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δ-CONVEX STRUCTURE ON RECTANGULAR METRIC SPACES CONCERNING KANNAN-TYPE CONTRACTION AND REICH-TYPE CONTRACTION

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Abstract. In the present paper, we introduce the notation of δ-convex rectangular metric spaces with the help of convex structure. We investigate fixed point results concerning Kannan-type contraction and Reich-type contraction in such spaces. We also propound an ingenious example in reference of given new notion.

1. Introduction and Preliminaries

Banach Contraction Principle [3] is the key outcome of the fixed point theory which has been handed down in many different directions of mathematics. In 1989, Bakhtin [2], Czerwik [9] established the concept of b-metric spaces and in 2000, [7] introduced rectangular metric spaces. Since then many scholar's have proposed a series of new fixed point theorems for different functions in rectangular metric spaces.

Next, Takahashi [14] introduced the conception of convexity in metric spaces and provided some fixed point results in convex metric spaces. Subsequently, Beg [4], Beg and Abbas [5, 6], Kim and Jin [8], Ding [10] and many others [1, 11, 12] obtained fixed point theorems in convex metric spaces and convex b metric spaces.

In this paper, we present an idea of δ-convex rectangular metric space. After that we obtain extended, improved, generalized and unified results for Kannan-type and Reich-type contraction mapping in δ-convex rectangular metric spaces. We also provide an example in reference of such spaces.

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Definition 1.1. Let C be a subset of the set of real numbers R and $\delta \in [0,1]$. Then C is called *δ-convex* if $\lambda x + (1-\lambda)\delta y \in C$ for all $x, y \in C$ and $\lambda, \delta \in [0,1]$.

Definition 1.2. Let C be a subset of the set of real numbers R and $\delta \in [0,1]$. A function $T : C \subset R \to R$ is called *δ-convex* if C is a δ-convex subset of R and

$$T(\lambda x + (1-\lambda)\delta y) \leq \lambda T(x) + (1-\lambda)\delta T(y);$$

for all $x, y \in C$ and $\lambda, \delta \in [0,1]$.

Definition 1.3 ([7])**.** Let H be a set and $H \neq \phi$. A function $d_r : H \times H \to [0, \infty]$ is said to be a *rectangular metric* if the following hold:

(d_{r_1}) $d_r(\phi, \psi) = 0$ if and only if $\phi = \psi$ for every $\phi, \psi \in H$;
(d_{r_2}) $d_r(\phi, \psi) = d_r(\psi, \phi)$ for every $\phi, \psi \in H$;
(d_{r_3}) $d_r(\phi, v) \leq d_r(\phi, p) + d_r(p, q) + d_r(q, \psi)\big]$ for every distinct $\phi, \psi, p, q \in H$.

The pair (H, d_r) is known a *rectangular metric space* (in short RMS).

Definition 1.4 ([7])**.** Let $\{\phi_n\}$ be a sequence in RMS (H, d_r).

(1) The sequence $\{\phi_n\}$ is said to be *convergent* in (H, d_r) if $\phi^* \in H$ exists such that $\lim_{n\to\infty} d_r(\phi_n, \phi^*) = 0$.
(2) The sequence $\{\phi_n\}$ is said to be *Cauchy* in RMS (H, d_r) if for every $\epsilon > 0$, there exists a positive integer n_0 such that $d_r(\phi_n, \phi_m) < \epsilon$ for all $n, m > n_0$.
(3) The RMS (H, d_r) is known a *complete RMS* if every Cauchy sequence is convergent in H.

Definition 1.5. Let H be a non-empty set and $I = [0,1]$. Define the mapping $d_r : H \times H \to [0, \infty]$. Let $w : H \times H \times J \times I \to H$ be a continuous function. Then w is said to be the *δ-convex structure* on H if,

$$d_r\big(t, w(\phi, \psi; \lambda, \delta)\big) \leq \lambda d_r(t, \phi) + (1-\lambda)\delta d_r(t, \psi)$$

for all $t \in H$ and $(\phi, \psi; \lambda, \delta) \in H \times H \times J \times I$ where $J \subseteq I$.

Definition 1.6. Let $w : H \times H \times J \times I \to H$ be a δ-convex structure on a rectangular metric space (H, d_r) and $I = [0,1]$. Then (H, d_r, w) is called a *δ-convex rectangular metric space* (In short $\delta - CRMS$).

