{S}+\frac{\beta}{2}\overline{\mathcal{L}}_{\xi}g+\mu\xi^{\flat}\otimes\xi^{\flat}+\rho\eta\otimes\eta+\lambda g\\&=a\alpha g+b\alpha\eta\otimes\eta+\mu\eta\otimes\eta+\rho\eta\otimes\eta+\lambda g\\&=(a\alpha+\lambda)g+(b\alpha+\mu+\rho)\eta\otimes\eta.\end{aligned}$$

From the above equation, M admits a generalized η-Ricci soliton $(g,\xi,\alpha,\beta,\mu,\rho,\lambda)$ with respect to the Schouten-van Kampen connection if $\lambda=-a\alpha$ and $\rho=-b\alpha-\mu$.

Hence, we can state the following theorem:

Theorem 3.3. *Suppose that M is an η-Einstein quasi-Sasakian 3-dimensional manifold with respect to the Schouten-van Kampen connection, that is, $\bar{S}=ag+b\eta\otimes\eta$ for some constants a and b on M. Then the manifold M satisfies a generalized η-Ricci soliton $(g,\xi,\alpha,\beta,\mu,-b\alpha-\mu,-a\alpha)$ with respect to the Schouten-van Kampen connection.*

Applying (10) in (18), we obtain

$$S(X,Y)+(\varphi X)[\sigma]\eta(Y)-2\sigma^2\eta(X)\eta(Y)=\lambda(-g(X,Y)+\eta(X)\eta(Y)) \tag{19}$$

for any vector fields X, Y. Substituting (6) in (19), we get

$$(\frac{r}{2}-\sigma^2+\lambda)g(X,Y)+(\sigma^2-\frac{r}{2}-\lambda)\eta(X)\eta(Y)-(\varphi Y)[\sigma]\eta(X)=0$$

for any vector fields X, Y. We plug $X=\xi$ in the above equation to yield

$$(\varphi Y)[\sigma]=0. \tag{20}$$

Replacing φY instead of Y in (20) and using $\xi[\sigma]=0$, we infer

$$d\sigma(Y)=0,$$

that is σ is constant. Thus, we can state the following theorem:

Theorem 3.4. *Suppose that M is a quasi-Sasakian 3-dimensional manifold. If M satisfies the generalized η-Ricci soliton $(g,\xi,\alpha,\beta,\mu,\rho,\lambda)$ with respect to the Schouten-van Kampen connection such that $\alpha\neq 0$ and $\mu+\rho=-\lambda$ then M is a σ-Sasakian manifold.*

In a generalized η-Ricci soliton $(g,\xi,\alpha,\beta,\mu,\rho,\lambda)$ with respect to the Schouten-van Kampen connection, if $V=\nabla\psi$, where $\psi\in C^{\infty}(M)$, then g is called a generalized gradient η-Ricci soliton. In this case, we have

$$\begin{aligned}&\overline{\mathcal{L}}_V g(X,Y)=\overline{\mathcal{L}}_{\nabla\psi}g(X,Y)=\mathcal{L}_{\nabla\psi}g(X,Y)=2\mathrm{Hess}\psi(X,Y),\\&V^{\flat}(X)=(\nabla\psi)^{\flat}(X)=g(\nabla\psi,X)=d\psi(X)\end{aligned}$$

for any vector fields X, Y. Hence, the equation (11) becomes

$$\alpha\bar{S}+\beta\mathrm{Hess}\psi+\mu d\psi\otimes d\psi+\rho\eta\otimes\eta+\lambda g=0. \tag{21}$$

