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SOLUTIONS OF FRACTIONAL ORDER TIME-VARYING LINEAR DYNAMICAL SYSTEMS USING THE RESIDUAL POWER SERIES METHOD

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Abstract. In this paper, the fractional order time-varying linear dynamical systems are investigated by using a residual power series method. A residual power series method (RPSM) is constructed for this problem. The exact solution is obtained by the Laplace transform method and the analytical solution is calculated via the residual power series method (RPSM). As an application, some examples are tested to show the accuracy and efficacy of the proposed methods. The obtained result showed that the proposed methods are effective and accurate for this type of problem.

1. Introduction

Partial differential equations (PDEs) and the fractional order partial differential equations (FPDEs) have wide applications in many branches of science. Mathematical models of various physical, chemical, biological, or environmental processes often include non-classical conditions. Many different methods are used to investigate the approximate solution and the exact solution to the PDEs and FPDEs, among these, are the homotopy perturbation method [14], the modified quintic B-spline Crank-Nicolson collocation method [20], the Daftar-Gaji-Jafaris method and the Laplace transforms collocation method [17], finite difference method [18], Adomian decomposition method [22], Laplace transforms method [7], Fourier series method [3], power series method [1], Sumudu transforms method [6] and the Adams-Bashforth numerical method [21]. Various types of problems have been solved with exact and analytical solutions using the RPSM method. Finally, a residual power series method was applied to obtain the approximate analytical solution for PDEs [15]. Furthermore, the dynamical systems connected to biology phenomena are described by PDEs involving dissipation actions in [12-15].

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In this work, the fractional order time varying linear dynamical system

$$(1)\quad \begin{cases} \frac{\partial^{3\alpha} w(t,x)}{\partial t^{3\alpha}} + \frac{\partial^{2\alpha} w(t,x)}{\partial t^{2\alpha}} + \frac{\partial^{\alpha} w(t,x)}{\partial t^{\alpha}} - \frac{\partial^{2} w(t,x)}{\partial x^{2}} = f(t,x), 0 < x < L, \\ 0 < t < T, 0 < \alpha \leq 1 \\ w(0,x) = v_1(x), w_t(0,x) = v_2(x), w_{tt}(0,x) = v_3, 0 \leq x \leq L, \\ w(t,0) = k_1(t), 0 \leq t \leq T, w(t,L) = k_2(t), 0 \leq t \leq T. \end{cases}$$

is investigated. Here, $f, v_{1,} v_2$ and v_3 are known continuous functions and $w(t,x)$ is an unknown function to be found. The Laplace transform method and the residual power series method will be used to find exact and analytical solution of this problem. This study differs from earlier studies by examining the solution to this problem using the suggested method. The linear time-varying system model is critical for many ostensibly nonlinear systems, including mixers and capacitor switching filters, where the signal path is designed to be linear but can change in response to other inputs like local oscillators and clocks. The main advantage of the Caputo approach is that the initial conditions defined for Caputo-derived fractional differential equations are the same as those specified for integer-order differential equations. Therefore, according to recent studies in the literature, the Caputo partial derivative factor is preferred more than the Riemann-Liouville derivative in analytic and numerical solutions to fractional differential equations. The exact solutions of the partial differential equations of fractional order are very rare in the literature, various methods have been developed for finding approximate and analytical solutions, ,including the Homotopy Perturbation General Transform Method (HPGTM) [16], the fractional residual power series method [17], the Homotopy perturbation Sawi transform method (HPSTM) [18], the Laplace Residual Power Series Method [19], and the Crank–Nicholson difference scheme [20]. One of these methods is "Residual power series method", which will be tested in this study. The residual power series method is mainly based on the general formula of Taylor series and the residual error function. A new analytical solution is being investigated. The residual power series method is designed to reveal the reliability and fast convergence ability by comparing the solutions obtained with the present method with the exact solution.

