

ITERATING A SYSTEM OF SET-VALUED VARIATIONAL INCLUSION PROBLEMS IN SEMI-INNER PRODUCT SPACES

SUMEERA SHAFI

ABSTRACT. In this paper, we introduce a new system of set-valued variational inclusion problems in semi-inner product spaces. We use resolvent operator technique to propose an iterative algorithm for computing the approximate solution of the system of set-valued variational inclusion problems. The results presented in this paper generalize, improve and unify many previously known results in the literature.

1. INTRODUCTION

Variational inequalities have been well inquired and theorized to distinct directions due to its huge collaboration with partial differential equations and optimization problems. An intimate conclusion of variational inequality problem is a variational inclusion problem which is of ruling concern. Numerous researchers used different tactics to establish iterative algorithms for solving various classes of variational inequality and variational inclusion problems. The method based on the resolvent operator technique is a conception of projection method and has been widely used to solve variational inclusion problems, see for example, [1-4, 7-9, 11, 13, 14, 19].

Inspired by recent research works in this area, in this paper, we consider a system of set-valued variational inclusion problems (in short, SSVIP) in 2-uniformly smooth Banach space. Further, using $H - \eta$ -accretive mapping, we establish an iterative algorithm for recognizing the approximate solution of the system of variational inclusions and check the convergence of sequences generated by iterative algorithm.

Received by the editors January 06, 2022. Revised September 13. Accepted September 20, 2022.
2010 *Mathematics Subject Classification.* 47H09, 47J20, 49J40.

Key words and phrases. system of set-valued variational inclusion problems, $H - \eta$ -accretive mappings, 2-uniformly smooth Banach spaces, generalized adjoint operator, iterative algorithm, convergence analysis.

2. RESOLVENT OPERATOR AND FORMULATION OF PROBLEM

Let X be a real 2-uniformly smooth Banach space equipped with norm $\|\cdot\|$ and a semi-inner product $[\cdot, \cdot]$. Let $C(X)$ be the family of all nonempty compact subsets of X and 2^X be the power set of X .

We need the following definitions and results from the literature.

Definition 2.1 ([15]). Let X be a vector space over the field F of real or complex numbers. A functional $[\cdot, \cdot] : X \times X \rightarrow F$ is called a *semi-inner product* if it satisfies the following:

- (i) $[x + y, z] = [x, z] + [y, z]$, $\forall x, y, z \in X$,
- (ii) $[\lambda x, y] = \lambda[x, y]$, $\forall \lambda \in F$ and $x, y \in X$,
- (iii) $[x, x] > 0$, for $x \neq 0$,
- (iv) $|[x, y]|^2 \leq [x, x][y, y]$.

The pair $(X, [\cdot, \cdot])$ is called a *semi-inner product space*.

We observe that $\|x\| = [x, x]^{\frac{1}{2}}$ is a norm on X . Hence every semi-inner product space is a normed linear space. On the other hand, in a normed linear space, one can generate semi-inner product in infinitely many different ways. Giles [10] had proved that if the underlying space X is a uniformly convex smooth Banach space then it is possible to find a semi-inner product, uniquely. Also the unique semi-inner product has the following nice properties:

- (i) $[x, y] = 0$ if and only if y is orthogonal to x , that is if and only if $\|y\| \leq \|y + \lambda x\|$, \forall scalars λ .
- (ii) Generalized Riesz representation theorem: If f is a continuous linear functional on X then there is a unique vector $y \in X$ such that $f(x) = [x, y]$, $\forall x \in X$.
- (iii) The semi-inner product is continuous, that is for each $x, y \in X$, we have $\text{Re}[y, x + \lambda y] \rightarrow \text{Re}[y, x]$ as $\lambda \rightarrow 0$.

The sequence space l^p , $p > 1$ and the function space L^p , $p > 1$ are uniformly convex smooth Banach spaces. So one can define semi-inner product on these spaces, uniquely.

Example 2.2 ([20]). The real sequence space l^p for $1 < p < \infty$ is a semi-inner product space with the semi-inner product defined by

$$[x, y] = \frac{1}{\|y\|_p^{p-2}} \sum_i x_i y_i |y_i|^{p-2}, \quad x, y \in l^p.$$

Example 2.3 ([10, 20]). The real Banach space $L^p(X, \mu)$ for $1 < p < \infty$ is a semi-inner product space with the semi-inner product defined by

$$[f, g] = \frac{1}{\|g\|_p^{p-2}} \int_X f(x)|g(x)|^{p-1} \operatorname{sgn}(g(x)) d\mu, \quad f, g \in L^p.$$

Definition 2.4 ([20, 21]). Let X be a real Banach space. Then:

(i) The modulus of smoothness of X is defined as

$$\rho_X(t) = \sup \left\{ \frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1, \|y\| = t, t > 0 \right\}.$$

(ii) X is said to be *uniformly smooth* if $\lim_{t \rightarrow 0} \frac{\rho_X(t)}{t} = 0$.

(iii) X is said to be *p -uniformly smooth* if there exists a positive real constant c such that $\rho_X(t) \leq c t^p$, $p > 1$. Clearly, X is 2-uniformly smooth if there exists a positive real constant c such that $\rho_X(t) \leq c t^2$.

Lemma 2.5 ([20, 21]). Let $p > 1$ be a real number and X be a smooth Banach space. Then the following statements are equivalent:

(i) X is 2-uniformly smooth.

(ii) There is a constant $c > 0$ such that for every $x, y \in X$, the following inequality holds

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, f_x \rangle + c\|y\|^2,$$

where $f_x \in J(x)$ and $J(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 \text{ and } \|x^*\| = \|x\|\}$ is the normalized duality mapping.

Remark 2.6 ([20]). Every normed linear space is a semi-inner product space (see [15]). In fact by Hahn Banach theorem, for each $x \in X$, there exists atleast one functional $f_x \in X^*$ such that $\langle x, f_x \rangle = \|x\|^2$. Given any such mapping f from X into X^* , we can verify that $[y, x] = \langle y, f_x \rangle$ defines a semi-inner product. Hence we can write (ii) of above Lemma as

$$\|x+y\|^2 \leq \|x\|^2 + 2[y, x] + c\|y\|^2, \quad \forall x, y \in X.$$

The constant c is chosen with best possible minimum value. We call c , as the constant of smoothness of X .

Example 2.7 ([20]). The function space L^p is 2-uniformly smooth for $p \geq 2$ and it is p -uniformly smooth for $1 < p < 2$. If $2 \leq p < \infty$, then we have for all $x, y \in L^p$,

$$\|x + y\|^2 \leq \|x\|^2 + 2[y, x] + (p - 1)\|y\|^2.$$

Here the constant of smoothness is $p - 1$.

Definition 2.8 ([16, 20]). Let X be a real 2-uniformly smooth Banach space. A mapping $T : X \rightarrow X$ is said to be:

- (i) *monotone*, if $[Tx - Ty, x - y] \geq 0, \forall x, y \in X$,
- (ii) *strictly monotone*, if $[Tx - Ty, x - y] \geq 0, \forall x, y \in X$, and equality holds if and only if $x = y$,
- (iii) *r -strongly monotone* if there exists a positive constant $r > 0$ such that

$$[Tx - Ty, x - y] \geq r\|x - y\|^2, \forall x, y \in X,$$

- (iv) *δ -Lipschitz continuous*, if there exists a constant $\delta > 0$ such that

$$\|T(x) - T(y)\| \leq \delta\|x - y\|, \forall x, y \in X,$$

- (v) *η -monotone*, if $[Tx - Ty, \eta(x, y)] \geq 0, \forall x, y \in X$,
- (vi) *strictly η -monotone*, if $[Tx - Ty, \eta(x, y)] \geq 0, \forall x, y \in X$, and equality holds if and only if $x = y$,
- (vii) *r -strongly η -monotone* if there exists a positive constant $r > 0$ such that

$$[Tx - Ty, \eta(x, y)] \geq r\|x - y\|^2, \forall x, y \in X,$$

- (viii) *μ -cocoercive* if there exists a constant $\mu > 0$ such that

$$[Tx - Ty, x - y] \geq \mu\|Tx - Ty\|^2, \forall x, y \in X.$$

Definition 2.9. Let X be a 2-uniformly smooth Banach space. Let $H : X \rightarrow X$, $\eta : X \times X \rightarrow X$ be single-valued mappings and $M : X \times X \rightarrow 2^X$ be multi-valued mapping. Then

- (i) H is said to be *η -accretive*, if

$$\left[Hx - Hy, \eta(x, y) \right] \geq 0, \forall x, y \in X.$$

- (ii) H is said to be *strictly η -accretive*, if H is η -accretive and equality holds if and only if $x = y$.
- (iii) H is said to be *r -strongly η -accretive* if there exists a constant $r > 0$ such that

$$\left[Hx - Hy, \eta(x, y) \right] \geq r\|x - y\|^2, \forall x, y \in X.$$

(iv) η is said to be *m-Lipschitz continuous*, if there exists a constant $m > 0$ such that

$$\|\eta(x, y)\| \leq m\|x - y\|, \forall x, y \in X.$$

(v) M is said to be η -accretive in the first argument if

$$[u - v, \eta(x, y)] \geq 0, \forall x, y \in X, \forall u \in M(x, t), v \in M(y, t), \text{ for each fixed } t \in X.$$

(vi) M is said to be *strictly η -accretive*, if M is η -accretive in the first argument and equality holds if and only if $x = y$.

