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A CLASS OF STRUCTURED FRAMES IN FINITE DIMENSIONAL HILBERT SPACES

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Abstract. We introduce a special class of structured frames having single generators in finite dimensional Hilbert spaces. We call them as pseudo B-Gabor like frames and present a characterisation of the frame operators associated with these frames. The concept of Gabor semi-frames is also introduced and some significant properties of the associated semi-frame operators are discussed.

1. Introduction

The fast growing theory of Hilbert space frames clenches its own space in both pure and applied mathematics due to its wide applications. Time- frequency analysis of signals in $L^2(\mathbb{R})$, as suggested by Dennis Gabor in *Theory of Communication* [7], requires a special system of the form $\{E_{mb}T_{na}g : m, n \in \mathbb{Z}\}$, where $g \in L^2(\mathbb{R})$ and E_{mb}, T_{na} $(m, n \in \mathbb{Z}, a, b > 0)$ are the Modulation and Translation operators respectively. This perspective has become the benchmark for the spectral analysis associated with various time-frequency methods.

Frames in Hilbert spaces were introduced in 1952 by Duffin and Schaeffer [4] in their study of non harmonic Fourier series. In 1980's, Janssen designed it an independent topic of mathematical investigation by his outstanding work [10]. The gravity of the theory of frames in modern signal processing and time frequency analysis is now well accepted [8].

Frames were brought to life by Daubechies, Grossmann and Meyer in 1986 with the fundamental works [3] and put forth the idea of combining Gabor analysis with frame theory. Gabor analysis aims at representing functions (signals) $f \in L^2(\mathbb{R})$ as superpositions of translated and modulated versions of a fixed function $g \in L^2(\mathbb{R})$.

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A foremost object in frame theory, both from the theoretical and applications point of view, is the frame operator associated with a given frame. On the other hand, frames corresponding to a given *nice* operator on the space has substantial practical importance. In particular, Gabor frame operators, which are very special in their construction, are acquiring notable research attention and are of interest in this paper too, in the general perspective of finite dimensional Hilbert spaces.

Section 2 provides some basic definitions and results which are very essential for this article. pseudo B-Gabor like frames in finite dimensional Hilbert spaces and their frame operators are discussed in Section 3. Section 4 focuses on Gabor semi-frame operators associated with Gabor semi-frames. Our basic references for both abstract frame theory and the theory of Gabor frames are [2] and [8].

2. Preliminaries

A countable sequence $\{f_k\}_{k=1}^{\infty}$ of elements in a Hilbert space $\mathcal{H}$ is said to be a *frame* in $\mathcal{H}$, if there are constants $A, B > 0$, such that

$$A\|f\|^2 \;\le\; \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 \;\le\; B\|f\|^2 \quad \forall\ f \in \mathcal{H}.$$

These constants A, B are called *frame bounds.* If $A = B$, then the frame $\{f_k\}_{k=1}^{\infty}$ is called a *tight frame* and is called a *parseval frame or normalised tight frame* when $A = B = 1$. Whenever a sequence $\{f_k\}_{k=1}^{\infty}$ of elements in $\mathcal{H}$ satisfies the upper frame inequality, then it is said to be a *Bessel sequence* or a *semi-frame sequence* and $\{f_k\}_{k=1}^{\infty}$ is a *frame sequence* if it is a frame for $\overline{span}\{f_k\}_{k=1}^{\infty}$.

If $\{f_k\}_{k=1}^{\infty}$ is a frame in a Hilbert space $\mathcal{H}$, then the map S defined by $Sf = \sum_{k=1}^{\infty}\langle f, f_k\rangle f_k$ for all $f \in \mathcal{H}$ is a bounded linear operator on $\mathcal{H}$, called the *frame operator* associated with the frame $\{f_k\}_{k=1}^{\infty}$. The frame operator of a tight frame is a scalar multiple of the identity operator and that of a normalised tight frame is the identity operator [2].

Remark 2.1. Let $\{f_k\}_{k=1}^{\infty}$ be a frame with frame operator S and frame bounds A, B in a Hilbert space $\mathcal{H}$. Then,

(i) S is bounded, invertible, self-adjoint and positive.
(ii) $\{S^{-1}f_k\}_{k=1}^{\infty}$ is a frame with frame bounds B^{-1}, A^{-1} and $\{S^{-1/2}f_k\}_{k=1}^{\infty}$ is a normalised tight frame.

(iii) If A and B are the optimal frame bounds for $\{f_k\}_{k=1}^{\infty}$, then the bounds B^{-1}, A^{-1} are the optimal frame bounds for $\{S^{-1}f_k\}_{k=1}^{\infty}$. The frame operator for $\{S^{-1}f_k\}_{k=1}^{\infty}$ is S^{-1}. Further we have $B^{-1}I \leq S^{-1} \leq A^{-1}I$.

Let $\{f_k\}_{k=1}^{\infty}$ be a frame with frame operator S in $\mathcal{H}$, then the frame $\{S^{-1}f_k\}_{k=1}^{\infty}$ is called the *(canonical) dual frame* of the frame $\{f_k\}_{k=1}^{\infty}$. As follows, every frame in $\mathcal{H}$ admits the *frame decomposition* in two ways.

Theorem 2.2. *Let $\{f_k\}_{k=1}^{\infty}$ be a frame with frame operator S in a Hilbert space $\mathcal{H}$. Then for all $f \in \mathcal{H}$, $f = \sum_{k=1}^{\infty}\langle f, S^{-1}f_k\rangle f_k$ and $f = \sum_{k=1}^{\infty}\langle f, f_k\rangle S^{-1}f_k$. Both the series converge unconditionally for all $f \in \mathcal{H}$.*

Gabor analysis uses two important unitary operators, namely, the translation and modulation operators.