Now, we will give some basic definitions:

Definition 1.1. [19] *Let $\beta, n \in N$, $n-1 \leq \alpha \leq n$, and $\beta \geq [|\alpha|]$. Then we define*

$$D^{\alpha}(x^{\beta}) = \frac{x^{\beta-\alpha}}{\Gamma(\beta-\alpha+1)}\Gamma(\beta+1).$$

Definition 1.2. *Time dependent α-order $D_t^{\alpha} w(t,x)$ Caputo fractional derivative for $n-1 < \alpha \leq n$, is defined in* [19], *as follows*

$$D_t^{\alpha} w(t,x) = \frac{\partial^{\alpha} w(t,x)}{\partial t^{\alpha}} = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{1}{(t-p)^{\alpha-n+1}} \frac{\partial^{\alpha} w(p,x)}{\partial p^{\alpha}} \partial p,$$

and for $\alpha = n \in N$, *it is defined as*

$$D_t^{\alpha} w(t,x) = \frac{\partial^{\alpha} w(t,x)}{\partial t^{\alpha}} = \frac{\partial^{n} w(t,x)}{\partial t^{n}}.$$

Now, we will try to find the exact solution of the following example for the problem (1), by using the Laplace transform method.

Example 1.3. *Consider the third-order fractional partial differential equation:*

$$(2) \quad \begin{cases} \frac{\partial^{3\alpha} w(t,x)}{\partial t^{3\alpha}} + \frac{\partial^{2\alpha} w(t,x)}{\partial t^{2\alpha}} + \frac{\partial^{\alpha} w(t,x)}{\partial t^{\alpha}} - \frac{\partial^{2} w(t,x)}{\partial x^{2}} = f(t,x), \\ f(t,x) = (\frac{6t^{3-3\alpha}}{\Gamma(4-3\alpha)} + \frac{6t^{3-2\alpha}}{\Gamma(4-2\alpha)} + \frac{6t^{3-\alpha}}{\Gamma(4-\alpha)} + t^3) \cos x, \\ 0 < x < \frac{\pi}{2}, 0 < t < 1, 0 < \alpha \leq 1, \\ w(0,x) = w_t(0,x) = w_{tt}(0,x) = 0, \\ w(t,0) = w(t,\frac{\pi}{2}) = 0. \end{cases}$$

To find the exact solution of equation (2) firstly, we take the Laplace transform to both sides of equation (2), we have

$$(3) \quad \left(\frac{\partial^{3\alpha} w(t,x)}{\partial t^{3\alpha}}\right) + \left(\frac{\partial^{2\alpha} w(t,x)}{\partial t^{2\alpha}}\right) + \left(\frac{\partial^{\alpha} w(t,x)}{\partial t^{\alpha}}\right) - \left(\frac{\partial^{2} w(t,x)}{\partial x^{2}}\right) = (f(t,x)),$$

$$\mathcal{L}\left(\frac{\partial^{3\alpha} w(t,x)}{\partial t^{3\alpha}}\right) + \mathcal{L}\left(\frac{\partial^{2\alpha} w(t,x)}{\partial t^{2\alpha}}\right) + \mathcal{L}\left(\frac{\partial^{\alpha} w(t,x)}{\partial t^{\alpha}}\right) - \mathcal{L}\left(\frac{\partial^{2} w(t,x)}{\partial x^{2}}\right) =$$

$$(4) \quad \mathcal{L}\left((\frac{6t^{3-3\alpha}}{\Gamma(4-3\alpha)} + \frac{6t^{3-2\alpha}}{\Gamma(4-2\alpha)} + \frac{6t^{3-\alpha}}{\Gamma(4-\alpha)} + t^3) \cos x\right),$$

by using the initial conditions, we obtain

$$s^{3\alpha} w(s,x) + s^{2\alpha} w(s,x) + s^{\alpha} w(s,x) - w_{xx}(s,x)$$

$$(5) \quad = \left\{\frac{6\Gamma(4-3\alpha)}{s^{4-3\alpha}\Gamma(4-3\alpha)} + \frac{6\Gamma(4-2\alpha)}{s^{4-2\alpha}\Gamma(4-2\alpha)} - \frac{6\Gamma(4-\alpha)}{s^{4-\alpha}\Gamma(4-\alpha)} + \frac{6}{s^4}\right\} \cos x,$$