(vii) μ -strongly η -accretive if there exists a positive constant $\mu > 0$ such that

$$[u - v, \eta(x, y)] \geq \mu\|x - y\|^2, \forall x, y \in X, u \in M(x, t), v \in M(y, t).$$

Definition 2.10. Let $S : X \times X \times X \rightarrow X$ be a single-valued mapping. Then the mapping S is called

(i) (ξ, γ) -relaxed cocoercive in the second argument if there exists a constant $\xi, \gamma > 0$ such that

$$[S(x, y, z) - S(x, y', z), y - y'] \geq -\xi\|S(x, y, z) - S(x, y', z)\|^2 + \gamma\|y - y'\|^2, \forall x, y, y', z \in X.$$

(ii) σ -relaxed accretive in the third argument if there exists a constant $\sigma > 0$ such that

$$[S(x, y, z) - S(x, y, z'), z - z'] \geq -\sigma\|z - z'\|^2, \forall x, y, z, z' \in X.$$

(iii) δ -strongly monotone in the first argument if there exists a constant $\delta > 0$ such that

$$[S(x, y, z) - S(x', y, z), x - x'] \geq \delta\|x - x'\|^2, \forall x, x', y, z \in X.$$

Definition 2.11. Let $H : X \rightarrow X, \eta : X \times X \rightarrow X$ be single-valued mappings, $M : X \times X \rightarrow 2^X$ be a multi-valued mapping, then M is said to be $H - \eta$ -accretive mapping if for each fixed $t \in X, M(\cdot, t)$ is η -accretive in the first argument and $(H + \rho M(\cdot, t))X = X, \forall \rho > 0$.

Theorem 2.12. Let $H : X \rightarrow X, \eta : X \times X \rightarrow X$ be single-valued mappings. Let $H : X \rightarrow X$ be s -strongly η -accretive, $M : X \times X \rightarrow 2^X$ be $H - \eta$ -accretive mapping. If the following inequality: $[u - v, \eta(x, y)] \geq 0$, holds $\forall (y, v) \in \text{Graph}(M(\cdot, t))$, then $(x, u) \in \text{Graph}(M(\cdot, t))$, where $\text{Graph}(M(\cdot, t)) := \{(x, u) \in X \times X : u \in M(x, t)\}$.

Proof. Suppose, on the contrary that there exists some $(x_0, u_0) \notin \text{Graph}(M(., t))$ such that

$$(2.1) \quad \left[u_0 - v, \eta(x_0, y) \right] \geq 0, \quad \forall (y, v) \in \text{Graph}(M(., t)).$$

Since M is $H - \eta$ -accretive operator,

$$\implies (H + \rho M(., t))(X) = X \quad \forall \rho > 0 \text{ and for each fixed } t \in X.$$

Therefore, there exists $(x_1, u_1) \in \text{Graph}(M(., t))$ such that

$$(2.2) \quad H(x_1) + \rho u_1 = H(x_0) + \rho u_0 \in X.$$

Combining (2.1) and (2.2), we have

$$\begin{aligned} 0 &\leq \rho \left[u_0 - u_1, \eta(x_0, x_1) \right] \\ &= - \left[H(x_0) - H(x_1), \eta(x_0, x_1) \right] \\ &\leq -s \|x_0 - x_1\|^2 \leq 0. \end{aligned}$$

Since $s > 0$, therefore $x_1 = x_0$.

Hence, it follows from (2.2) that $u_1 = u_0$, a contradiction. This completes the proof. \square

Theorem 2.13. *Let $H : X \rightarrow X$, $\eta : X \times X \rightarrow X$ be single-valued mappings. Let $H : X \rightarrow X$ be s -strongly η -accretive, $M : X \times X \rightarrow 2^X$ be $H - \eta$ -accretive mappings. Then the mapping $(H + \rho M(., t))^{-1}$ is single-valued, $\forall \rho > 0$.*

Proof. For any $z \in X$, let $x, y \in (H + \rho M(., t))^{-1}(z)$. It follows that

$$\frac{1}{\rho}(z - H(x)) \in M(x, t),$$

and

$$\frac{1}{\rho}(z - H(y)) \in M(y, t).$$

Since $M(., t)$ is η -accretive in the first argument and H is s -strongly η -accretive, we have

$$\begin{aligned} 0 &\leq \left[\frac{1}{\rho}(z - H(x)) - \frac{1}{\rho}(z - H(y)), \eta(x, y) \right] \\ &= -\frac{1}{\rho} \left[H(x) - H(y), \eta(x, y) \right] \\ &\leq -\frac{1}{\rho} s \|x - y\|^2, \end{aligned}$$

which implies that

$$\frac{1}{\rho}s\|x - y\|^2 \leq 0.$$

Since $s > 0$, therefore $x = y$ and so $(H + \rho M(\cdot, t))^{-1}$ is single-valued. This completes the proof. \square

Definition 2.14. Let $H : X \rightarrow X$, $\eta : X \times X \rightarrow X$ be single-valued mappings. Let $H : X \rightarrow X$ be s -strongly η -accretive, $M : X \times X \rightarrow 2^X$ be $H - \eta$ -accretive mappings. Then for each fixed $t \in X$, the resolvent operator $R_{\rho,\eta}^{H,M(\cdot,t)} : X \rightarrow X$ is defined by

$$R_{\rho,\eta}^{H,M(\cdot,t)}(x) = (H + \rho M(\cdot, t))^{-1}(x), \forall x \in X.$$

Now, we prove that the resolvent operator defined above is Lipschitz continuous.

Theorem 2.15. Let $\eta : X \times X \rightarrow X$ be m -Lipschitz continuous mapping. Let $M : X \times X \rightarrow 2^X$ be $H - \eta$ -accretive mapping and H be s -strongly η -accretive. Then for each fixed $t \in X$ the resolvent operator of M , $R_{\rho,\eta}^{H,M(\cdot,t)}(x) = (H + \rho M(\cdot, t))^{-1}(x)$ is $\frac{m}{s}$ -Lipschitz continuous, that is,

$$\left\| R_{\rho,\eta}^{H,M(\cdot,t)}(x) - R_{\rho,\eta}^{H,M(\cdot,t)}(y) \right\| \leq \frac{m}{s}\|x - y\|, \forall x, y, t \in X.$$

Proof. Let $x, y \in X$. Then from Definition 2.14, it follows that

$$R_{\rho,\eta}^{H,M(\cdot,t)}(x) = (H + \rho M(\cdot, t))^{-1}(x),$$

and

$$R_{\rho,\eta}^{H,M(\cdot,t)}(y) = (H + \rho M(\cdot, t))^{-1}(y),$$

and so

$$\frac{1}{\rho} \left(x - H(R_{\rho,\eta}^{H,M(\cdot,t)}(x)) \right) \in M \left(R_{\rho,\eta}^{H,M(\cdot,t)}(x), t \right),$$

and

$$\frac{1}{\rho} \left(y - H(R_{\rho,\eta}^{H,M(\cdot,t)}(y)) \right) \in M \left(R_{\rho,\eta}^{H,M(\cdot,t)}(y), t \right).$$

For the sake of brevity, let $A(x) = R_{\rho,\eta}^{H,M(\cdot,t)}(x)$, $A(y) = R_{\rho,\eta}^{H,M(\cdot,t)}(y)$.