From the property of Lie derivative, we conclude

$$(\mathcal{L}_{\xi}(\mathcal{L}_X g))(Y,\xi)=\xi((\mathcal{L}_X g)(Y,\xi))-\mathcal{L}_X g(\mathcal{L}_{\xi}Y,\xi)-\mathcal{L}_X g(Y,\mathcal{L}_{\xi}\xi)$$

for any vector fields X, Y. Since $\mathcal{L}_\xi Y = [\xi, Y]$ and $\mathcal{L}_\xi \xi = 0$, we deduce

$$(\mathcal{L}_\xi(\mathcal{L}_X g))(Y,\xi) = g(\nabla_\xi\nabla_Y X,\xi)+g(Y,\nabla_\xi\nabla_\xi X)-g(\nabla_{[\xi,Y]}X,\xi)+g(\nabla_\xi X,\nabla_Y\xi)$$

for any vector fields X, Y. We have $\nabla_\xi\xi = -\sigma\varphi\xi = 0$, so that we get

$$\begin{aligned}(\mathcal{L}_\xi(\mathcal{L}_X g))(Y,\xi) &= g(\nabla_\xi\nabla_Y X,\xi) + g(Y,\nabla_\xi\nabla_\xi X) - g(\nabla_{[\xi,Y]}X,\xi)\\ &\quad +Yg(\nabla_\xi X,\xi) - g(\nabla_Y\nabla_\xi X,\xi)\end{aligned}$$

for any vector fields X, Y. By definition of Riemannian curvature, we have

$$(\mathcal{L}_\xi(\mathcal{L}_X g))(Y,\xi) = g(R(\xi,Y)X,\xi) + g(Y,\nabla_\xi\nabla_\xi X) + Yg(\nabla_\xi X,\xi)$$

for any vector fields X, Y. The equation (5) implies that

(22)
$$(\mathcal{L}_\xi(\mathcal{L}_X g))(Y,\xi) = \sigma^2 g(X,Y) - \sigma^2\eta(X)\eta(Y) + g(Y,\nabla_\xi\nabla_\xi X) + Yg(\nabla_\xi X,\xi)$$

for any vector fields X, Y. Also, by a direct computation we have

$$\begin{aligned}\mathcal{L}_\xi(d\psi\otimes d\psi)(Y,\xi) &= \xi((d\psi\otimes d\psi)(Y,\xi)) - (d\psi\otimes d\psi)(\mathcal{L}_\xi Y,\xi)\\ &\quad -(d\psi\otimes d\psi)(Y,\mathcal{L}_\xi\xi)\\ &= \xi(Y[\psi]\xi[\psi]) - [\xi,Y][\psi]\xi[\psi] - Y[\psi][\xi,\xi][\psi]\\ &= \xi[Y[\psi]]\xi[\psi] + Y[\psi]\xi[\xi[\psi]] - [\xi,Y][\psi]\xi[\psi]\end{aligned}$$

for any vector field Y. Since $[\xi, Y][\psi] = \xi[Y[\psi]] - Y[\xi[\psi]]$, we obtain

$$\mathcal{L}_\xi(d\psi\otimes d\psi)(Y,\xi) = Y[\xi[\psi]]\xi[\psi] + Y[\psi]\xi[\xi[\psi]] \tag{23}$$

for any vector field Y. Using (7), (10) and (21), we have

$$\begin{aligned}-\beta\mathrm{Hess}\psi(Y,\xi) &= \alpha\bar{S}(Y,\xi) + \mu d\psi\otimes d\psi(Y,\xi) + (\rho+\lambda)\eta(Y)\\ &= -\alpha d\sigma(\varphi Y) + \mu d\psi(\xi)d\psi(Y) + (\rho+\lambda)\eta(Y)\end{aligned}$$

for any vector field Y. By definition of Hessψ, we conclude that

$$-\beta g(\nabla_\xi\nabla\psi, Y) = \alpha g(\varphi(\nabla\sigma),Y) + \mu d\psi(\xi)g(\nabla\psi,Y) + (\rho+\lambda)\eta(Y)$$

for any vector field Y. Therefore,

$$-\beta\nabla_\xi\nabla\psi = \mu d\psi(\xi)\nabla\psi + (\rho+\lambda)\xi. \tag{24}$$

Putting $X = \nabla\psi$ in (22) and considering $\eta(Y) = 0$, we get

$$2(\mathcal{L}_\xi(\mathrm{Hess}\psi))(Y,\xi) = \sigma^2 g(\nabla\psi,Y) + g(Y,\nabla_\xi\nabla_\xi\nabla\psi) + Yg(\nabla_\xi\nabla\psi,\xi). \tag{25}$$