we can rewrite the equation (5), as follows

$$(6) \quad s^{3\alpha} w(s,x) + s^{2\alpha} w(s,x) + s^{\alpha} w(s,x) - w_{xx}(s,x) = \left\{\frac{6}{s^{4-3\alpha}} + \frac{6}{s^{4-2\alpha}} - \frac{6}{s^{4-\alpha}} + \frac{6}{s^4}\right\} \cos x.$$

Equation (6) is called the second order nonlinear homogeneous equation. Suppose that the solution of equation (6) is of the following form:

$$(7) \quad w(s,x) = w^c(s,x) + w^p(s,x).$$

The homogeneous part of equation (6) is

$$(8) \quad \left(s^{3\alpha} + s^{2\alpha} + s^{\alpha}\right).w(s,x) - w_{xx}(s,x) = 0.$$

To solve this equation, it is possible to take

$$(9) \quad w^c(s,x) = c.e^{mx}.$$

If the partial derivative of the equation (9), with respect to x is written in the form of equation (8), then the characteristic equation is;

$$s^{3\alpha} + s^{2\alpha} + s^{\alpha} - m^2 = 0. \tag{10}$$

From equation (10), we can obtain the roots of equation (8), then we get the solution to the homogeneous part of equation (6), us follows

$$w^c(s,x) = c_1 e^{x\sqrt{s^{3\alpha}+s^{2\alpha}+s^{\alpha}}} + c_2 e^{-x\sqrt{s^{3\alpha}+s^{2\alpha}+s^{\alpha}}}. \tag{11}$$

To solve the non-homogeneous part of the equation (6) as follows

$$w^p(s,x) = A(s)\cos x \tag{12}$$

firstly take the derivatives of equation (12), and substitute in equation (6), then we get

$$\left(s^{3\alpha} + s^{2\alpha} + s^{\alpha} + 1\right).A(s)\cos x = \frac{6}{s^4}\left\{s^{3\alpha} + s^{2\alpha} + s^{\alpha} + 1\right\}\cos x, \tag{13}$$

simplify the above equation, then we get

$$A(s) = \frac{6}{s^4}. \tag{14}$$

Substituting the value of $A(s)$ in (12), it gives

$$w^p(s,x) = \frac{6}{s^4}\cos x. \tag{15}$$

Substituting (11) and (15) into equation (7), we have

(16)

$$w(s,x) = w^c(s,x) + w^p(s,x) = c_1 e^{x\sqrt{s^{3\alpha}+s^{2\alpha}+s^{\alpha}}} + c_2 e^{-x\sqrt{s^{3\alpha}+s^{2\alpha}+s^{\alpha}}} + \frac{6}{s^4}\cos x.$$

If the limit value in equation (2) is used instead of conditions, then it is clear that the values of c_1 and c_2 are 0. Therefore the solution is

$$w(s,x) = \frac{6}{s^4}\cos x, \tag{17}$$

take the inverse Laplace transform of both sides of equation (17), then it gives the exact solution

$$w(t,x) = L^{-1}(w(s,x)) = L^{-1}(\frac{6}{s^4}\cos x) = t^3\cos x. \tag{18}$$

2. Constructing the residual power series method for the proposed model

This section includes the discussion of the fundamental scheme of RPSM to solve problem (1). The solution of problems (1) in RPSM can be explained by expanding the power series around the initial point $t = 0$, and suppose that the solutions take the expansion

$$w(t,x) = \sum_{i=0}^{\infty} f_i(x) t^i q \tag{19}$$

firstly, we define the $w_k(t,x)$ to denote k. term of the solution $w(t,x)$ is given as

$$w_k(t,x) = \sum_{i=0}^{\infty} f_i(x) t^i, i = 0, 1, 2, ..., \tag{20}$$

for the solution of $w(t,x)$, where $k = 1, 2, 3, ..., w(t,x)$ clearly satisfies the initial conditions. Therefore, the solution of the equation $w(t,x)$ depending on the initial value conditions, the 0^{th} residual power series is taken as

$$w(0,x) = v_1(x) = f_0(x), w_t(0,x) = v_2(x) = f_1(x), w_{tt}(0,x) = v_3 = f_2(x). \tag{21}$$