Since $M(\cdot, t)$ is η -accretive operator in the first argument, we have

$$0 \leq \left[\frac{1}{\rho} \left(x - H(A(x)) \right) - \frac{1}{\rho} \left(y - H(A(y)) \right), \eta \left(A(x), A(y) \right) \right]$$

or,

$$\left[x - y, \eta \left(A(x), A(y) \right) \right] \geq \left[H(A(x)) - H(A(y)), \eta \left(A(x), A(y) \right) \right].$$

Since H is s -strongly η -accretive, we have

$$\begin{aligned} & \|x - y\| \left\| \eta \left(A(x), A(y) \right) \right\| \\ & \geq \left[x - y, \eta \left(A(x), A(y) \right) \right] \\ & \geq \left[H(A(x)) - H(A(y)), \eta \left(A(x), A(y) \right) \right] \\ & \geq s \|A(x) - A(y)\|^2, \end{aligned}$$

therefore,

$$\|x - y\| m \|A(x) - A(y)\| \geq s \|A(x) - A(y)\|^2.$$

This implies

$$\|A(x) - A(y)\| \leq \frac{m}{s} \|x - y\|,$$

or,

$$\|R_{\rho, \eta}^{H, M(\cdot, t)}(x) - R_{\rho, \eta}^{H, M(\cdot, t)}(y)\| \leq \frac{m}{s} \|x - y\|.$$

This completes the proof. \square

Definition 2.16. The Hausdorff metric $\mathcal{D}(\cdot, \cdot)$ on $CB(X)$, is defined by

$$\mathcal{D}(A, B) = \max \left\{ \sup_{u \in A} \inf_{v \in B} d(u, v), \sup_{v \in B} \inf_{u \in A} d(u, v) \right\}, \quad A, B \in CB(X),$$

where $d(\cdot, \cdot)$ is the induced metric on X and $CB(X)$ denotes the family of all nonempty closed and bounded subsets of X .

Definition 2.17 ([6]). A set-valued mapping $T : X \rightarrow CB(X)$ is said to be γ - \mathcal{D} -Lipschitz continuous, if there exists a constant $\gamma > 0$ such that

$$\mathcal{D}(T(x), T(y)) \leq \gamma \|x - y\|, \quad \forall x, y \in X.$$

Theorem 2.18 ([17]). Let $T : X \rightarrow CB(X)$ be a set-valued mapping on X and (X, d) be a complete metric space. Then:

- (i) For any given $\xi > 0$ and for any given $u, v \in X$ and $x \in T(u)$, there exists $y \in T(v)$ such that

$$d(x, y) \leq (1 + \xi) \mathcal{D}(T(u), T(v));$$

- (ii) If $T : X \rightarrow C(X)$, then (i) holds for $\xi = 0$, (where $C(X)$ denotes the family of all nonempty compact subsets of X).

Definition 2.19 ([18]). Let Y be a semi-inner product space and let $T : X \rightarrow Y$ be an arbitrary operator. Then the generalized adjoint operator T^+ of an operator T is defined as follows: The domain $D(T^+)$ of T^+ consists of those $y \in Y$ for which there exists $z \in X$ such that

$$\left[Tx, y \right]_Y = \left[x, z \right]_X$$

for each $x \in X$ and $z = T^+y$.

Remark 2.20. T^+ is an operator from $D(T^+)$ into X with the nonempty domain $D(T^+)$, since $0 \in D(T^+)$. Hence $T^+(0) = 0$. As it is observed in [5] that if X and Y are Hilbert spaces then the generalized adjoint operator is the usual adjoint operator. In general, T^+ is not linear even for T is a bounded linear operator.

Proposition 2.21. *Let X and Y be 2-uniformly convex smooth Banach spaces and let $T : X \rightarrow Y$ be a bounded linear operator. Then*

- (i) $D(T^+) = Y$
- (ii) T^+ is bounded, and it holds that

$$\|T^+y\| \leq \|T\| \|y\|, \forall y \in Y.$$

Now, we formulate our main problem.

For each $i = 1, 2$, let $S_i, N_i : X_i \times X_i \times X_i \rightarrow X_i$, $H_i : X_i \rightarrow X_i$, $\eta_i : X_i \times X_i \rightarrow X_i$ be single-valued mappings. Let $A_i, B_i, F_i : X_i \rightarrow C(X_i)$ be set-valued mappings. Suppose that $M_i : X_i \times X_i \rightarrow 2^{X_i}$ is $H_i - \eta_i$ -accretive mapping and let $G : X_1 \rightarrow X_2$ be a bounded linear operator such that $x_2 = Gx_1 \in X_2, y_2 = Gy_1 \in X_2$. Then we consider the following system of set-valued variational inclusion problems (in short, SSVIP): Find $(x_i, y_i) \in X_i \times X_i, u_i \in A_i(y_i), v_i \in B_i(y_i), w_i \in F_i(y_i), u'_i \in A_i(x_i), v'_i \in B_i(x_i), w'_i \in F_i(x_i)$ such that

$$(2.3) \left. \begin{aligned} 0 \in H_1(x_1) - H_1(y_1) + \rho_1 \{ S_1(u_1, v_1, w_1) + M_1(x_1, y_1) \} \\ 0 \in H_2(x_2) - H_2(y_2) + \rho_2 \{ S_2(u_2, v_2, w_2) + M_2(x_2, y_2) \} \\ 0 \in H_1(y_1) - H_1(x_1) + \rho_1 \{ N_1(u'_1, v'_1, w'_1) + M_1(y_1, x_1) \} \\ 0 \in H_2(y_2) - H_2(x_2) + \rho_2 \{ N_2(u'_2, v'_2, w'_2) + M_2(y_2, x_2) \}, \forall \rho_1, \rho_2 > 0. \end{aligned} \right\}$$

SPECIAL CASES:

I. If in problem (2.3) $X_i \equiv X$ (a real Hilbert space), $M_i = M : X \times X \rightarrow 2^X$ be a maximal η -monotone mapping, $S_i = S, N_i = N$ such that $S, N : X \rightarrow X$ then problem (2.3) reduces to the following problem: Find $(x, y) \in X \times X, u \in A(y), u' \in A(x)$ such that

$$(2.4) \quad \left. \begin{aligned} 0 \in H(x) - H(y) + \rho\{S(u) + M(x, y)\} \\ 0 \in H(y) - H(x) + \rho\{N(u') + M(y, x)\}, \quad \forall \rho > 0. \end{aligned} \right\}$$

This type of problem has been considered and studied by Kazmi and Bhat [12].

3. ITERATIVE ALGORITHM

First, we give the following technical lemma:

Lemma 3.1. *Let X_1, X_2 be 2-uniformly smooth Banach spaces. Let for each $i \in \{1, 2\}$, S_i, N_i, H_i, η_i be single-valued mappings, $G : X_1 \rightarrow X_2$ be a bounded linear operator and $M_i : X_i \times X_i \rightarrow 2^{X_i}$ be $H_i - \eta_i$ -accretive mappings. Then $(x_i, y_i, u_i, v_i, w_i, u'_i, v'_i, w'_i)$ where $(x_i, y_i) \in X_i \times X_i, u_i \in A_i(y_i), v_i \in B_i(y_i), w_i \in F_i(y_i), u'_i \in A_i(x_i), v'_i \in B_i(x_i), w'_i \in F_i(x_i)$ with $x_2 = Gx_1, y_2 = Gy_1$ is a solution of (2.3) if and only if $(x_i, y_i, u_i, v_i, w_i, u'_i, v'_i, w'_i)$ satisfies*

$$(3.1) \quad \left. \begin{aligned} x_1 &= R_{\rho_1, \eta_1}^{H_1, M_1(\cdot, y_1)} \left\{ H_1(y_1) - \rho_1 S_1(u_1, v_1, w_1) \right\} \\ x_2 &= R_{\rho_2, \eta_2}^{H_2, M_2(\cdot, y_2)} \left\{ H_2(y_2) - \rho_2 S_2(u_2, v_2, w_2) \right\} \\ y_1 &= R_{\rho_1, \eta_1}^{H_1, M_1(\cdot, x_1)} \left\{ H_1(x_1) - \rho_1 N_1(u'_1, v'_1, w'_1) \right\} \\ y_2 &= R_{\rho_2, \eta_2}^{H_2, M_2(\cdot, x_2)} \left\{ H_2(x_2) - \rho_2 N_2(u'_2, v'_2, w'_2) \right\}. \end{aligned} \right\}$$

where $R_{\rho_i, \eta_i}^{H_i, M_i(\cdot, y_i)} = \left(H_i + \rho_i M_i(\cdot, y_i) \right)^{-1}, R_{\rho_i, \eta_i}^{H_i, M_i(\cdot, x_i)} = \left(H_i + \rho_i M_i(\cdot, x_i) \right)^{-1}$ are the resolvent operators.