Definition 1.7. Let (H, d_r, w) be a $\delta - CRMS$ with a function $T : H \to H$. Then for $\phi_n \in H$ and $\alpha_n \in [0,1]$, a generalized Mann's iteration sequence $\{\phi_n\}$ is defined as

$$\phi_{n+1} = w(\phi_n, T\phi_n; \alpha_n, \delta),\ n \in N,$$

where N is a set of natural numbers.

Example 1.8. Let $H = A \cup B$, where $A = \left\{\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}\right\}$ and $B = [1, 2]$. Define $d_r : H \times H \to [0, +\infty)$ such that $d_r(\phi, \psi) = d_r(\psi, \phi)$, for all $\phi, \psi \in H$ and

$$\begin{cases} d_r\left(\frac{1}{2}, \frac{3}{4}\right) &= d_r\left(\frac{5}{6}, \frac{7}{8}\right) = 0.3 \\ d_r\left(\frac{1}{2}, \frac{7}{8}\right) &= d_r\left(\frac{3}{4}, \frac{5}{6}\right) = 0.2 \\ d_r\left(\frac{1}{2}, \frac{5}{6}\right) &= d_r\left(\frac{7}{8}, \frac{3}{4}\right) = 0.6 \\ d_r\left(\frac{1}{2}, \frac{1}{2}\right) &= d_r\left(\frac{3}{4}, \frac{3}{4}\right) = d_r\left(\frac{5}{6}, \frac{5}{6}\right) = d_r\left(\frac{7}{8}, \frac{7}{8}\right) = 0 \end{cases}$$

and $d_r\big(\phi, \psi\big) = |\phi - \psi|$ if $\phi, \psi \in B$ or $\phi \in A, \psi \in B$ or $\phi \in B, \psi \in A$.

It is clear that d_r does not satisfy the triangle inequality on A. Indeed,

$$0.6 = d\Big(\frac{1}{2}, \frac{5}{6}\Big) \geq d\Big(\frac{1}{2}, \frac{3}{4}\Big) + d\Big(\frac{3}{4}, \frac{5}{6}\Big) = 0.3 + 0.2 = 0.5$$

Note that d_r satisfies the rectangular inequality. Hence (H, d_r) is an RMS.

Let us define the function $w : H \times H \times \{\frac{1}{2}\} \times \{1\} \to H$ by

$$w(\phi, \psi; \alpha, \delta) = \frac{\phi + \psi}{2}.$$

Let $t, \phi, \psi \in H$, we get

$$\begin{aligned} d_r\big(t, w(\phi, \psi; \alpha), \delta\big) &= d_r\left(t, \frac{(\phi + \psi)}{2}\right) \\ &= \left|t - \frac{(\phi + \psi)}{2}\right| \\ &= \left|\frac{(2t - \phi - \psi)}{2}\right| = \left|\frac{(t - \phi)}{2} + \frac{(t - \psi)}{2}\right| \\ &\leq 2^{-1}|t - \phi| + 2^{-1}|t - \psi| \\ &= \alpha d_r(t, \phi) + (1 - \alpha)\delta d_r(t, \psi). \end{aligned}$$

Hence (H, d_r, w) is $\delta - CRMS$ with $\alpha = 2^{-1}$ and $\delta = 1$.

2. Main Results

Theorem 2.1. *Let (H, d_r, w) be a complete δ-convex rectangular metric space. Let a contraction mapping $T : H \to H$ satisfy the condition*

$$d_r(T\phi, T\psi) \leq \rho\Big[d_r(\phi, \psi)\Big] \quad \text{for } \phi, \psi \in H \tag{2.1}$$

and for some $\rho \in [0, 1)$. Take $\phi_0 \in H$ such that $d_r(\phi_0, T\phi_0) = M < \infty$ and define $\phi_n = w(\phi_{n-1}, T\phi_{n-1}; \alpha_{n-1}, \delta)$ for $n \in \mathbb{N}$ and $\alpha_{n-1} \in [0, 1)$. Then T has a unique fixed point in H.