On the N-dimensional Hilbert space $l^2(\mathbb{Z}_N)$ of complex functions on $\mathbb{Z}_N$ (equipped with the standard inner product), the *Translation operator* T_k, $k \in \mathbb{Z}_N$ and the *Modulation operator* M_l, $l \in \mathbb{Z}_N$ are defined by

$$(T_k g)(j) = g(j-k) \text{ and } (M_l g)(j) = e^{2\pi i l j/N} g(j)$$

where $j = 0, 1, 2, ..., N-1$ and $g \in l^2(\mathbb{Z}_N)$.

Gabor type frame operators on the finite dimensional Hilbert space $l^2(\mathbb{Z}_N)$ and their characterisation were discussed in [13]. In this paper, we present a similar discussion in the context of general finite dimensional Hilbert spaces.

3. Pseudo Gabor like Frames and Operators in Finite Hilbert Spaces

In our discussions, $\mathcal{H}$ and $\mathcal{K}$ will denote finite dimensional Hilbert spaces. More precisely we take $dim\mathcal{H} = N$. Also through out this paper, we denote $\Lambda = \Lambda_1 \times \Lambda_2$, where Λ_1, Λ_2 are subgroups of $\mathbb{Z}_N$ with $|\Lambda| \geq N$ (unless otherwise specified). We begin with a simple observation, analogous to Corollary 5.3.2 in [2], on the interplay of bounded linear operators between two separable Hilbert spaces.

Lemma 3.1. *Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces. Then every surjective bounded linear operator $A : \mathcal{H} \to \mathcal{K}$ maps frames in $\mathcal{H}$ to frames in $\mathcal{K}$. In particular, every invertible bounded linear operator between two separable Hilbert spaces maps frames in one to frames in the other.*

Lemma 3.1 motivates to look at the aspects of the images of Gabor frames under invertible bounded linear operators $B : l^2(\mathbb{Z}_N) \to \mathcal{H}$. Since

$$\begin{aligned} B(\{M_l T_k g \ : (k,l) \in \Lambda\}) &= \{B M_l T_k g \ : (k,l) \in \Lambda\} \\ &= \{B M_l B^{-1} B T_k B^{-1} B g \ : (k,l) \in \Lambda\} \\ &= \{(B M_l B^{-1})(B T_k B^{-1})(Bg) \ : (k,l) \in \Lambda\} \end{aligned}$$

they are generated by the action of a family of operators $\{M_l^B T_k^B \ : (k,l) \in \Lambda\}$ on a single generator Bg, where $M_l^B = BM_lB^{-1}$ and $T_k^B = BT_kB^{-1}$. Thus, such image frames are structured frames in $\mathcal{H}$. We will formulate the following definitions which will be useful in our further discussions.

Definition. For an invertible bounded linear operator $B : l^2(\mathbb{Z}_N) \to \mathcal{H}$ and $k \in \mathbb{Z}_N$, a *B-translation* T_k^B on $\mathcal{H}$ is defined by $T_k^B = BT_kB^{-1}$ and for $l \in \mathbb{Z}_N$, *B-modulation* M_l^B on $\mathcal{H}$ is defined by $M_l^B = BM_lB^{-1}$ where T_k and M_l are respectively the translation and modulation operators on $l^2(\mathbb{Z}_N)$.

The family $\{M_l^B T_k^B g \ : (k,l) \in \Lambda\}$ generated by $g \in \mathcal{H}$ is called a *pseudo B-Gabor like system* in $\mathcal{H}$. Such a system is called a *pseudo B-Gabor like frame (pseudo B-Gabor like Bessel sequence)* if it forms a frame (Bessel sequence) in $\mathcal{H}$.

A frame $\mathcal{G}$ in $\mathcal{H}$ is called a *pseudo Gabor like frame*, if $\mathcal{G}$ is a pseudo B-Gabor like frame for some invertible operator $B : l^2(\mathbb{Z}_N) \to \mathcal{H}$.

Gabor type unitary systems discussed in [9] were defined by generalising the remarkable property $T_aE_b = e^{-2\pi iab}E_bT_a$ of the pair (T_a, E_b) of translation and modulation operators on $L^2(\mathbb{R})$. Interestingly, for the generalisation of the system of operators generated by the combination T_aE_b, we need not stick on to the unitary system. Instead, a system of invertible operators can be considered, as Proposition 3.2 below suggests.

Proposition 3.2. *Let $B : l^2(\mathbb{Z}_N) \to \mathcal{H}$ be invertible. Then the following statements hold.*

(i) $M_l^B T_k^B = e^{2\pi ilk/N} T_k^B M_l^B$ *for all* $(k,l) \in \Lambda$.

(ii) $\{M_l^B T_k^B g \ : (k,l) \in \Lambda\}$ *is a pseudo B-Gabor like frame in $\mathcal{H}$ if and only if the family* $\{T_k^B M_l^B g \ : (k,l) \in \Lambda\}$ *is also a frame in $\mathcal{H}$.*

Proof. For all $f \in l^2(\mathbb{Z}_N)$, the commutator relation [1]
$T_kM_lf(j) = e^{-2\pi ilk/N}M_lT_kf(j)$ holds for all $j \in \mathbb{Z}_N$.