Proof. Let $(x_i, y_i, u_i, v_i, w_i, u'_i, v'_i, w'_i)$ is a solution of (2.3), then we have

$$\begin{aligned} x_i &= R_{\rho_i, \eta_i}^{H_i, M_i(\cdot, y_i)} \left\{ H_i(y_i) - \rho_i S_i(u_i, v_i, w_i) \right\} \\ \iff x_i &= \left(H_i + \rho_i M_i(\cdot, y_i) \right)^{-1} \left\{ H_i(y_i) - \rho_i S_i(u_i, v_i, w_i) \right\} \\ \iff H_i(x_i) + \rho_i M_i(x_i, y_i) &= \left\{ H_i(y_i) - \rho_i S_i(u_i, v_i, w_i) \right\} \\ \iff 0 \in H_i(x_i) - H_i(y_i) + \rho_i \{ S_i(u_i, v_i, w_i) + M_i(x_i, y_i) \} \end{aligned}$$

Proceeding likewise by using (3.1), we have

$$y_i = R_{\rho_i, \eta_i}^{H_i, M_i(\cdot, x_i)} \left\{ H_i(x_i) - \rho_i N_i(u'_i, v'_i, w'_i) \right\}$$

$$\iff 0 \in H_i(y_i) - H_i(x_i) + \rho_i \{ N_i(u'_i, v'_i, w'_i) + M_i(y_i, x_i) \}.$$

□

Lemma 3.1 allows us to suggest the following iterative algorithm for finding the approximate solution of (2.3).

Iterative Algorithm 3.2. For each $i = \{1, 2\}$ given $\{x_i^0, y_i^0, u_i^0, v_i^0, w_i^0, u_i^0, v_i^0, w_i^0\}$ where $x_i^0 \in X_i, y_i^0 \in X_i, u_i^0 \in A_i(y_i^0), v_i^0 \in B_i(y_i^0), w_i^0 \in F_i(y_i^0), u_i^0 \in A_i(x_i^0), v_i^0 \in B_i(x_i^0), w_i^0 \in F_i(x_i^0)$ compute the sequences $\{x_i^n, y_i^n, u_i^n, v_i^n, w_i^n, u_i^n, v_i^n, w_i^n\}$ defined by the iterative schemes

$$p_1^n = R_{\rho_1, \eta_1}^{H_1, M_1(\cdot, y_1^n)} \left\{ H_1(y_1^n) - \rho_1 S_1(u_1^n, v_1^n, w_1^n) \right\}$$

$$p_2^n = R_{\rho_2, \eta_2}^{H_2, M_2(\cdot, y_2^n)} \left\{ H_2(y_2^n) - \rho_2 S_2(u_2^n, v_2^n, w_2^n) \right\}$$

$$r_1^n = R_{\rho_1, \eta_1}^{H_1, M_1(\cdot, x_1^n)} \left\{ H_1(x_1^n) - \rho_1 N_1(u_1^n, v_1^n, w_1^n) \right\}$$

$$r_2^n = R_{\rho_2, \eta_2}^{H_2, M_2(\cdot, x_2^n)} \left\{ H_2(x_2^n) - \rho_2 N_2(u_2^n, v_2^n, w_2^n) \right\}$$

and

$$x_1^{n+1} = (1 - \beta^n)x_1^n + \beta^n(p_1^n + \mu G^+(p_2^n - Gp_1^n))$$

$$y_1^{n+1} = (1 - \beta^n)y_1^n + \beta^n(r_1^n + \mu G^+(r_2^n - Gr_1^n))$$

for all $n = 0, 1, 2, \dots$ and $\rho_1, \rho_2, \mu > 0$, where G^+ is the generalized adjoint operator of G and $x_2^n = Gx_1^n$ and $y_2^n = Gy_1^n$ for all n .

4. EXISTENCE OF SOLUTION AND CONVERGENCE ANALYSIS

Theorem 4.1. For each $i \in \{1, 2\}$, let X_i be a 2-uniformly smooth Banach space with k as constant of smoothness. Let $S_i : X_i \times X_i \times X_i \rightarrow X_i$ be δ_i -strongly monotone in the first argument, σ_i -relaxed accretive in the third argument and (ξ_i, γ_i) -relaxed cocoercive w.r.t H_i in second argument. Let $N_i : X_i \times X_i \times X_i \rightarrow X_i$ be λ_i -strongly monotone in the first argument, ν_i -relaxed accretive in the third argument and (τ_i, ϵ_i) -relaxed cocoercive w.r.t H_i in second argument. Let S_i be $L_{S_{i_1}}, L_{S_{i_2}}$ and $L_{S_{i_3}}$ -Lipschitz continuous in the first, second and third arguments respectively and N_i be $L_{N_{i_1}}, L_{N_{i_2}}$ and $L_{N_{i_3}}$ -Lipschitz continuous in the first, second and third arguments respectively. Let $A_i, B_i, F_i : X_i \rightarrow C(X_i)$ be set-valued mappings such that A_i is $l_{A_i} - \mathcal{D}$ -Lipschitz continuous, B_i is $l_{B_i} - \mathcal{D}$ -Lipschitz continuous

and F_i is $l_{F_i} - \mathcal{D}$ -Lipschitz continuous. Let H_i be L_{H_i} -Lipschitz continuous. Let $G : X_1 \rightarrow X_2$ be bounded linear operator such that $x_2 = Gx_1, y_2 = Gy_1$. In addition if

$$(4.1) \quad \left. \begin{aligned} & \left\| R_{\rho_i, \eta_i}^{H_i, M_i(\cdot, y_i^n)}(z_i) - R_{\rho_i, \eta_i}^{H_i, M_i(\cdot, y_i)}(z_i) \right\| \leq t_i \|y_i^n - y_i\|, \quad \forall z_i \in X_i \\ & \left\| R_{\rho_i, \eta_i}^{H_i, M_i(\cdot, x_i^n)}(z'_i) - R_{\rho_i, \eta_i}^{H_i, M_i(\cdot, x_i)}(z'_i) \right\| \leq t'_i \|x_i^n - x_i\|, \quad \forall z'_i \in X_i. \end{aligned} \right\}$$

$$(4.2) \quad \left. \begin{aligned} & 0 < \Phi_1 + \mu \|G^+\| \|G\| (\Phi_1 + \Phi_3) < 1 \\ & 0 < \Phi_2 + \mu \|G^+\| \|G\| (\Phi_2 + \Phi_4) < 1, \end{aligned} \right\}$$