On the other hand, the first condition in equation (20) is satisfied. After that, the first approximate solutions of the residual power method with $w(x,t)$ should be

$$w_1(t,x) = f_0(x) + f_1(x)t + f_2(x)t^2. \tag{22}$$

Therefore, to find the value of the coefficients of the residual power series method, the expansion series (20) for $k = 3, 4, 5, ...$ can be reformulated as

$$w_k(t,x) = f_0(x) + f_1(x)t + f_2(x)t^2 + \sum_{i=3}^{k} f_i(x) t^i. \tag{23}$$

In the expansion of solution for equation (1), the remaining functions of the equation (23) should be defined as:

$$Resw(t,x) = \frac{\partial^{3\alpha} w(t,x)}{\partial t^{3\alpha}} + \frac{\partial^{2\alpha} w(t,x)}{\partial t^{2\alpha}} + \frac{\partial^{\alpha} w(t,x)}{\partial t^{\alpha}} - \frac{\partial^2 w(t,x)}{\partial x^2} - f(t,x). \tag{24}$$

Therefore, the residual functions of the k.term of $Resw_k(t,x)$, is of the following form

$$Resw_k(t,x) = \frac{\partial^{3\alpha} w_k(t,x)}{\partial t^{3\alpha}} + \frac{\partial^{2\alpha} w_k(t,x)}{\partial t^{2\alpha}} + \frac{\partial^{\alpha} w_k(t,x)}{\partial t^{\alpha}} - \frac{\partial^2 w_k(t,x)}{\partial x^2} - f(t,x) \tag{25}$$

for $k = 1, 2, 3, ...$. It is clear that $Resw_k(t,x) = 0$ and $\lim_{x\to\infty} Resw_k(t,x) = Resw_k(t,x)$ for $x \in [0, X]$ and $t \geq 0$. Therefore, it can be written for $t = 0$, $s = 0$, $\frac{\partial^s}{\partial t^s} Resw_k(t,x) = 0$. For more details (see[15, 2]).

To obtain the $f_i(x)$ coefficients, where $i = 2, 3, 4, ..., k$. Then, the following operations will be applied for the values of equation (25)

$$\frac{\partial^s Resw_k(t,x)}{\partial t^s} = 0.$$

Applying the derivative formula and substituting the expression for $t = 0$, then the following equation will be solved for the coefficients

$$Resw_k(t,x) = 0, t = 0, s = 2,3,4,,,,k. \tag{26}$$

In this way, all coefficients in the power series can be found.

3. Applications of the RPSM to the fractional order time-varying linear dynamical systems

In this section, we will apply the residual power series method for solving fractional order time varying linear dynamical system, to show the accuracy and efficacy of the proposed method. For this purpose, we consider the following third order fractional partial differential equation

$$\begin{cases} \frac{\partial^{3\alpha} w(t,x)}{\partial t^{3\alpha}} + \frac{\partial^{2\alpha} w(t,x)}{\partial t^{2\alpha}} + \frac{\partial^{\alpha} w(t,x)}{\partial t^{\alpha}} - \frac{\partial^{2} w(t,x)}{\partial x^{2}} = \\ \quad \left(\frac{6t^{3-3\alpha}}{\Gamma(4-3\alpha)} + \frac{6t^{3-2\alpha}}{\Gamma(4-2\alpha)} + \frac{6t^{3-\alpha}}{\Gamma(4-\alpha)} + t^3\right)\cos x, \\ 0 < x < \frac{\pi}{2}, 0 < t < 1, 0 < \alpha \leq 1, \\ w(0,x) = w_t(0,x) = w_{tt}(0,x) = 0, \\ w(t,0) = w(t,\frac{\pi}{2}) = 0. \end{cases} \tag{27}$$