Hence for any $x \in \mathcal{H}$ and for a bounded invertible operator $B : l^2(\mathbb{Z}_N) \to \mathcal{H}$,

$$\begin{aligned} M_l^B T_k^B x &= B M_l T_k B^{-1} x \\ &= B e^{2\pi i l k/N} T_k M_l B^{-1} x \\ &= e^{2\pi i l k/N} B T_k M_l B^{-1} x \\ &= e^{2\pi i l k/N} T_k^B M_l^B x. \end{aligned}$$

Thus $M_l^B T_k^B = e^{2\pi i l k/N} T_k^B M_l^B$ for all $(k, l) \in \Lambda$, proving (i).
Since $\{M_l^B T_k^B g \ : (k, l) \in \Lambda\}$ is a frame in $\mathcal{H}$ and $e^{2\pi i l k/N}$ is of absolute value 1, the necessary frame inequality for the collection $\{T_k^B M_l^B g \ : (k, l) \in \Lambda\}$ follows immediately from that of $\{M_l^B T_k^B g \ : (k, l) \in \Lambda\}$. The reverse implication follows likewise. □

Forthcoming proposition gives a connection between pseudo B-Gabor like frame in $\mathcal{H}$ and Gabor frame in $l^2(\mathbb{Z}_N)$.

Proposition 3.3. *The family $\{M_l^B T_k^B g : (k, l) \in \Lambda\}$ forms a pseudo B-Gabor like frame in $\mathcal{H}$ if and only if $\{M_l T_k B^{-1} g : (k, l) \in \Lambda\}$ forms a Gabor frame in $l^2(\mathbb{Z}_N)$.*

Proof. Since
$$\begin{aligned} \{M_l^B T_k^B g : (k, l) \in \Lambda\} &= \{B M_l B^{-1} B T_k B^{-1} g : (k, l) \in \Lambda\} \\ &= B(\{M_l T_k B^{-1} g : (k, l) \in \Lambda\}), \end{aligned}$$
the proof follows for both the cases of implications from Lemma 3.1. □

Remark 3.4. The existence of a Gabor frame in $l^2(\mathbb{Z}_N)$ has been established by Jim Laurence [11] for prime values of N. Romanos-Digenes Malikiosis established this for any $N \geq 4$ [12]. Consequently, for any positive integer N, there is a Gabor frame in $l^2(\mathbb{Z}_N)$ of the form $\{M_l T_k g : (k, l) \in \Lambda_1 \times \Lambda_2\}$ where $|\Lambda_1 \times \Lambda_2| \geq N$ and Λ_1, Λ_2 are subgroups of $\mathbb{Z}_N$. We combine these facts together as Proposition 3.5, which will be used in sequel [13].

Proposition 3.5. *For any two subgroups Λ_1 and Λ_2 of $\mathbb{Z}_N$ with $|\Lambda_1 \times \Lambda_2| \geq N$, there is a tight Gabor frame in $l^2(\mathbb{Z}_N)$ with the identity operator as its frame operator.*

For each invertible bounded linear map $B : l^2(\mathbb{Z}_N) \to \mathcal{H}$, the bounded linear operator BB^* on $\mathcal{H}$ is positive and invertible. Hence it becomes a frame operator of some frame in $\mathcal{H}$ [5]. Interestingly, this frame operator corresponds to a pseudo B-Gabor like frame in $\mathcal{H}$.

Proposition 3.6. *For any subset* $\Lambda = \Lambda_1 \times \Lambda_2$ *of* $\mathbb{Z}_N \times \mathbb{Z}_N$ *where* Λ_1, Λ_2 *are subgroups of* $\mathbb{Z}_N$*, with* $|\Lambda| \geq N$*, there is always a pseudo* B*-Gabor like frame in* $\mathcal{H}$ *with* BB^* *as its frame operator.*

Proof. Let $\Lambda = \Lambda_1 \times \Lambda_2$ be a subset of $\mathbb{Z}_N \times \mathbb{Z}_N$ with $|\Lambda| \geq N$, where Λ_1 and Λ_2 are subgroups of $\mathbb{Z}_N$. By Proposition 3.5, there is a tight Gabor frame $\{M_l T_k g : (k,l) \in \Lambda_1 \times \Lambda_2\}$ in $l^2(\mathbb{Z}_N)$ with the identity operator as its frame operator. If $B : l^2(\mathbb{Z}_N) \to \mathcal{H}$ is a bounded invertible linear map, then the image $B(\{M_l T_k g : (k,l) \in \Lambda_1 \times \Lambda_2\}) = \{M_l^B T_k^B Bg \ : (k,l) \in \Lambda_1 \times \Lambda_2\}$ is a pseudo B-Gabor like frame in $\mathcal{H}$ with frame operator $BIB^* = BB^*$. □

Apart from the positivity and invertibility of the Gabor frame operators on $l^2(\mathbb{Z}_N)$, their commutativity with some specific modulation and translation operators were significant in characterising the Gabor frame operators [6, 13]. Here we look at the similar situation in the context of pseudo B-Gabor like frames.