where

$$\begin{aligned} \Phi_1 & := \left\{ \frac{m_1}{s_1} \left\{ \sqrt{1 + k\rho_1^2 L_{S_{11}}^2 l_{A_1}^2 - 2\rho_1 \delta_1} + \sqrt{1 + k\rho_1^2 L_{S_{13}}^2 l_{F_1}^2 - 2\rho_1 \sigma_1} \right. \right. \\ & \quad \left. \left. + \sqrt{L_{H_1}^2 + k\rho_1^2 L_{S_{12}}^2 l_{B_1}^2 - 2\rho_1 (-\xi_1 L_{S_{12}}^2 l_{B_1}^2 + \gamma_1 L_{H_1}^2)} \right\} + t_1 \right\}. \\ \Phi_2 & := \left\{ \frac{m_1}{s_1} \left\{ \sqrt{1 + k\rho_1^2 L_{N_{11}}^2 l_{A_1}^2 - 2\rho_1 \lambda_1} + \sqrt{1 + k\rho_1^2 L_{N_{13}}^2 l_{F_1}^2 - 2\rho_1 \nu_1} \right. \right. \\ & \quad \left. \left. + \sqrt{L_{H_1}^2 + k\rho_1^2 L_{N_{12}}^2 l_{B_1}^2 - 2\rho_1 (-\tau_1 L_{N_{12}}^2 l_{B_1}^2 + \epsilon_1 L_{H_1}^2)} \right\} + t'_1 \right\}. \\ \Phi_3 & := \left\{ \frac{m_2}{s_2} \left\{ \sqrt{1 + k\rho_2^2 L_{S_{21}}^2 l_{A_2}^2 - 2\rho_2 \delta_2} + \sqrt{1 + k\rho_2^2 L_{S_{23}}^2 l_{F_2}^2 - 2\rho_2 \sigma_2} \right. \right. \\ & \quad \left. \left. + \sqrt{L_{H_2}^2 + k\rho_2^2 L_{S_{22}}^2 l_{B_2}^2 - 2\rho_2 (-\xi_2 L_{S_{22}}^2 l_{B_2}^2 + \gamma_2 L_{H_2}^2)} \right\} + t_2 \right\} \\ \Phi_4 & := \left\{ \frac{m_2}{s_2} \left\{ \sqrt{1 + k\rho_2^2 L_{N_{21}}^2 l_{A_2}^2 - 2\rho_2 \lambda_2} + \sqrt{1 + k\rho_2^2 L_{N_{23}}^2 l_{F_2}^2 - 2\rho_2 \nu_2} \right. \right. \\ & \quad \left. \left. + \sqrt{L_{H_2}^2 + k\rho_2^2 L_{N_{22}}^2 l_{B_2}^2 - 2\rho_2 (-\tau_2 L_{N_{22}}^2 l_{B_2}^2 + \epsilon_2 L_{H_2}^2)} \right\} + t'_2 \right\}. \end{aligned}$$

Then the sequences $\{x_i^n\}, \{y_i^n\}, \{u_i^n\}, \{v_i^n\}, \{w_i^n\}, \{u_i^m\}, \{v_i^m\}, \{w_i^m\}$ generated by above iterative algorithm converges strongly to $(x_i, y_i, u_i, v_i, w_i, u_i^m, v_i^m, w_i^m)$ where $(x_i, y_i, u_i, v_i, w_i, u_i^m, v_i^m, w_i^m)$ is a solution of problem (2.3).

Proof. From Lemma 3.1, Iterative Algorithm 3.2, (4.1) and by using Theorem 2.15, it follows that

$$\begin{aligned}
 \left\| p_1^n - x_1 \right\| &= \left\| R_{\rho_1, \eta_1}^{H_1, M_1(\cdot, y_1^n)} \left\{ H_1(y_1^n) - \rho_1 S_1(u_1^n, v_1^n, w_1^n) \right\} \right. \\
 &\quad \left. - R_{\rho_1, \eta_1}^{H_1, M_1(\cdot, y_1)} \left\{ H_1(y_1) - \rho_1 S_1(u_1, v_1, w_1) \right\} \right\| \\
 &\quad + \left\| R_{\rho_1, \eta_1}^{H_1, M_1(\cdot, y_1^n)} \left\{ H_1(y_1) - \rho_1 S_1(u_1, v_1, w_1) \right\} \right. \\
 &\quad \left. - R_{\rho_1, \eta_1}^{H_1, M_1(\cdot, y_1)} \left\{ H_1(y_1) - \rho_1 S_1(u_1, v_1, w_1) \right\} \right\| \\
 &\leq \frac{m_1}{s_1} \left\| \left(H_1(y_1^n) - H_1(y_1) \right) - \rho_1 \left\{ S_1(u_1^n, v_1^n, w_1^n) - S_1(u_1, v_1, w_1) \right\} \right\| \\
 (4.3) \quad &\quad + t_1 \|y_1^n - y_1\|.
 \end{aligned}$$

Now, from (4.3) we can have

$$\begin{aligned}
 &\left\| \left(H_1(y_1^n) - H_1(y_1) \right) - \rho_1 \left\{ S_1(u_1^n, v_1^n, w_1^n) - S_1(u_1, v_1, w_1) \right\} \right\| \\
 &\leq \left\| \rho_1 \left\{ S_1(u_1^n, v_1^n, w_1^n) - S_1(u_1, v_1^n, w_1^n) \right\} - (y_1^n - y_1) \right\| \\
 &\quad + \left\| \rho_1 \left\{ S_1(u_1, v_1^n, w_1^n) - S_1(u_1, v_1, w_1) \right\} + (y_1^n - y_1) \right\| \\
 (4.4) \quad &\quad + \left\| \rho_1 \left\{ S_1(u_1, v_1^n, w_1) - S_1(u_1, v_1, w_1) \right\} - (H_1(y_1^n) - H_1(y_1)) \right\|.
 \end{aligned}$$

Using Remark 2.6, δ_1 -strongly monotonicity of S_1 in the first argument, $L_{S_{1_1}}$ -Lipschitz continuity of S_1 in the first argument and $l_{A_1} - \mathcal{D}$ -Lipschitz continuity of A_1 , it follows that

$$\begin{aligned}
 &\left\| \rho_1 \left\{ S_1(u_1^n, v_1^n, w_1^n) - S_1(u_1, v_1^n, w_1^n) \right\} - (y_1^n - y_1) \right\|^2 \\
 &\leq \|y_1^n - y_1\|^2 + k\rho_1^2 \left\| S_1(u_1^n, v_1^n, w_1^n) - S_1(u_1, v_1^n, w_1^n) \right\|^2 \\
 &\quad - 2\rho_1 \left[S_1(u_1^n, v_1^n, w_1^n) - S_1(u_1, v_1^n, w_1^n), y_1^n - y_1 \right] \\
 &\leq \left(1 + k\rho_1^2 L_{S_{1_1}}^2 l_{A_1}^2 - 2\rho_1 \delta_1 \right) \|y_1^n - y_1\|^2.
 \end{aligned}$$

This implies

$$\begin{aligned}
 &\left\| \rho_1 \left\{ S_1(u_1^n, v_1^n, w_1^n) - S_1(u_1, v_1^n, w_1^n) \right\} - (y_1^n - y_1) \right\| \\
 (4.5) \quad &\leq \sqrt{1 + k\rho_1^2 L_{S_{1_1}}^2 l_{A_1}^2 - 2\rho_1 \delta_1} \|y_1^n - y_1\|.
 \end{aligned}$$

Again, using σ_1 -relaxed accretivity of S_1 in third argument, $L_{S_{1_3}}$ -Lipschitz continuity of S_1 in the third argument, $l_{F_1} - \mathcal{D}$ -Lipschitz continuity of F_1 and Remark 2.6, we

have

$$\begin{aligned}
& \left\| \rho_1 \left\{ S_1(u_1, v_1^n, w_1^n) - S_1(u_1, v_1^n, w_1) \right\} + (y_1^n - y_1) \right\|^2 \\
& \leq \|y_1^n - y_1\|^2 + k\rho_1^2 \left\| S_1(u_1, v_1^n, w_1^n) - S_1(u_1, v_1^n, w_1) \right\|^2 \\
& \quad + 2\rho_1 \left[S_1(u_1, v_1^n, w_1^n) - S_1(u_1, v_1^n, w_1), y_1^n - y_1 \right] \\
& \leq \left(1 + k\rho_1^2 L_{S_1^3}^2 l_{F_1}^2 - 2\rho_1\sigma_1 \right) \|y_1^n - y_1\|^2.
\end{aligned}$$

This implies

$$\begin{aligned}
(4.6) \quad & \left\| \rho_1 \left\{ S_1(u_1, v_1^n, w_1^n) - S_1(u_1, v_1^n, w_1) \right\} + (y_1^n - y_1) \right\| \\
& \leq \sqrt{1 + k\rho_1^2 L_{S_1^3}^2 l_{F_1}^2 - 2\rho_1\sigma_1} \|y_1^n - y_1\|.
\end{aligned}$$