Applying (24) to (25), we arrive at

$$\begin{aligned}&-2\beta(\mathcal{L}_\xi(\mathrm{Hess}\psi))(Y,\xi)\\ =\ &-\beta\sigma^2 Y[\psi] + \alpha g(\nabla_\xi\varphi(\nabla\sigma),Y) + \mu d\psi(\xi)g(\nabla_\xi\nabla\psi,Y)\\ &+\mu\xi[d\psi(\xi)]g(\nabla\psi,Y) + 2\mu Y[d\psi(\xi)]g(\nabla\psi,\xi) + Y[\lambda].\end{aligned}$$

Also, by considering $\eta(Y)=0$, we have

$$\begin{aligned}(\mathcal{L}_\xi\bar{S})(Y,\xi) &= \xi[\bar{S}(Y,\xi)]-\bar{S}(\mathcal{L}_\xi Y,\xi)-\bar{S}(Y,\mathcal{L}_\xi\xi)\\ &= \xi(\varphi(Y)[\sigma])-\varphi(\mathcal{L}_\xi Y)[\sigma]+2\sigma^2\eta(\mathcal{L}_\xi Y)\\ &= -\xi(g(Y,\varphi(\nabla\sigma)))+g(\nabla_\xi Y-\nabla_Y\xi,\varphi(\nabla\sigma))\\ &\quad +2\sigma^2 g(\xi,\nabla_\xi Y-\nabla_Y\xi)\\ &= -g(Y,\nabla_\xi\varphi(\nabla\sigma))+\sigma(Y,\nabla\sigma).\end{aligned}$$

If σ and λ are two constants and $\beta\neq 0$, then $(\mathcal{L}_\xi\bar{S})(Y,\xi)=0$ and

$$\begin{aligned}-2\beta(\mathcal{L}_\xi(\mathrm{Hess}\psi))(Y,\xi) &= -\beta\sigma^2 Y[\psi]+\mu d\psi(\xi)g(\nabla_\xi\nabla\psi,Y)\\ &\quad +\mu\xi[d\psi(\xi)]g(\nabla\psi,Y)+2\mu Y[d\psi(\xi)]g(\nabla\psi,\xi)\\ &= -\beta\sigma^2 Y[\psi]-\frac{\mu^2}{\beta}(d\psi(\xi))^2 g(\nabla\psi,Y)\\ &\quad +\mu\xi[d\psi(\xi)]g(\nabla\psi,Y)+2\mu Y[d\psi(\xi)]g(\nabla\psi,\xi).\end{aligned} \tag{26}$$

Taking the Lie derivative of the generalized η-Ricci soliton equation (21) yields

$$-2\beta(\mathcal{L}_\xi(\mathrm{Hess}\psi))(Y,\xi)=2\mu\mathcal{L}_\xi(d\psi\otimes d\psi)(Y,\xi). \tag{27}$$

Hence, from equations (23), (26) and (27), we infer

$$\begin{aligned}&-\beta\sigma^2 Y[\psi]-\frac{\mu^2}{\beta}(d\psi(\xi))^2 g(\nabla\psi,Y)+\mu\xi[d\psi(\xi)]g(\nabla\psi,Y)\\ &+2\mu Y[d\psi(\xi)]g(\nabla\psi,\xi)-2\mu\left(Y[\xi[\psi]]\xi[\psi]+Y[\psi]\xi[\xi[\psi]]\right)=0.\end{aligned} \tag{28}$$

We have

$$\xi[\xi[\psi]]=\xi[g(\xi,\nabla\psi)]=g(\xi,\nabla_\xi\nabla\psi)=-\frac{1}{\beta}\left(\mu(d\psi(\xi))^2+\rho+\lambda\right). \tag{29}$$

Substituting (29) into (28), we get

$$\left(\beta^2\sigma^2+(-\mu+\mu^2)(d\psi(\xi))^2-\mu(\rho+\lambda)\right)Y[\psi]=0.$$

If $\mu\in\{0,1\}$ and $\beta^2\sigma^2-\mu(\rho+\lambda)\neq 0$, then $Y[\psi]=0$, i.e., $\nabla\psi$ is parallel to ξ. Thus $\nabla\psi=0$ as $D=\ker\eta$ is nowhere integrable, i.e., ψ is a constant function. Hence, we state the following theorem:

Theorem 3.5. *Let M be a quasi-Sasakian 3-dimensional manifold bearing a generalized gradient η-Ricci soliton associated to the Schouten-van Kampen connection (21) with $\beta\neq 0$, $\mu=0$ or 1, $\beta^2\sigma^2-\mu(\rho+\lambda)\neq 0$. Let σ and λ be two constants. Then ψ is a constant function and M is an η-Einstein manifold.*

Definition 3.6. *A vector field V is said to a conformal Killing vector field if*

$$(\mathcal{L}_V g)(X,Y)=2hg(X,Y) \tag{30}$$

for any vector fields X,Y, where h is some function on M. The conformal Killing vector field V is called

- *proper when h is not constant,*

- *homothetic vector field when h is a constant, and*
- *Killing vector field when $h = 0$.*

Let V be a conformal Killing vector field satisfying (30). By (30), (10), and (11), we have

$$\alpha(S(X,Y) + (\varphi X)[\sigma]\eta(Y) - 2\sigma^2\eta(X)\eta(Y)) + \beta hg(X,Y)$$
$$+\mu V^\flat(X)V^\flat(Y) + \rho\eta(X)\eta(Y) + \lambda g(X,Y) = 0$$

for any vector fields X, Y. By inserting $Y = \xi$ in the above equation, we get

$$g(\beta h\xi + \mu\eta(V)V + \rho\xi + \lambda\xi, X) = 0,$$

for any vector field X. Since X is an arbitrary vector field, we have the following theorem:

Theorem 3.7. *If the metric g of a quasi-Sasakian 3-dimensional manifold satisfies the generalized η-Ricci soliton associated to the Schouten-van Kampen connection (11) $(g, V, \alpha, \beta, \mu, \rho, \lambda)$, where V is conformally Killing vector field, that is, $\mathcal{L}_V g = 2hg$, then*

$$(\beta h + \rho + \lambda)\xi + \mu\eta(V)V = 0.$$

Definition 3.8. *A nonvanishing vector field V on a pseudo-Riemannian manifold (M, g) is called torse-forming* [41] *if*

$$\nabla_X V = fX + \omega(X)V, \tag{31}$$

for any vector field X, where ∇ is the Levi-Civita connection of g, f is a smooth function, and ω is a 1-form. The vector field V is called

- *concircular* [16, 40] *whenever in the equation (31) the 1-form ω vanishes identically,*
- *concurrent* [33, 42] *if in equation (31) the 1-form ω vanishes identically and $f = 1$,*
- *parallel vector field if in equation (31) $f = \omega = 0$, and*
- *torqued vector field* [15] *if in equation (31) $\omega(V) = 0$.*

Let $(g, V, \alpha, \beta, \mu, \rho, \lambda)$ be a generalized η-Ricci soliton on a quasi-Sasakian 3-dimensional manifold associated to the Schouten-van Kampen connection, where V is a torse-forming vector filed satisfying (31). Then

$$\begin{aligned}\alpha(S(X,Y) + (\varphi X)[\sigma]\eta(Y) - 2\sigma^2\eta(X)\eta(Y)) + (\mathcal{L}_V g)(X,Y)\\ +\mu V^\flat(X)V^\flat(Y) + \rho\eta(X)\eta(Y) + \lambda g(X,Y) = 0\end{aligned} \tag{32}$$

for any vector fields X, Y. On the other hand,

$$(\mathcal{L}_V g)(X,Y) = 2fg(X,Y) + \omega(X)g(V,Y) + \omega(Y)g(V,X) \tag{33}$$

for any vector fields X, Y. Applying (33) to (32), we arrive at

$$\alpha S(X,Y) + \alpha((\varphi X)[\sigma]\eta(Y) - 2\sigma^2\eta(X)\eta(Y)) + [\beta f + \lambda]\, g(X,Y)$$
$$+\rho\eta(X)\eta(Y) + \frac{\beta}{2}\,[\omega(X)g(V,Y) + \omega(Y)g(V,X)] + \mu g(V,X)g(V,Y) = 0$$

for any vector fields X, Y. We take the contraction of the above equation over X and Y to obtain

$$\alpha r + \alpha \sum_{i=1}^{3} g(e_i, \varphi(\nabla\sigma)) - 2\sigma^2\alpha + 3\left[\beta f + \lambda\right] + \rho + \beta\omega(V) + \mu|V|^2 = 0.$$