Using the residual power series method, firstly by using the formula (19) on each parts of (27), then we obtain the following equations

$$\begin{aligned} w_x(t,x) &= \sum_{i=0}^{\infty} \frac{\partial f_i(x) t^i}{\partial x}, \\ w_{xx}(t,x) &= \sum_{i=0}^{\infty} \frac{\partial^2 f_i(x) t^i}{\partial x^2}, \\ D_t^{3\alpha} w(t,x) &= \sum_{i=0}^{\infty} f_i(x) \frac{t^{i-3\alpha}}{\Gamma(i+1-3\alpha)} \Gamma(i+1), \\ D_t^{2\alpha} w(t,x) &= \sum_{i=0}^{\infty} f_i(x) \frac{t^{i-2\alpha}}{\Gamma(i+1-2\alpha)} \Gamma(i+1), \\ D_t^{\alpha} w(t,x) &= \sum_{i=0}^{\infty} f_i(x) \frac{t^{i-\alpha}}{\Gamma(i+1-\alpha)} \Gamma(i+1). \end{aligned}$$

Now we express the k.terms of $w(t,x)$ for $i = 0,1,2,...$,

$$w(t,x) = \sum_{i=0}^{k} f_i(x) t^i,$$

using the above formula, we get the following terms

$$
\begin{aligned}
w_x(t,x) &= \sum_{i=0}^{k} \frac{\partial f_i(x) t^i}{\partial x}, \\
w_{xx}(t,x) &= \sum_{i=0}^{k} \frac{\partial^2 f_i(x) t^i}{\partial x^2}, \\
D_t^{3\alpha} w(t,x) &= \sum_{i=3}^{k} f_i(x) \frac{t^{i-3\alpha}}{\Gamma(i+1-3\alpha)} \Gamma(i+1), \\
D_t^{2\alpha} w(t,x) &= \sum_{i=2}^{k} f_i(x) \frac{t^{i-2\alpha}}{\Gamma(i+1-2\alpha)} \Gamma(i+1), \\
D_t^{\alpha} w(t,x) &= \sum_{i=1}^{k} f_i(x) \frac{t^{i-\alpha}}{\Gamma(i+1-\alpha)} \Gamma(i+1).
\end{aligned}
$$

The solution of $w(t,x)$ of the residual power series with 0^{th}, depending on the initial conditions are

$$
\begin{aligned}
w(0,x) &= 0 = f_o(x), \\
w_t(0,x) &= 0 = f_1(x), \\
w_{tt}(0,x) &= 0 = f_2(x),
\end{aligned}
$$

using the formula (25), when $i = 0, 1, 2, ...$in the residual iteration method, then equation (27) becomes:

$$
\begin{aligned}
Resw_k(t,x) &= \left(\sum_{i=3}^{k} f_i(x) \frac{t^{i-3\alpha}}{\Gamma(i+1-3\alpha)} \Gamma(i+1) \right) + \left(\sum_{i=2}^{k} f_i(x) \frac{t^{i-2\alpha}}{\Gamma(i+1-2\alpha)} \Gamma(i+1) \right) \\
&+ \left(\sum_{i=1}^{k} f_i(x) \frac{t^{i-\alpha}}{\Gamma(i+1-\alpha)} \Gamma(i+1) \right) - \left(\sum_{i=0}^{k} \frac{\partial^2 f_i(x) t^i}{\partial x^2} \right) \\
&-\left(\frac{6t^{3-3\alpha}}{\Gamma(4-3\alpha)} + \frac{6t^{3-2\alpha}}{\Gamma(4-2\alpha)} + \frac{6t^{3-\alpha}}{\Gamma(4-\alpha)} + t^3 \right) \cos x.
\end{aligned} \tag{28}
$$