Again by using (ξ_1, γ_1) -relaxed cocoercivity of S_1 w.r.t H_1 in second argument, $L_{S_{12}}$ -Lipschitz continuity of S_1 in the second argument, $l_{B_1} - \mathcal{D}$ -Lipschitz continuity of B_1 , L_{H_1} -Lipschitz continuity of H_1 and using Remark 2.6, we have

$$\begin{aligned}
& \left\| \rho_1 \left\{ S_1(u_1, v_1^n, w_1) - S_1(u_1, v_1, w_1) \right\} - (H_1(y_1^n) - H_1(y_1)) \right\|^2 \\
& \leq \left\| H_1(y_1^n) - H_1(y_1) \right\|^2 + k\rho_1^2 \left\| S_1(u_1, v_1^n, w_1) - S_1(u_1, v_1, w_1) \right\|^2 \\
& \quad - 2\rho_1 \left[S_1(u_1, v_1^n, w_1) - S_1(u_1, v_1, w_1), H_1(y_1^n) - H_1(y_1) \right] \\
& \leq L_{H_1}^2 \|y_1^n - y_1\|^2 + k\rho_1^2 L_{S_{12}}^2 l_{B_1}^2 \|y_1^n - y_1\|^2 \\
& \quad - 2\rho_1 \left(-\xi_1 \left\| S_1(u_1, v_1^n, w_1) - S_1(u_1, v_1, w_1) \right\|^2 + \gamma_1 \left\| H_1(y_1^n) - H_1(y_1) \right\|^2 \right) \\
& \leq \left(L_{H_1}^2 + k\rho_1^2 L_{S_{12}}^2 l_{B_1}^2 - 2\rho_1(-\xi_1 L_{S_{12}}^2 l_{B_1}^2 + \gamma_1 L_{H_1}^2) \right) \|y_1^n - y_1\|^2.
\end{aligned}$$

This implies

$$\begin{aligned}
(4.7) \quad & \left\| \rho_1 \left\{ S_1(u_1, v_1^n, w_1) - S_1(u_1, v_1, w_1) \right\} - (H_1(y_1^n) - H_1(y_1)) \right\| \\
& \leq \sqrt{L_{H_1}^2 + k\rho_1^2 L_{S_{12}}^2 l_{B_1}^2 - 2\rho_1(-\xi_1 L_{S_{12}}^2 l_{B_1}^2 + \gamma_1 L_{H_1}^2)} \|y_1^n - y_1\|.
\end{aligned}$$

Therefore combining (4.3)-(4.7), we have

$$\begin{aligned}
& \left\| p_1^n - x_1 \right\| \\
& \leq \left\{ \frac{m_1}{s_1} \left\{ \sqrt{1 + k\rho_1^2 L_{S_{11}}^2 l_{A_1}^2 - 2\rho_1\delta_1} + \sqrt{1 + k\rho_1^2 L_{S_{13}}^2 l_{F_1}^2 - 2\rho_1\sigma_1} \right. \right.
\end{aligned}$$

$$\begin{aligned}
 & + \sqrt{L_{H_1}^2 + k\rho_1^2 L_{S_{1_2}}^2 l_{B_1}^2 - 2\rho_1(-\xi_1 L_{S_{1_2}}^2 l_{B_1}^2 + \gamma_1 L_{H_1}^2)} \Big\} + t_1 \Big\} \|y_1^n - y_1\| \\
 (4.8) \quad & \leq \Phi_1 \|y_1^n - y_1\|,
 \end{aligned}$$

where Φ_1 is defined by (4.2).

Again from Lemma 3.1, Iterative Algorithm 3.2, (4.1) and by using Theorem 2.15, it follows that

$$\begin{aligned}
 & \left\| r_1^n - y_1 \right\| \\
 & = \left\| R_{\rho_1, \eta_1}^{H_1, M_1(\cdot, x_1^n)} \left\{ H_1(x_1^n) - \rho_1 N_1(u_1^m, v_1^m, w_1^m) \right\} \right. \\
 & \quad - R_{\rho_1, \eta_1}^{H_1, M_1(\cdot, x_1^n)} \left\{ H_1(x_1) - \rho_1 N_1(u_1', v_1', w_1') \right\} \Big\| \\
 & \quad + \left\| R_{\rho_1, \eta_1}^{H_1, M_1(\cdot, x_1^n)} \left\{ H_1(x_1) - \rho_1 N_1(u_1', v_1', w_1') \right\} \right. \\
 & \quad \left. - R_{\rho_1, \eta_1}^{H_1, M_1(\cdot, x_1)} \left\{ H_1(x_1) - \rho_1 N_1(u_1', v_1', w_1') \right\} \right\| \\
 & \leq \frac{m_1}{s_1} \left\| \left(H_1(x_1^n) - H_1(x_1) \right) - \rho_1 \left\{ N_1(u_1^m, v_1^m, w_1^m) - N_1(u_1', v_1', w_1') \right\} \right\| \\
 (4.9) \quad & + t_1' \|x_1^n - x_1\|.
 \end{aligned}$$

Now, we have

$$\begin{aligned}
 & \left\| \left(H_1(x_1^n) - H_1(x_1) \right) - \rho_1 \left\{ N_1(u_1^m, v_1^m, w_1^m) - N_1(u_1', v_1', w_1') \right\} \right\| \\
 & \leq \left\| \rho_1 \left\{ N_1(u_1^m, v_1^m, w_1^m) - N_1(u_1', v_1^m, w_1^m) \right\} - (x_1^n - x_1) \right\| \\
 & \quad + \left\| \rho_1 \left\{ N_1(u_1', v_1^m, w_1^m) - N_1(u_1', v_1^m, w_1') \right\} + (x_1^n - x_1) \right\| \\
 (4.10) \quad & + \left\| \rho_1 \left\{ N_1(u_1', v_1^m, w_1') - N_1(u_1', v_1', w_1') \right\} - (H_1(x_1^n) - H_1(x_1)) \right\|.
 \end{aligned}$$

Using Remark 2.6, λ_1 -strongly monotonicity of N_1 in the first argument, $L_{N_{1_1}}$ -Lipschitz continuity of N_1 in the first argument and $l_{A_1} - \mathcal{D}$ -Lipschitz continuity of A_1 , it follows that

$$\begin{aligned}
 & \left\| \rho_1 \left\{ N_1(u_1^m, v_1^m, w_1^m) - N_1(u_1', v_1^m, w_1^m) \right\} - (x_1^n - x_1) \right\|^2 \\
 & \leq \|x_1^n - x_1\|^2 + k\rho_1^2 \left\| N_1(u_1^m, v_1^m, w_1^m) - N_1(u_1', v_1^m, w_1^m) \right\|^2 \\
 & \quad - 2\rho_1 \left[N_1(u_1^m, v_1^m, w_1^m) - N_1(u_1', v_1^m, w_1^m), x_1^n - x_1 \right] \\
 & \leq \left(1 + k\rho_1^2 L_{N_{1_1}}^2 l_{A_1}^2 - 2\rho_1 \lambda_1 \right) \|x_1^n - x_1\|^2.
 \end{aligned}$$

This implies

$$(4.11) \quad \begin{aligned} & \left\| \rho_1 \left\{ N_1(u_1^m, v_1^m, w_1^m) - N_1(u_1', v_1^m, w_1^m) \right\} - (x_1^n - x_1) \right\| \\ & \leq \sqrt{1 + k\rho_1^2 L_{N_1}^2 l_{A_1}^2 - 2\rho_1 \lambda_1} \|x_1^n - x_1\|. \end{aligned}$$

Again, since N_1 is ν_1 -relaxed accretive in third argument, L_{N_1} -Lipschitz continuity of N_1 in the third argument and l_{F_1} - \mathcal{D} -Lipschitz continuity of F_1 and using Remark 2.6, we have

$$\begin{aligned} & \left\| \rho_1 \left\{ N_1(u_1', v_1^m, w_1^m) - N_1(u_1', v_1^m, w_1') \right\} + (x_1^n - x_1) \right\|^2 \\ & \leq \|x_1^n - x_1\|^2 + k\rho_1^2 \left\| N_1(u_1', v_1^m, w_1^m) - N_1(u_1', v_1^m, w_1') \right\|^2 \\ & \quad + 2\rho_1 \left[N_1(u_1', v_1^m, w_1^m) - N_1(u_1', v_1^m, w_1'), x_1^n - x_1 \right] \\ & \leq \left(1 + k\rho_1^2 L_{N_1}^2 l_{F_1}^2 - 2\rho_1 \nu_1 \right) \|x_1^n - x_1\|^2. \end{aligned}$$

This implies

$$(4.12) \quad \begin{aligned} & \left\| \rho_1 \left\{ N_1(u_1', v_1^m, w_1^m) - N_1(u_1', v_1^m, w_1') \right\} + (x_1^n - x_1) \right\| \\ & \leq \sqrt{1 + k\rho_1^2 L_{N_1}^2 l_{F_1}^2 - 2\rho_1 \nu_1} \|x_1^n - x_1\|. \end{aligned}$$