Therefore we have the following theorem:

Theorem 3.9. *If the metric g of a quasi-Sasakian 3-dimensional manifold satisfies the generalized η-Ricci soliton $(g, V, \alpha, \beta, \mu, \rho, \lambda)$, where V is torse-forming vector filed satisfying (31), then*

$$\lambda = -\frac{1}{3}\left[\alpha r + \alpha \sum_{i=1}^{3} g(e_i, \varphi(\nabla\sigma)) - 2\sigma^2\alpha + \rho + \beta\omega(V) + \mu|V|^2\right] - \beta f.$$

4. Example

In this section, we give an example of a quasi-Sasakian 3-dimensional manifold with respect to the Schouten-van Kampen connection such that it admits in a generalized η-Ricci soliton associated to the Schouten-van Kampen connection.

Example 4.1. *Let (x, y, z) be the standard coordinates in $\mathbb{R}^3$ and $M = \{(x, y, z) \in \mathbb{R}^3 | (x, y, z) \neq (0, 0, 0)\}$. We consider the linearly independent vector fields*

$$e_1 = \frac{\partial}{\partial x} - y\frac{\partial}{\partial z}, \qquad e_2 = \frac{\partial}{\partial y}, \qquad e_3 = \frac{\partial}{\partial z}.$$

We define the metric g by

$$g(e_i, e_j) = \begin{cases} 1, & \text{if } i = j \text{ and } i, j \in \{1, 2, 3\}, \\ 0, & \text{otherwise.} \end{cases}$$

We define an almost contact structure (φ, ξ, η) on M by

$$\xi = e_3, \quad \eta(X) = g(X, e_3), \quad \varphi = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

for any vector field X. Note the relations $\varphi^2(X) = -X + \eta(X)\xi$, $\eta(\xi) = 1$, and $g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$ hold for any vector fields X, Y. Thus $(M, \varphi, \xi, \eta, g)$ defines an almost contact structure on M. We have

[,]	e_1	e_2	e_3
e_1	0	e_3	0
e_2	$-e_3$	0	0
e_3	0	0	0

The Levi-Civita connection ∇ of M is determined by

$$\nabla_{e_i}e_j = \begin{pmatrix} 0 & \frac{1}{2}e_3 & -\frac{1}{2}e_2 \\ -\frac{1}{2}e_3 & 0 & \frac{1}{2}e_1 \\ -\frac{1}{2}e_2 & \frac{1}{2}e_1 & 0 \end{pmatrix}.$$

We see that the structure (φ, ξ, η) satisfies the formula $\nabla_X \xi = -\sigma\varphi X$ for $\sigma = -\frac{1}{2}$. Thus, (M, ϕ, ξ, η, g) becomes a quasi-Sasakian 3-dimensional manifold. Now, using (8) we get the Schouten-van- Kampen connection on M as follows:

$$\bar{\nabla}_{e_i}e_j = \begin{pmatrix} 0 & (\frac{1}{2}+\sigma)e_3 & -(\frac{1}{2}+\sigma)e_2 \\ -(\frac{1}{2}+\sigma)e_3 & 0 & (\frac{1}{2}+\sigma))e_1 \\ -\frac{1}{2}e_2 & \frac{1}{2}e_1 & 0 \end{pmatrix}.$$

The nonvanishing components of curvature tensor with respect to the Schouten-van Kampen connection are:

$$\bar{R}(e_1, e_2)e_1 = \frac{1}{2}e_2, \ \bar{R}(e_1, e_2)e_2 = -\frac{1}{2}e_1,$$
$$\bar{R}(e_1, e_3)e_1 = -(\frac{1}{4} + \frac{\sigma}{2})e_3, \ \bar{R}(e_1, e_3)e_2 = -(\frac{1}{4} + \frac{\sigma}{2})e_3.$$