Theorem 3.7. *The following are equivalent for a given invertible bounded linear operator* $B : l^2(\mathbb{Z}_N) \to \mathcal{H}$*.*

i) B^*B *is a Gabor frame operator on* $l^2(\mathbb{Z}_N)$*.*

ii) Every pseudo B*-Gabor like frame operator on* $\mathcal{H}$ *commutes with its involved* B*-modulations and* B*-translations.*

iii) There exists a Parseval pseudo B*-Gabor like frame* $\mathcal{G}_{\Lambda_1 \times \Lambda_2}$ *in* $\mathcal{H}$ *for every subgroups* Λ_1*,* Λ_2 *of* $\mathbb{Z}_N$ *with* $|\Lambda_1 \times \Lambda_2| \geq N$*.*

Proof. i) ⇒ ii): Assume that S is the frame operator of a pseudo B-Gabor like frame $\{M_l^B T_k^B g \ : (k,l) \in \Lambda\}$ in $\mathcal{H}$. Then $B^{-1} : \mathcal{H} \to l^2(\mathbb{Z}_N)$ maps this frame to the Gabor frame $\{M_l T_k B^{-1} g \ : (k,l) \in \Lambda\}$ whose frame operator is $B^{-1}S(B^{-1})^*$. Hence the operator $B^{-1}S(B^{-1})^*$ commutes with M_l and T_k for all $(k,l) \in \Lambda$. Now, assuming (i), we obtain

$$\begin{aligned} SM_l^B &= SBM_lB^{-1} = S(B^{-1})^*(B^*B)M_lB^{-1} \\ &= S(B^{-1})^*M_l(B^*B)B^{-1} = S(B^{-1})^*M_lB^* \\ &= BB^{-1}S(B^{-1})^*M_lB^* = BM_l(B^{-1}S(B^{-1})^*)B^* \\ &= BM_lB^{-1}S = M_l^BS \quad \text{for all} \ \ l \in \Lambda_2. \end{aligned}$$

Similarly $ST_k^B = T_k^BS$. This proves ii).

ii) ⇒ iii): For any two subgroups Λ_1, Λ_2 of $\mathbb{Z}_N$ with $|\Lambda_1 \times \Lambda_2| \geq N$, there is always a Gabor frame $\mathcal{G}$ in $l^2(\mathbb{Z}_N)$ and hence there is a pseudo B-Gabor like frame

$\mathcal{P} = B(\mathcal{G})$ in $\mathcal{H}$. Since by (ii), the frame operator S of such a pseudo B-Gabor like frame $\mathcal{P}$ commutes with its involved B-modulations and B-translations, so does the operator $S^{-1/2}$. Hence the image frame $\mathcal{G}_{\Lambda_1\times\Lambda_2} = S^{-1/2}(\mathcal{P})$ will be a Parseval pseudo B-Gabor like frame in $\mathcal{H}$ with frame operator as identity operator.

iii) $\Rightarrow$ i): Let $\mathcal{G}_{\Lambda_1\times\Lambda_2}$ be a Parseval pseudo B-Gabor like frame in $\mathcal{H}$ for subgroups Λ_1, Λ_2 of $\mathbb{Z}_N$ with $|\Lambda_1 \times \Lambda_2| \geq N$. Then $B^{-1}(\mathcal{G}_{\Lambda_1\times\Lambda_2})$ will be a Gabor frame in $l^2(\mathbb{Z}_N)$ with frame operator $B^{-1}I(B^{-1})^* = (B^*B)^{-1}$. Thus $(B^*B)^{-1}$ commutes with M_l and T_k for all $(k,l) \in \Lambda = \Lambda_1 \times \Lambda_2$ and hence its inverse B^*B also has this property. □

Thus, each B as above has a specific control in terms of the bounded linear operator B^*B on $l^2(\mathbb{Z}_N)$ for yielding Parseval pseudo B-Gabor like frames in $\mathcal{H}$ as well as pseudo B-Gabor like frames having canonical dual frames with same structure. Such frames are more similar to Gabor frames in $l^2(\mathbb{Z}_N)$. In view of the above discussions we give a new definition which is suitable for identifying the structures more specifically.

Definition. A pseudo B-Gabor like frame $\{M_l^B T_k^B Bg \ : (k,l) \in \Lambda_1 \times \Lambda_2\}$ in a separable Hilbert space $\mathcal{H}$ is said to be a *pseudo B-Gabor frame* if B^*B is a Gabor frame operator on $l^2(\mathbb{Z}_N)$. The frame operator of a pseudo B-Gabor frame is called a *pseudo B-Gabor frame operator.*

Now we look at the canonical dual frame of pseudo B-Gabor frames in $\mathcal{H}$. An important consequence of Theorem 3.7 is the following.

Theorem 3.8. *Let $B : l^2(\mathbb{Z}_N) \to \mathcal{H}$ be an invertible map such that B^*B is a Gabor frame operator on $l^2(\mathbb{Z}_N)$. Then for any given Gabor frame $\{M_l T_k g \ : (k,l) \in \Lambda_1 \times \Lambda_2\}$ in $l^2(\mathbb{Z}_N)$ with frame operator S, the canonical dual frame of the pseudo B-Gabor frame $\{M_l^B T_k^B Bg \ : (k,l) \in \Lambda\}$ in $\mathcal{H}$ is again a pseudo B-Gabor frame with generator $CS^{-1}g$ where $C = (B^*)^{-1}$. Also, this pseudo B-Gabor frame has a dual pseudo C-Gabor frame with same generator $CS^{-1}g$.*

Proof. Let $B : l^2(\mathbb{Z}_N) \to \mathcal{H}$ be an invertible map. Take $C = (B^*)^{-1}$, then $C : l^2(\mathbb{Z}_N) \to \mathcal{H}$ is also an invertible map and
$C^*C = ((B^*)^{-1})^*(B^*)^{-1} = B^{-1}(B^*)^{-1} = (B^*B)^{-1}$.
Since B^*B is a Gabor frame operator on $l^2(\mathbb{Z}_N)$ so is its inverse $(B^*B)^{-1}$. Thus C^*C is also a Gabor frame operator on $l^2(\mathbb{Z}_N)$.