Again by using (τ_1, ϵ_1) -relaxed cocoercivity of N_1 w.r.t H_1 in second argument, L_{N_1} -Lipschitz continuity of N_1 in the second argument, l_{B_1} - \mathcal{D} -Lipschitz continuity of B_1 , L_{H_1} -Lipschitz continuity of H_1 and using Remark 2.6, we have

$$\begin{aligned} & \left\| \rho_1 \left\{ N_1(u_1', v_1^m, w_1') - N_1(u_1', v_1', w_1') \right\} - (H_1(x_1^n) - H_1(x_1)) \right\|^2 \\ & \leq \left\| H_1(x_1^n) - H_1(x_1) \right\|^2 + k\rho_1^2 \left\| N_1(u_1', v_1^m, w_1') - N_1(u_1', v_1', w_1') \right\|^2 \\ & \quad - 2\rho_1 \left[N_1(u_1', v_1^m, w_1') - N_1(u_1', v_1', w_1'), H_1(x_1^n) - H_1(x_1) \right] \\ & \leq \left\| H_1(x_1^n) - H_1(x_1) \right\|^2 + k\rho_1^2 \left\| N_1(u_1', v_1^m, w_1') - N_1(u_1', v_1', w_1') \right\|^2 \\ & \quad - 2\rho_1 \left(-\tau_1 \left\| N_1(u_1', v_1^m, w_1') - N_1(u_1', v_1', w_1') \right\|^2 + \epsilon_1 \left\| H_1(x_1^n) - H_1(x_1) \right\|^2 \right) \\ & \leq L_{H_1}^2 \|x_1^n - x_1\|^2 + k\rho_1^2 L_{N_1}^2 l_{B_1}^2 \|x_1^n - x_1\|^2 - 2\rho_1 (-\tau_1 L_{N_1}^2 l_{B_1}^2 + \epsilon_1 L_{H_1}^2) \|x_1^n - x_1\|^2 \\ & \leq \left(L_{H_1}^2 + k\rho_1^2 L_{N_1}^2 l_{B_1}^2 - 2\rho_1 (-\tau_1 L_{N_1}^2 l_{B_1}^2 + \epsilon_1 L_{H_1}^2) \right) \|x_1^n - x_1\|^2. \end{aligned}$$

This implies

$$(4.13) \quad \begin{aligned} & \left\| \rho_1 \left\{ N_1(u'_1, v'_1, w'_1) - N_1(u'_1, v'_1, w'_1) \right\} - (H_1(x_1^n) - H_1(x_1)) \right\| \\ & \leq \sqrt{L_{H_1}^2 + k\rho_1^2 L_{N_{1_2}}^2 l_{B_1}^2 - 2\rho_1(-\tau_1 L_{N_{1_2}}^2 l_{B_1}^2 + \epsilon_1 L_{H_1}^2)} \|x_1^n - x_1\|. \end{aligned}$$

Therefore combining (4.9)-(4.13), we have

$$(4.14) \quad \begin{aligned} & \left\| r_1^n - y_1 \right\| \\ & \leq \left\{ \frac{m_1}{s_1} \left\{ \sqrt{1 + k\rho_1^2 L_{N_{1_1}}^2 l_{A_1}^2 - 2\rho_1 \lambda_1} + \sqrt{1 + k\rho_1^2 L_{N_{1_3}}^2 l_{F_1}^2 - 2\rho_1 \nu_1} \right. \right. \\ & \quad \left. \left. + \sqrt{L_{H_1}^2 + k\rho_1^2 L_{N_{1_2}}^2 l_{B_1}^2 - 2\rho_1(-\tau_1 L_{N_{1_2}}^2 l_{B_1}^2 + \epsilon_1 L_{H_1}^2)} \right\} + t'_1 \right\} \|x_1^n - x_1\| \\ & \leq \Phi_2 \|x_1^n - x_1\|. \end{aligned}$$

Similarly, following the same procedure as in (4.3)-(4.8) and (4.9)-(4.14), we have

$$(4.15) \quad \begin{aligned} & \left\| p_2^n - x_2 \right\| \\ & \leq \left\{ \frac{m_2}{s_2} \left\{ \sqrt{1 + k\rho_2^2 L_{S_{2_1}}^2 l_{A_2}^2 - 2\rho_2 \delta_2} + \sqrt{1 + k\rho_2^2 L_{S_{2_3}}^2 l_{F_2}^2 - 2\rho_2 \sigma_2} \right. \right. \\ & \quad \left. \left. + \sqrt{L_{H_2}^2 + k\rho_2^2 L_{S_{2_2}}^2 l_{B_2}^2 - 2\rho_2(-\xi_2 L_{S_{2_2}}^2 l_{B_2}^2 + \gamma_2 L_{H_2}^2)} \right\} + t_2 \right\} \|y_2^n - y_2\| \\ & \leq \Phi_3 \|y_2^n - y_2\|, \end{aligned}$$

and

$$(4.16) \quad \begin{aligned} & \left\| r_2^n - y_2 \right\| \\ & \leq \left\{ \frac{m_2}{s_2} \left\{ \sqrt{1 + k\rho_2^2 L_{N_{2_1}}^2 l_{A_2}^2 - 2\rho_2 \lambda_2} \right. \right. \\ & \quad \left. \left. + \sqrt{1 + k\rho_2^2 L_{N_{2_3}}^2 l_{F_2}^2 - 2\rho_2 \nu_2} \right. \right. \\ & \quad \left. \left. + \sqrt{L_{H_2}^2 + k\rho_2^2 L_{N_{2_2}}^2 l_{B_2}^2 - 2\rho_2(-\tau_2 L_{N_{2_2}}^2 l_{B_2}^2 + \epsilon_2 L_{H_2}^2)} \right\} + t'_2 \right\} \|x_2^n - x_2\| \\ & \leq \Phi_4 \|x_2^n - x_2\|, \end{aligned}$$

where Φ_2, Φ_3, Φ_4 is defined by (4.2).

Now, from (4.8),(4.15) and using the fact that G^+ is bounded, we have

$$\begin{aligned}
(4.17) \quad & \|x_1^{n+1} - x_1\| \\
& \leq (1 - \beta^n)\|x_1^n - x_1\| + \beta^n\|p_1^n - x_1 + \mu G^+(p_2^n - Gp_1^n)\| \\
& \leq (1 - \beta^n)\|x_1^n - x_1\| + \beta^n\|p_1^n - x_1\| + \beta^n\mu\|G^+\| \|p_2^n - Gp_1^n\| \\
& \leq (1 - \beta^n)\|x_1^n - x_1\| + \beta^n\Phi_1\|y_1^n - y_1\| + \beta^n\mu\|G^+\| \left(\|p_2^n - x_2 - Gp_1^n + x_2\| \right) \\
& \leq (1 - \beta^n)\|x_1^n - x_1\| + \beta^n\Phi_1\|y_1^n - y_1\| + \beta^n\mu\|G^+\| \left(\|p_2^n - x_2\| + \|G\|\|p_1^n - x_1\| \right) \\
& \leq (1 - \beta^n)\|x_1^n - x_1\| + \beta^n\Phi_1\|y_1^n - y_1\| + \beta^n\mu\|G^+\| \left(\Phi_3\|y_2^n - y_2\| + \|G\|\Phi_1\|y_1^n - y_1\| \right) \\
& \leq (1 - \beta^n)\|x_1^n - x_1\| + \beta^n\Phi_1\|y_1^n - y_1\| + \beta^n\mu\|G^+\| \|G\| \left(\Phi_3\|y_1^n - y_1\| + \Phi_1\|y_1^n - y_1\| \right) \\
& \leq (1 - \beta^n)\|x_1^n - x_1\| + \beta^n \left(\Phi_1 + \mu\|G^+\| \|G\|(\Phi_1 + \Phi_3) \right) \|y_1^n - y_1\|.
\end{aligned}$$

Similarly, using the boundedness of G^+ , (4.14),(4.16) and following the same process as in (4.17), we obtain

$$\begin{aligned}
(4.18) \quad & \|y_1^{n+1} - y_1^n\| \\
& \leq (1 - \beta^n)\|y_1^n - y_1\| + \beta^n \left(\Phi_2 + \mu\|G^+\| \|G\|(\Phi_2 + \Phi_4) \right) \|x_1^n - x_1\|.
\end{aligned}$$

Now define a norm $\|\cdot\|_*$ on $X_1 \times X_2$ by $\|(x, y)\|_* = \|x\| + \|y\|$, $(x, y) \in X_1 \times X_2$.