Hence, we obtain

$$\bar{S} = \begin{pmatrix} \frac{\sigma}{2}-\frac{1}{4} & 0 & 0 \\ 0 & \frac{\sigma}{2}-\frac{1}{4} & 0 \\ 0 & 0 & 0 \end{pmatrix} = (\frac{\sigma}{2} - \frac{1}{4})(g - \eta \otimes \eta).$$

If we consider $V = \xi$, then $\mathcal{L}_V g = 0$. Therefore $(g, \xi, \alpha, \beta, \mu, \rho = \alpha(\frac{\sigma}{2} - \frac{1}{4}) - \mu, \lambda = -\alpha(\frac{\sigma}{2} - \frac{1}{4}))$ is a generalized η-Ricci soliton on manifold M with respect to the Schouten-van Kampen connection.

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We declare that we do not have any commercial or associative interest that represents a conflict of interest in connection with the work submitted.

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Author's contributions

All authors contributed equally in the preparation of this manuscript.

References

[1] B. S. Acharya, A-O'Farrell Figurea, C. M. Hull, and B. J. Spence, *Branes at Canonical singularities and holography*, Adv. Theor. Math. Phys. **2** (1999), 1249–1286.

[2] I. Agricola and T. Friedrich, *Killing spinors in super gravity with 4-fluxes*, Class. Quant. Grav. **20** (2003), 4707–4717.

[3] S. Azami, *Generalized Ricci solitons of three-dimensional Lorentzian Lie groups associated canonical connections and Kobayashi-Nomizu connections*, J. Nonlinear Math. Phys. **30** (2023), 1–33.

[4] A. M. Blaga, *Cononical connections on para-Kenmotsu manifolds*, Novi Sad J. Math. **45** (2015), no. 2, 131–142.

[5] A. M. Blaga, *η-Ricci solitons on Lorentzian para-Sasakian manifolds*, Filomat **30** (2016), no. 2, 489–496.

[6] A. M. Blaga, *η-Ricci solitons on para-Kenmotsu manifolds*, Balkan J. Geom. Appl. **20** (2015), 1–13.

[7] A. M. Blaga, *Torse-forming η-Ricci solitons in almost paracontact η-Einstein geometry*, Filomat,**31** (2017), no. 2, 499–504.

[8] D. E. Blair, *Contact manifolds in Riemannian geometry*, Lecture Notes in Mathematics Vol 509. Springer-Verlag, Berlin-New York, 1976.

[9] D. E. Blair, *Riemannian geometry of contact and symplectic manifolds*, Progress in Mathematics, Vol. 203. Birkhauser Boston Inc. 2002.

[10] D. E. Blair, *The theory of quasi-Sasakian structure*, J. Differential Geo. **1** (1967), 331–345.

[11] C. Calin and M. Crasmareanu, *From the Eisenhart problem to Ricci solitons in f-Kenmotsu manifolds*, Bull. Malays. Math. Soc. **33** (2010), no. 3, 361–368.

[12] G. Calvaruso, *Three-dimensional homogeneous generalized Ricci solitons*, Mediterr. J. Math. **14** (2017), no. 5, 1–21.

[13] T. Chave and G. Valent, *Quasi-Einstein metrics and their renormalizability properties*, Helv. Phys. Acta. **69** (1996) 344–347.

[14] T. Chave and G. Valent, *On a class of compact and non-compact quasi-Einstein metrics and their renormalizability properties*, Nuclear Phys. B. **478** (1996) 758–778.

[15] B. Y. Chen, *Classification of torqued vector fields and its applications to Ricci solitons*, Kragujevac J. Math. **41** (2017), no. 2, 239–250.

[16] B. Y. Chen, *A simple characterization of generalized Robertson-Walker space-times*, Gen. Relativity Gravitation, **46** (2014), no. 12, Article ID 1833.

[17] J. T. Cho and M. Kimura, *Ricci solitons and real hypersurfaces in a complex space form*, Tohoku Math. J. **61** (2009), no. 2, 205–212.