Now, for a given Gabor frame $\mathcal{G} = \{M_l T_k g \ : (k,l) \in \Lambda_1 \times \Lambda_2\}$ in $l^2(\mathbb{Z}_N)$ with frame operator S, the frame operator of the pseudo B-Gabor frame $B(\mathcal{G}) = \{M_l^B T_k^B Bg \ : (k,l) \in \Lambda_1 \times \Lambda_2\}$ is BSB^*. Hence the canonical dual frame of $B(\mathcal{G})$ is $(BSB^*)^{-1}(B(\mathcal{G}))$
$= (BSB^*)^{-1}(\{M_l^B T_k^B Bg \ : (k,l) \in \Lambda_1 \times \Lambda_2\}$
$= \{M_l^B T_k^B (BSB^*)^{-1} Bg \ : (k,l) \in \Lambda_1 \times \Lambda_2\}$, by Theorem 3.7 (ii)
$= \{M_l^B T_k^B (B^*)^{-1} S^{-1} g \ : (k,l) \in \Lambda_1 \times \Lambda_2\}$
$= \{M_l^B T_k^B C S^{-1} g \ : (k,l) \in \Lambda_1 \times \Lambda_2\}$, since $C = (B^*)^{-1}$.
Thus, the canonical dual frame of the pseudo B-Gabor frame $B(\mathcal{G})$ in $\mathcal{H}$ is again a pseudo B-Gabor frame with generator $CS^{-1}g$ and same generating set $\Lambda = \Lambda_1 \times \Lambda_2$.

Now, mapping the canonical dual Gabor frame $S^{-1}(\mathcal{G}) = \{M_l T_k S^{-1} g \ : (k,l) \in \Lambda\}$ of $\mathcal{G}$ by C, we obtain the pseudo C-Gabor frame $\{M_l^C T_k^C C S^{-1} g \ : (k,l) \in \Lambda\}$. Frame operator of this frame is $CS^{-1}C^* = (B^*)^{-1}S^{-1}((B^*)^{-1})^* = (B^*)^{-1}S^{-1}(B^{-1})$ $= (BSB^*)^{-1}$, the canonical dual frame operator of $B(\mathcal{G})$.

Thus both the frames $\{M_l^B T_k^B C S^{-1} g \ : (k,l) \in \Lambda\}$ and $\{M_l^C T_k^C C S^{-1} g \ : (k,l) \in \Lambda\}$ are dual frames of $B(\mathcal{G})$ with common generator $CS^{-1}g$. □

Obviously, when the map $B : l^2(\mathbb{Z}_N) \to \mathcal{H}$ is unitary, we have $C = (B^*)^{-1} = B$ so that the above frames are precisely the same.

The following example is a specific situation of Theorem 3.8. If $B,\ C : l^2(\mathbb{Z}_N) \to \mathcal{H}$ are invertible with $C = (B^*)^{-1}$, then $\{M_l^B T_k^B C S^{-1} g : (k,l) \in \Lambda\}$ and $\{M_l^C T_k^C C S^{-1} g \ : \ (k,l) \in \Lambda\}$ are respectively, pseudo B-Gabor frame and pseudo C-Gabor frame on $\mathcal{H}$ with same generator $CS^{-1}g$ and same frame operator $(BSB^*)^{-1}$.

Example 3.9. Consider a prime number N. For $\alpha \neq 0$ in $\mathbb{Z}_N$, the dilation operator on $l^2(\mathbb{Z}_N)$ defined by $D_\alpha(f(m)) = \alpha^{-1} f(k_m)$; where $\alpha m \equiv k_m (mod\ N)$, for each $m \in \mathbb{Z}_N$, is a unitary operator and $D_\alpha^* = D_\alpha^{-1} = D_{\alpha^{-1}} = D_{\frac{1}{\alpha}}$.
Direct calculation proves that the operators, dilation D_α, translation T_k and modulation M_l in the finite dimensional space $l^2(\mathbb{Z}_N)$ satisfy the following relations.
$T_k D_\alpha = D_\alpha T_{\alpha^{-1}k}$ and $D_\alpha M_l = M_{\alpha l} D_\alpha$.
For $\beta, \gamma \in \mathbb{C} - \{0\}$, with $\mid \beta \mid \neq \mid \gamma \mid$ define for all $m \in \mathbb{Z}_N$,

$$\phi_{\beta,\gamma}(m) = \begin{cases} \beta & \text{if } m \text{ is odd} \\ \gamma & \text{if } m \text{ is even} \end{cases}$$

The multiplication operators $M_{\phi_{\beta,\gamma}}$ on $l^2(\mathbb{Z}_N)$ is defined by $M_{\phi_{\beta,\gamma}}(f) = \phi_{\beta,\gamma}.f = \phi_{\beta,\gamma}(m).f(m)$; for all $m \in \mathbb{Z}_N$, is invertible with inverse $M_{\phi_{\beta,\gamma}}^{-1} = M_{\phi_{\frac{1}{\beta},\frac{1}{\gamma}}}$. Further the adjoint $M_{\phi_{\beta,\gamma}}^* = M_{\phi_{\overline{\beta},\overline{\gamma}}}$ so that $M_{\phi_{\beta,\gamma}}^* M_{\phi_{\beta,\gamma}} = M_{\phi_{|\beta|^2,|\gamma|^2}}$. Also we can see that