Proof. For any $n \in N$, we have

$$\begin{aligned} d_r\big(\phi_n, \phi_{n+1}\big) &= d_r\big(\phi_n, w(\phi_n, T\phi_n; \alpha_n, \delta)\big) \\ &= \alpha_n d_r\big(\phi_n, \phi_n\big) + (1 - \alpha_n)\delta d_r\big(\phi_n, T\phi_n\big) \end{aligned}$$

$$d_r\big(\phi_n, \phi_{n+1}\big) \leq (1 - \alpha_n)\delta d_r\big(\phi_n, T\phi_n\big). \tag{2.2}$$

Now using rectangular inequality, we obtain

$$\begin{aligned} d_r\big(\phi_n, T\phi_n\big) &= d_r\big(w(\phi_{n-1}, T\phi_{n-1}; \alpha_{n-1}, \delta), T\phi_n\big) \\ &\leq (\alpha_{n-1})d_r\big(\phi_{n-1}, T\phi_n\big) + (1 - \alpha_{n-1})\delta d_r\big(T\phi_{n-1}, T\phi_n\big) \\ &\leq (\alpha_{n-1})\Big\{d_r\big(\phi_{n-1}, \phi_n\big) + d_r\big(\phi_n, T\phi_{n-1}\big) + d_r\big(T\phi_{n-1}, T\phi_n\big)\Big\} \\ &\quad + (1 - \alpha_{n-1})\delta d_r\big(T\phi_{n-1}, T\phi_n\big) \\ &\leq \alpha_{n-1}\{(1 - \alpha_{n-1})\delta d_r\big(\phi_{n-1}, T\phi_{n-1}\big) \\ &\quad + d_r\big(w(\phi_{n-1}, T\phi_{n-1}; \alpha_{n-1}, \delta), T\phi_{n-1}\big)\} \\ &\quad + \{(1 - \alpha_{n-1})\delta + \alpha_{n-1}\}\rho d_r\big(\phi_{n-1}, \phi_n\big) \\ &\leq \alpha_{n-1}\{(1 - \alpha_{n-1})\delta d_r(\phi_{n-1}, T\phi_{n-1}) + \alpha_{n-1}d_r(\phi_{n-1}, T\phi_{n-1})\} \\ &\quad + \{(1 - \alpha_{n-1})\delta + \alpha_{n-1}\}\rho(1 - \alpha_{n-1})\delta d_r\big(\phi_{n-1}, T\phi_{n-1}\big) \\ &\leq \{\alpha_{n-1}(\sigma + \alpha_{n-1}) + (\sigma + \alpha_{n-1})\rho\sigma\}d_r\big(\phi_{n-1}, T\phi_{n-1}\big) \\ &\leq (\sigma + \alpha_{n-1})(\alpha_{n-1} + \rho\sigma)d_r\big(\phi_{n-1}, T\phi_{n-1}\big) \end{aligned}$$

$$d_r\big(\phi_n, T\phi_n\big) \leq \lambda_{n-1}d_r\big(\phi_{n-1}, T\phi_{n-1}\big) \tag{2.3}$$

where $\sigma = (1 - \alpha_{n-1})\delta \leq 1$ and $\lambda_{n-1} = (\sigma + \alpha_{n-1})(\alpha_{n-1} + \rho\sigma) \leq 1$ for $\alpha_{n-1} \in [0, 1)$ and $\rho \in [0, 1)$.

Thus $d_r\big(\phi_n, T\phi_n\big)$ is a decreasing sequence of positive reals. Hence we get $\gamma \geq 0$ such that

$$\lim_{n\to\infty} d_r\big(\phi_n, T\phi_n\big) = \gamma.$$

If possible, take $\gamma > 0$ and Letting $n \to \infty$ in (2.3), we obtain $\gamma \le \lambda_{n-1}\gamma < \gamma$, which is a contradiction. Therefore, we get $\gamma = 0$ i.e

$$\lim_{n\to\infty} d_r\big(\phi_n, T\phi_n\big) = 0.$$

Moreover, by inequality (2.2), we obtain

$$d_r\big(\phi_n, \phi_{n+1}\big) \le (1-\alpha_n)\delta d_r\big(\phi_n, T\phi_n\big) < d_r\big(\phi_n, T\phi_n\big).$$