We can show that $(X_1 \times X_1, \|\cdot\|_*)$ is a Banach space.

By making use of (4.17) and (4.18), we have the following estimate

$$\begin{aligned}
(4.19) \quad & \|(x_1^{n+1}, y_1^{n+1}) - (x_1, y_1)\|_* \\
& = \|x_1^{n+1} - x_1\| + \|y_1^{n+1} - y_1\| \\
& \leq (1 - \beta^n)(\|x_1^n - x_1\| + \|y_1^n - y_1\|) \\
& \quad + \beta^n \left(\Phi_1 + \mu\|G^+\| \|G\|(\Phi_1 + \Phi_3) \right) \|y_1^n - y_1\| \\
& \quad + \beta^n \left(\Phi_2 + \mu\|G^+\| \|G\|(\Phi_2 + \Phi_4) \right) \|x_1^n - x_1\| \\
& \leq (1 - \beta^n)(\|x_1^n - x_1\| + \|y_1^n - y_1\|) \\
& \quad + \beta^n \max \{ \tilde{h}_1, \tilde{h}_2 \} \{ \|x_1^n - x_1\| + \|y_1^n - y_1\| \} \\
& = (1 - \beta^n(1 - \Phi)) \|(x_1^n, y_1^n) - (x_1, y_1)\|_*,
\end{aligned}$$

where $\Phi = \max \{ \tilde{h}_1, \tilde{h}_2 \}$ and

$$\tilde{h}_1 = \Phi_1 + \mu\|G^+\| \|G\|(\Phi_1 + \Phi_3)$$

or,

$$\begin{aligned} \bar{h}_1 &= \Phi_1 + b(\Phi_1 + \Phi_3) \\ \bar{h}_2 &= \Phi_2 + \mu \|G^+\| \|G\| (\Phi_2 + \Phi_4) \end{aligned}$$

or,

$$\bar{h}_2 = \Phi_2 + b(\Phi_2 + \Phi_4)$$

where $b = \mu \|G^+\| \|G\|$. Thus it follows that

$$(4.20) \quad \begin{aligned} & \| (x_1^{n+1}, y_1^{n+1}) - (x_1, y_1) \|_* \\ & < \Pi_{r=1}^n (1 - \beta^r (1 - \Phi)) \| (x_1^0, y_1^0) - (x_1, y_1) \|_* . \end{aligned}$$

It follows from (4.2) that $\Phi \in (0, 1)$. Since $\sum_{n=1}^{\infty} \beta^n = \infty$ it follows that

$$\lim_{n \rightarrow \infty} \Pi_{r=1}^n (1 - \beta^r (1 - \Phi)) = 0.$$

Thus, it follows from (4.20) that $\{(x_1^{n+1}, y_1^{n+1})\}$ converges strongly to (x_1, y_1) as $n \rightarrow \infty$, that is $x_1^n \rightarrow x_1$ and $y_1^n \rightarrow y_1$ as $n \rightarrow \infty$. Moreover, it follows from (4.8), (4.14) that $p_1^n \rightarrow x_1$ and $r_1^n \rightarrow y_1$ as $n \rightarrow \infty$. Therefore, it follows from (4.15), (4.16) respectively that $p_2^n \rightarrow x_2 = Gx_1$ and $r_2^n \rightarrow y_2 = Gy_1$ as $n \rightarrow \infty$. This completes the proof. \square

5. CONCLUSION

System of variational inclusions can be viewed as natural and innovative generalizations of the system of variational inequalities. Two of the most difficult and important problems related to inclusions are the establishment of generalized inclusions and the development of an iterative algorithm. In this article a new system of set-valued variational inclusion problems is introduced and studied which is more general than many existing system of variational inclusions in the literature. An iterative algorithm is established to approximate the solution of our system, and convergence criteria is also discussed.

We remark that our results are new and useful for further research and one can extend these results in higher dimensional spaces. Much more work is needed in all these areas to develop a sound basis for applications of the system of set-valued variational inclusion problems in engineering and physical sciences.

REFERENCES

1. Adly, S.: Perturbed algorithms and sensitivity analysis for a general class of variational inclusions. *J. Math. Anal. Appl.* **201** (1996), 609-630.
2. Ahmad, I., Rahaman, M. & Ahmad, R.: Relaxed resolvent operator for solving variational inclusion problem. *Stat. Optim. Inf. Comput.* **4** (2016), 183-193.
3. Bhat, M.I. & Shafi, S.: H -Monotone Operators with an Application to a System of Nonlinear Implicit Variational Inclusions. *Caspian Journal of Applied Mathematics, Ecology and Economics* **4** (2016), no. 2, 22-39.
4. Bhat, M.I. & Shafi, S.: An Iterative Algorithm for a System of Generalized Mixed Variational Inclusion Problems. *Int. J. Modern Math. Sci.* (2017), 1-18.
5. Censor, Y. & Elfving, T.: A multiprojection algorithm using Bregman projections in product space. *Numerical Algorithms* **8** (1994), 221-239.
6. Chang, S.S., Kim, J.K. & Kim, K.H.: On the existence and iterative approximation problems of solutions for set-valued variational inclusions in Banach spaces. *J. Math. Anal. Appl.* **268** (2002), 89-108.
7. Ding, X.P.: Perturbed proximal point algorithm for generalized quasi-variational inclusions. *J. Math. Anal. Appl.* **210** (1997), no. 1, 88-101.
8. Ding, X.P. & Luo, C.L.: Perturbed proximal point algorithms for general quasi-variational-like inclusions. *J. Comput. Appl. Math.* **113** (2000), no. 1-2, 153-165.
9. Fang, Y.P. & Huang, N.J.: H -monotone operator and resolvent operator technique for variational inclusions. *Appl. Math. Comput.* **145** (2003), 795-803.
10. Giles, J.R.: Classes of semi-inner product spaces. *Trans. Am. Math. Soc.* **129** (1967), 436-446.
11. Huang, N.J.: A new class of generalized set-valued implicit variational inclusions in Banach spaces with an application. *Comput. Math. Appl.* **41** (2001), 937-943.
12. Kazmi, K.R. & Bhat, M.I.: Iterative algorithm for a system of nonlinear variational-like inclusions. *Comput. Math. Appl.* **48** (2004), no. 12, 1929-1935.
13. Lan, H.Y., Kim, J.H. & Cho, Y.J.: On a new system of nonlinear A -monotone multi-valued variational inclusions. *J. Math. Anal. Appl.* **327** (2007), 481-493.
14. Lee, C.H., Ansari, Q.H. & Yao, J.C.: A perturbed algorithm for strongly nonlinear variational like inclusions. *Bull. Aust. Math. Soc.* **62** (2000), 417-426.
15. Lumer, G.: Semi inner product spaces. *Trans. Am. Math. Soc.* **100** (1961), 29-43.
16. Luo, X.P. & Huang, N.J.: (H, ϕ) - η -monotone operators in Banach spaces with an application to variational inclusions. *Appl. Math. Comput.* **216** (2010), 1131-1139.
17. Nadler, S.B.: Multivalued contraction mappings. *Pacific J. Math.* **30** (1969), 475-488.
18. Pap, E. & Pavlovic, R., Adjoint theorem on Semi-inner product spaces of type (P). *Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat.* **25** (1995), no. 1, 39-46.

19. Sahu, N.K., Mahato, N.K. & Mohapatra, R.N.: System of nonlinear variational inclusion problems with (A, η) -maximal monotonicity in Banach spaces. *Stat. Optim. Inf. Comput.* **5** (2017), 244-261.
20. Sahu, N.K., Mohapatra, R.N., Nahak, C. & Nanda, S.: Approximation solvability of a class of A -monotone implicit variational inclusion problems in semi-inner product spaces. *Appl. Math. Comput.* **236** (2014), 109-117.
21. Xu, H.K.: Inequalities in Banach spaces with applications. *Nonlinear Anal.* **16** (1991), no. 12, 1127-1138.

PROFESSOR: DEPARTMENT OF MATHEMATICS, S.P. COLLEGE, CLUSTER UNIVERSITY, SRINAGAR-190001, INDIA

Email address: `sumeera.shafi@gmail.com`