multiplication operator $M_{\phi_{\beta,\gamma}}$ commutes with translation T_k and modulation M_l for $(k,l) \in \Lambda$.
Now for all $f \in l^2(\mathbb{Z}_N)$ define, $B : l^2(\mathbb{Z}_N) \to l^2(\mathbb{Z}_N)$ by $B(f) = (M_{\phi_{\beta,\gamma}} D_\alpha)(f)$ observe that, $B^* = (M_{\phi_{\beta,\gamma}} D_\alpha)^* = D^*_\alpha M^*_{\phi_{\beta,\gamma}} = D_{\alpha^{-1}} M_{\phi_{\overline{\beta},\overline{\gamma}}}$ and
$B^*B = D_{\alpha^{-1}} M_{\phi_{\overline{\beta},\overline{\gamma}}} M_{\phi_{\beta,\gamma}} D_\alpha = D_{\alpha^{-1}} M_{\phi_{|\beta|^2,|\gamma|^2}} D_\alpha$.
It can be easily verified that B^*B commutes with modulation M_l and translation T_k. Further, B^*B is positive and invertible. Hence B^*B is a Gabor frame operator on $l^2(\mathbb{Z}_N)$. Again we see that,

$$\begin{aligned}
BM_lB^{-1}f(m) &= BM_lD_{\alpha^{-1}}M_{\phi_{\frac{1}{\beta},\frac{1}{\gamma}}}f(m)\\
&= M_{\phi_{\beta,\gamma}}D_\alpha M_l D_{\alpha^{-1}}M_{\phi_{\frac{1}{\beta},\frac{1}{\gamma}}}f(m)\\
&= M_{\phi_{\beta,\gamma}}M_{\alpha l}M_{\phi_{\frac{1}{\beta},\frac{1}{\gamma}}}f(m)\\
&= M_{\alpha l}f(m),\\
BT_kB^{-1}f(m) &= BT_kD_{\alpha^{-1}}M_{\phi_{\frac{1}{\beta},\frac{1}{\gamma}}}f(m)\\
&= M_{\phi_{\beta,\gamma}}D_\alpha T_k D_{\alpha^{-1}}M_{\phi_{\frac{1}{\beta},\frac{1}{\gamma}}}f(m)\\
&= M_{\phi_{\beta,\gamma}}T_{\alpha k}M_{\phi_{\frac{1}{\beta},\frac{1}{\gamma}}}f(m)\\
&= M_{\phi_{\beta,\gamma}}\phi_{\frac{1}{\beta},\frac{1}{\gamma}}(m-\alpha k)f(m-\alpha k)\\
&= \phi_{\beta,\gamma}(m)\phi_{\frac{1}{\beta},\frac{1}{\gamma}}(m-\alpha k)f(m-\alpha k)
\end{aligned}$$

Taking $C = (B^*)^{-1}$ and by simple computations, we obtain
$C = (B^*)^{-1} = M_{\phi_{\frac{1}{\overline{\beta}},\frac{1}{\overline{\gamma}}}} D_\alpha$ and $C^{-1} = D_{\alpha^{-1}} M_{\phi_{\overline{\beta},\overline{\gamma}}}$.
Now,

$$\begin{aligned}
CM_lC^{-1}f(m) &= CM_lD_{\alpha^{-1}}M_{\phi_{\overline{\beta},\overline{\gamma}}}f(m)\\
&= M_{\phi_{\frac{1}{\overline{\beta}},\frac{1}{\overline{\gamma}}}}D_\alpha M_l D_{\alpha^{-1}}M_{\phi_{\overline{\beta},\overline{\gamma}}}f(m)\\
&= M_{\phi_{\frac{1}{\overline{\beta}},\frac{1}{\overline{\gamma}}}}M_{\alpha l}M_{\phi_{\overline{\beta},\overline{\gamma}}}f(m)\\
&= M_{\alpha l}f(m),\\
CT_kC^{-1}f(m) &= CT_kD_{\alpha^{-1}}M_{\phi_{\overline{\beta},\overline{\gamma}}}f(m)\\
&= M_{\phi_{\frac{1}{\overline{\beta}},\frac{1}{\overline{\gamma}}}}D_\alpha T_k D_{\alpha^{-1}}M_{\phi_{\overline{\beta},\overline{\gamma}}}f(m)\\
&= M_{\phi_{\frac{1}{\overline{\beta}},\frac{1}{\overline{\gamma}}}}T_{\alpha k}M_{\phi_{\overline{\beta},\overline{\gamma}}}f(m)\\
&= \phi_{\frac{1}{\overline{\beta}},\frac{1}{\overline{\gamma}}}(m)\phi_{\overline{\beta},\overline{\gamma}}(m-\alpha k)f(m-\alpha k).
\end{aligned}$$

Thus $BM_lB^{-1} = CM_lC^{-1}$, but $BT_kB^{-1} \neq CT_kC^{-1}$. Therefor, $M_l^BT_k^B \neq M_l^CT_k^C$. Hence for a given Gabor frame $\mathcal{G} = \{M_lT_kg : (k,l) \in \Lambda\}$ in $l^2(\mathbb{Z}_N)$ with frame operator S, $C\{M_lT_kS^{-1}g : (k,l) \in \Lambda\} = \{M_l^CT_k^CCS^{-1}g : (k,l) \in \Lambda\}$ and $\{M_l^BT_k^BCS^{-1}g : (k,l) \in \Lambda\}$ are different frames with same generator $CS^{-1}g$ and same frame operator $(BSB^*)^{-1}$.