[18] U. C. De and A. Sarkar,*On three-dimensional quasi-Sasakian manifolds*, SUT Journal of Mathematics **45** (2009), 59–71.

[19] U. C. De and A. K. Sengupta, *Notes on three-dimensional quasi-Sasakian manifolds*, Demonstratio Mathematica XXXVII (3) (2004), 655–660.

[20] T. Friedrichand and S. Ivanov, *Parallel spinors and connections with skew symmetric torsion in string theory*, Asian J. Math. **6** (2002), 303–336.

[21] R. S. Hamilton, *The Ricci flow on surfaces, Mathematics and general relativity*, Contemp. Math. Santa Cruz, CA, 1986, 71, American Math. Soc. 1988, 237–262.

[22] A. Haseeb, S. Pandey, and R. Prasad, *Some results on η-Ricci solitons in quasi-Sasakian 3-manifolds*, Commun. Korean Math. Soc. **36** (2021), no. 2, 377–387.

[23] D. Janssens and L. Vanhecke, *Almost contact structures and curvature tensors*, Kodai Math. J. **4** (1981), no. 1, 1–27.

[24] P. Majhi, U. C. De, and D. Kar, *η-Ricci Solitons on Sasakian 3-Manifolds*, Anal. de Vest Timisoara LV (2) (2017), 143–156.

[25] M. A. Mekki and A. M. Cherif, *Generalised Ricci solitons on Sasakian manifolds*, Kyungpook Math. J. **57** (2017), 677–682.

[26] P. Nurowski and M. Randall, *Generalized Ricci solitons*, J. Geom. Anal. **26** (2016), 1280–1345.

[27] Z. Olszak, *Normal almost contact metric manifolds of dimension 3*, Ann. Polon. Math. **47** (1986), 41–50.

[28] Z. Olszak, *On three-dimensional conformally flat quasi-Sasakian manifolds*, Period. Math. Hungar. **33** (1996), 105–113.

[29] S. Y. Perktas and A. Yildiz, *On Quasi-Sasakian 3-manifolds with respect to the Schouten-van- Kampen connection*, Int. Electron. J. Geom. **13** (2020), no. 2, 62–67.

[30] D. G. Prakasha and B. S. Hadimani, *η-Ricci solitons on para-Sasakian manifolds*, J. Geometry **108** (2017), 383–392.

[31] A. Sarkar, A. Sil, and D. Biswas, *A study on three-dimensional quasi-Sasakian amnifolds*, Indian J. Math. **59** (2017), 209–225.

[32] A. Sarkary, A. Silz, and A. K. Paul, *On three-dimensional quasi-Sasakian*, Applied Mathematics E-Notes **19** (2019), 55–64.

[33] J. A. Schouten, *Ricci Calculus*, Springer-Verlag, Berlin, 1954.

[34] M. D. Siddiqi, *Generalized η-Ricci solitons in trans Sasakian manifolds*, Eurasian bulltein of mathematics **1** (2018), no. 3, 107–116.

[35] A. F. Solovév, *On the curvature of the connection induced on a hyperdistribution in a Riemannian space*, Geom. Sb. **19** (1978), 12–23.

[36] A. F. Solovév, *The bending of hyperdistribution*, Geom. Sb. **20** (1979), 101–112.

[37] A. F. Solovév, *Second fundemamental form of a distribution*, Mat. Zametki **35** (1982), 139–146.

[38] S. Tanno, *Quasi-Sasakian structures of rank $2p+1$*, J. Differential Geom. **5** (1971), 317–324.

[39] M. Turana, C. Yetima, and S. K. Chaubey, *On quasi-Sasakian 3-manifolds admitting η-Ricci solitons*, Filomat **33** (2019), no. 15, 4923–4930.

[40] K. Yano, *Concircular geometry I. Concircular tranformations*, Proc. Imp. Acad. Tokyo **16** (1940), 195–200.

[41] K. Yano, *On the torse-forming directions in Riemannian spaces*, Proc. Imp. Acad. Tokyo **20** (1944), 340–345.

[42] K. Yano and B. Y. Chen, *On the concurrent vector fields of immersed manifolds*, Kodai Math. Sem. Rep. **23** (1971), no. 3, 343–350.

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