By using formula (22) and the above equation, then we obtain the following iterations

$$
w_1(t,x) = 0.
$$

If $k = 3$, then we have

$$\begin{aligned} Resw_3(t,x) &= \left(\sum_{i=3}^{3} f_i(x)\frac{t^{i-3\alpha}}{\Gamma(i+1-3\alpha)}\Gamma(i+1)\right) \\ &+ \left(\sum_{i=2}^{3} f_i(x)\frac{t^{i-2\alpha}}{\Gamma(i+1-2\alpha)}\Gamma(i+1)\right) \\ &+ \left(\sum_{i=1}^{3} f_i(x)\frac{t^{i-\alpha}}{\Gamma(i+1-\alpha)}\Gamma(i+1)\right) \\ &- \left(\sum_{i=0}^{3} \frac{\partial^2 f_i(x)t^3}{\partial x^2}\right) - (\frac{6t^{3-3\alpha}}{\Gamma(4-3\alpha)} + \frac{6t^{3-2\alpha}}{\Gamma(4-2\alpha)} + \frac{6t^{3-\alpha}}{\Gamma(4-\alpha)} + t^3)\cos x. \end{aligned}$$

Since $f_1(x) = f_2(x) = 0$, hence

$$\begin{aligned} Resu_3(t,x) &= \left(6f_3(x)\frac{t^{3-3\alpha}}{\Gamma(4-3\alpha)}\right) + \left(6f_3(x)\frac{t^{3-2\alpha}}{\Gamma(4-2\alpha)}\right) + \left(6f_3(x)\frac{t^{3-\alpha}}{\Gamma(4-\alpha)}\right) \end{aligned}$$

$$-\left(\frac{\partial^2 f_3(x)t^3}{\partial x^2}\right) = (\frac{6t^{3-3\alpha}}{\Gamma(4-3\alpha)} + \frac{6t^{3-2\alpha}}{\Gamma(4-2\alpha)} + \frac{6t^{3-\alpha}}{\Gamma(4-\alpha)} + t^3)\cos x, \tag{29}$$

if we rearranging the last equation;

$$\begin{aligned} &f_3(x)\left[\frac{6t^{3-3\alpha}}{\Gamma(4-3\alpha)} + \frac{6t^{3-2\alpha}}{\Gamma(4-2\alpha)} + \frac{6t^{3-\alpha}}{\Gamma(4-\alpha)}\right] - \left(\frac{\partial^2 f_3(x)t^3}{\partial x^2}\right) \\ &\quad = (\frac{6t^{3-3\alpha}}{\Gamma(4-3\alpha)} + \frac{6t^{3-2\alpha}}{\Gamma(4-2\alpha)} + \frac{6t^{3-\alpha}}{\Gamma(4-\alpha)} + t^3)\cos x. \end{aligned} \tag{30}$$

Since the first part will be zero for $t = 0$, so we have

$$-\left(\frac{\partial^2 f_3(x)t^3}{\partial x^2}\right) = t^3 \cos x,$$

integrating the above expression twice; we have

$$f_3(x) = \cos x. \tag{31}$$

Substituting the values of $f_0(x) = f_1(x) = f_2(x) = 0$, and equation (31) in formula (23), we obtain

$$w_2(t,x) = t^3 \cos x.$$

By the same way we can obtain $w_3(t,x)$, $w_4(t,x)$, $w_5(t,x)$ ..., and substituting in formula (19), then it gives the exact solution as

$$w(t,x) = t^3 \cos x.$$

4. Conclusion

In this study, the third-order time-varying fractional partial differential equation is discussed. The exact solution of the fractional partial differential equation is obtained by the Laplace transform method. The Residual Power Series method was constructed for the third-order fractional time-varying linear dynamical systems. The analytical solution was obtained by the proposed method. The result obtained by the Residual Power Series method is equivalent to the exact solution that we obtained by the Laplace transform method.

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