For Bessel sequences in $\{u_k\}_{k\in\mathbb{Z}}$ and $\{v_k\}_{k\in\mathbb{Z}}$ in $\mathcal{H}$ and $\mathcal{K}$ respectively, we can have a bounded linear operator $M : \mathcal{H} \to \mathcal{K}$ given by $M(x) = \sum_{k\in\mathbb{Z}} \langle x, v_k\rangle u_k$, where the series defining M converges for all $x \in \mathcal{H}$. The operator M is called the *mixed frame operator* associated with the Bessel sequences $\{u_k\}$ and $\{v_k\}$ [2].

Here is an interesting observation on the relation between mixed frame operators and invertible operators from $l^2(\mathbb{Z}_N)$ onto $\mathcal{H}$.

Proposition 3.10. *Every invertible operator $B : l^2(\mathbb{Z}_N) \to \mathcal{H}$ can be identified as a mixed frame operator.*

Proof. Let $B : l^2(\mathbb{Z}_N) \to \mathcal{H}$ be a bounded invertible map. Obviously, B maps any given Gabor frame $\mathcal{G} = \{M_lT_kg \ : (k,l) \in \Lambda\}$ in $l^2(\mathbb{Z}_N)$ to a pseudo B-Gabor like frame $B(\mathcal{G}) = \{M_l^BT_k^BBg \ : (k,l) \in \Lambda\}$ in $\mathcal{H}$. Let M be the mixed frame operator defined by $Mf = \sum_{(k,l)\in\Lambda} \langle f, M_lT_kg\rangle M_l^BT_k^BBg$, $f \in l^2(\mathbb{Z}_N)$.
Then for all $f \in l^2(\mathbb{Z}_N)$,

$$\begin{aligned} Mf &= \sum_{(k,l)\in\Lambda} \langle f, M_lT_kg\rangle BM_lB^{-1}BT_kB^{-1}Bg \\ &= B\sum_{(k,l)\in\Lambda} \langle f, M_lT_kg\rangle M_lT_kg \\ &= BS(f), \end{aligned}$$

where S is the frame operator of $\mathcal{G}$.
Thus, $M = BS$. Now, by choosing $\mathcal{G}$ as a Parseval Gabor frame in $l^2(\mathbb{Z}_N)$, we obtain $S = I_{l^2(\mathbb{Z}_N)}$ so that $B : l^2(\mathbb{Z}_N) \to \mathcal{H}$ is precisely a mixed frame operator. □

4. GABOR SEMI-FRAME OPERATORS IN FINITE DIMENSIONAL HILBERT SPACES

Let $\mathcal{H}$ be an N-dimensional Hilbert space. Any finite sequence of elements of $\mathcal{H}$ can be considered as a Bessel sequence in $\mathcal{H}$. Let $\{u_k\}_{k\in\Delta}$, $|\Delta| < \infty$ is such a sequence. Then there is a bounded linear positive operator S on $\mathcal{H}$ defined by

$S(x) = \sum_{k\in\Delta} \langle x, u_k\rangle u_k$ for all $x \in \mathcal{H}$. We call this operator as the *semi-frame operator* associated to $\{u_k\}_{k\in\Delta}$.

Analogously, the family $\mathcal{G}(g, \Delta) = \{M_l^B T_k^B g : (k, l) \in \Delta\}$ where $g \in \mathcal{H}$ and $\Delta \subseteq \mathbb{Z}_N \times \mathbb{Z}_N$, is a Bessel sequence in $\mathcal{H}$. Hence there is an associated semi-frame operator $S_{\mathcal{G},\Delta}$ corresponding to $\mathcal{G}(g, \Delta)$, called the *Gabor semi-frame operator* associated with the generating set Δ and generating function g.

Proposition 4.1. *Let $\Lambda = \Lambda_1 \times \Lambda_2 \subseteq \mathbb{Z}_N \times \mathbb{Z}_N$ be such that $\Lambda_1' = \Lambda_1 - r$ and $\Lambda_2' = \Lambda_2 - t$ are subgroups of $\mathbb{Z}_N$ for some $(r, t) \in \mathbb{Z}_N \times \mathbb{Z}_N$ with $|\Lambda| \geq N$. If S is the Gabor semi-frame operator on $\mathcal{H}$ associated with $\mathcal{G}(g, \Lambda)$ then there are Gabor semi-frame operators S_r and S_t on $\mathcal{H}$ such that $ST_r^{(B^*)^{-1}} = T_r^B S_r$ and $SM_t^{(B^*)^{-1}} = M_t^B S_t$. Moreover $S_r T_h^{(B^*)^{-1}} = T_h^B S_r$ for all $h \in \Lambda_1'$ and $S_t M_p^{(B^*)^{-1}} = M_p^B S_t$ for all $p \in \Lambda_2'$.*

Proof. Let S be the Gabor semi-frame operator of $\mathcal{G}(g, \Lambda)$ as given in the statement. Then for $x \in \mathcal{H}$, $S(x) = \sum_{(k,l)\in\Lambda_1\times\Lambda_2} \langle x, M_l^B T_k^B g\rangle M_l^B T_k^B g$

$$\begin{aligned}
ST_r^{(B^*)^{-1}}(x) &= \sum_{(k,l)\in\Lambda_1\times\Lambda_2} \langle T_r^{(B^*)^{-1}}(x), M_l^B T_k^B g\rangle M_l^B T_k^B g \\
&= \sum_{(k,l)\in\Lambda_1\times\Lambda_2} \langle x, (T_r^{(B^*)^{-1}})^* M_l^B T_k^B g\rangle M_l^B T_k^B g \\
&= \sum_{(k,l)\in\Lambda_1\times\Lambda_2} \langle x, BT_{-r}B^{-1}BM_lB^{-1}BT_kB^{-1}g\rangle M_l^B T_k^B g \\
&= \sum_{(k,l)\in\Lambda_1\times\Lambda_2} \langle x, BT_{-r}M_lT_kB^{-1}g\rangle M_l^B T_k^B g \\
&= \sum_{(k,l)\in\Lambda_1\times\Lambda_2} \langle x, Be^{2\pi ilr/N}M_lT_{k-r}B^{-1}g\rangle M_l^B T_k^B g.
\end{aligned}$$

Putting $k - r = k'$ we have $k = r + k'$, hence the above expression becomes;

$$\begin{aligned}
ST_r^{(B^*)^{-1}}(x) &= \sum_{(k',l)\in\Lambda_1'\times\Lambda_2} \langle x, Be^{2\pi ilr/N}M_lT_{k'}B^{-1}g\rangle M_l^B T_{r+k'}^B g \\
&= T_r^B \sum_{(k',l)\in\Lambda_1'\times\Lambda_2} \langle x, M_l^B T_{k'}^B g\rangle M_l^B T_{k'}^B g \\
&= T_r^B S_r(x),
\end{aligned}$$

where $S_r(x) = \sum_{(k',l)\in\Lambda_1'\times\Lambda_2} \langle x, M_l^B T_{k'}^B g\rangle M_l^B T_{k'}^B g$. Also we have,

$$\begin{aligned} SM_t^{(B^*)^{-1}}(x) &= \sum_{(k,l)\in\Lambda_1\times\Lambda_2} \langle M_t^{(B^*)^{-1}}(x), M_l^B T_k^B g\rangle M_l^B T_k^B g \\ &= \sum_{(k,l)\in\Lambda_1\times\Lambda_2} \langle x, (M_t^{(B^*)^{-1}})^* M_l^B T_k^B g\rangle M_l^B T_k^B g \\ &= \sum_{(k,l)\in\Lambda_1\times\Lambda_2} \langle x, BM_{-t}B^{-1}BM_lB^{-1}BT_kB^{-1}g\rangle M_l^B T_k^B g \\ &= \sum_{(k,l)\in\Lambda_1\times\Lambda_2} \langle x, BM_{l-t}T_kB^{-1}g\rangle M_l^B T_k^B g. \end{aligned}$$

Putting $l-t=l'$ we have $l=t+l'$, hence the above expression becomes;

$$\begin{aligned} SM_t^{(B^*)^{-1}}(x) &= \sum_{(k,l')\in\Lambda_1\times\Lambda_2'} \langle x, BM_l'T_kB^{-1}g\rangle M_{t+l'}^B T_k^B g \\ &= M_t^B \sum_{(k,l')\in\Lambda_1\times\Lambda_2'} \langle x, M_{l'}^B T_k^B g\rangle M_{l'}^B T_k^B g \\ &= M_t^B S_t(x), \end{aligned}$$

where $S_t(x) = \sum_{(k,l')\in\Lambda_1\times\Lambda_2'} \langle x, M_{l'}^B T_k^B g\rangle M_{l'}^B T_k^B g.$

Hence there are Gabor semi-frame operators S_r and S_t on $\mathcal{H}$ such that $ST_r^{(B^*)^{-1}} = T_r^B S_r$ and $SM_t^{(B^*)^{-1}} = M_t^B S_t$. Now for $h\in\Lambda_1'$,

$$\begin{aligned} S_rT_h^{(B^*)^{-1}}(x) &= \sum_{(k',l)\in\Lambda_1'\times\Lambda_2} \langle T_h^{(B^*)^{-1}}(x), M_l^B T_{k'}^B g\rangle M_l^B T_{k'}^B g \\ &= \sum_{(k',l)\in\Lambda_1'\times\Lambda_2} \langle (x), (T_h^{(B^*)^{-1}})^* M_l^B T_{k'}^B g\rangle M_l^B T_{k'}^B g \\ &= \sum_{(k',l)\in\Lambda_1'\times\Lambda_2} \langle (x), BT_{-h}M_lT_{k'}B^{-1}g\rangle M_l^B T_{k'}^B g \\ &= \sum_{(k',l)\in\Lambda_1'\times\Lambda_2} \langle (x), e^{2\pi ilh/N}BM_lT_{k'-h}B^{-1}g\rangle M_l^B T_{k'}^B g. \end{aligned}$$

Taking $k'-h=k''$,

$$\begin{aligned} S_rT_h^{(B^*)^{-1}}(x) &= \sum_{(k'',l)\in\Lambda_1'\times\Lambda_2} \langle (x), e^{2\pi ilh/N}BM_lT_{k''}B^{-1}g\rangle M_l^B T_{k''+h}^B g \\ &= T_h^B \sum_{(k'',l)\in\Lambda_1'\times\Lambda_2} \langle (x), M_l^B T_{k''}^B g\rangle M_l^B T_{k''}^B g \\ &= T_h^B S_r(x). \end{aligned}$$

Similarly, for each $p\in\Lambda_2'$, $S_tM_p^{(B^*)^{-1}} = M_p^B S_t.$ □

Remark 4.2. It can be noted that if S is a Gabor semi-frame operator as in Proposition 4.1, then the invertibility of S, S_r and S_t are equivalent. Also it can be seen that, If B is a unitary operator then these results coincide with results in [13].

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