That is,

$$\lim_{n\to\infty} d_r\big(\phi_n, \phi_{n+1}\big) = 0.$$

Next, we show that $\{\phi_n\}$ is a Cauchy sequence.
On contrary, suppose $\{\phi_n\}$ is not a Cauchy sequence, then there exists $\epsilon_0 > 0$ and we can find two sub sequences $\{\phi_{mi}\}$ and $\{\phi_{ni}\}$ of $\{\phi_n\}$ such that n_i is the littlest positive number for which $n_i > m_i > i$; $d_r\big(\phi_{mi}, \phi_{ni}\big) \ge \epsilon$.
This means

$$d_r\big(\phi_{mi}, \phi_{ni-1}\big) < \epsilon. \tag{2.4}$$

From equation (2.1) and using rectangular inequality we get

$$\epsilon \le d_r\big(\phi_{mi}, \phi_{ni}\big) \le d_r\big(\phi_{mi}, \phi_{ni-2}\big) + d_r\big(\phi_{ni-2}, \phi_{ni-1}\big) + d_r\big(\phi_{ni-1}, \phi_{ni}\big).$$

Letting $i \to \infty$ in the above inequality and using (2.4), we get

$$\lim_{i\to\infty} \sup d_r\big(\phi_{mi}, \phi_{ni}\big) \le \epsilon.$$

Now,

$$\begin{aligned}
d_r\big(\phi_{mi+1}, \phi_{ni}\big) &= d_r\big(w\big(\phi_{ni-1}, T\phi_{ni-1}; \alpha_{ni-1}, \delta\big), \phi_{mi+1}\big) \\
&= \alpha_{ni-1} d_r\big(\phi_{ni-1}, \phi_{mi+1}\big) + \big(1-\alpha_{ni-1}\big)\delta d_r\big(T\phi_{ni-1}, \phi_{mi+1}\big) \\
&\le \alpha_{ni-1} d_r\big(\phi_{ni-1}, \phi_{mi+1}\big) + \big(1-\alpha_{ni-1}\big)\delta\Big[d_r\big(T\phi_{ni-1}, T\phi_{mi}\big) \\
&\quad + d_r\big(T\phi_{mi}, T\phi_{mi+1}\big) + d_r\big(T\phi_{mi+1}, \phi_{mi+1}\big)\Big] \\
&\le \alpha_{ni-1} d_r\big(\phi_{ni-1}, \phi_{mi+1}\big) + \big(1-\alpha_{ni-1}\big)\delta\Big[\rho d_r(\phi_{ni-1}, \phi_{mi}) \\
&\quad + \rho d_r(\phi_{mi}, \phi_{mi+1}) + d_r\big(T\phi_{mi+1}, \phi_{mi+1}\big)\Big] \\
&\le \alpha_{ni-1}\Big[d_r\big(\phi_{ni-1}, \phi_{ni}\big) + d_r\big(\phi_{ni}, \phi_{mi}\big) + d_r\big(\phi_{mi}, \phi_{mi+1}\big)\Big] \\
&\quad + \big(1-\alpha_{ni-1}\big)\delta\Big[\rho d_r(\phi_{ni-1}, \phi_{mi}) + \rho d_r(\phi_{mi}, \phi_{mi+1}) \\
&\quad + d_r\big(T\phi_{mi+1}, \phi_{mi+1}\big)\Big].
\end{aligned}$$

Letting $i \to \infty$, we get

$$\lim_{i\to\infty} \sup d_r\big(\phi_{mi+1}, \phi_{ni}\big) < \epsilon$$

which is a contradiction. Thus $\{\phi_n\}$ is a Cauchy sequence in H. By completeness of H, there exists $\phi^* \in H$ such that

$$\lim_{n\to\infty} d_r\big(\phi_m, \phi^*\big) = 0.$$

Thus

$$\lim_{n\to\infty} d_r\big(\phi_n, \phi^*\big) = 0.$$

Now we will show that ϕ^* is a fixed point of T.
Applying rectangular inequality, we get

$$\begin{aligned} d_r\big(\phi^*, T\phi^*\big) &\le d_r\big(\phi^*, \phi_n\big) + d_r\big(\phi_n, T\phi_n\big) + d_r\big(T\phi_n, T\phi^*\big) \\ &\le d_r\big(\phi^*, \phi_n\big) + d_r\big(\phi_n, T\phi_n\big) + d_r\big(T\phi_n, T\phi^*\big) \\ &\le d_r\big(\phi^*, \phi_n\big) + d_r\big(\phi_n, T\phi_n\big) + \rho d_r\big(\phi_n, \phi^*\big) \\ &\le (1+\rho) d_r\big(\phi^*, \phi_n\big) + d_r\big(\phi_n, T\phi_n\big). \end{aligned}$$

Letting $n \to \infty$, we get $\lim_{n\to\infty} d_r\big(\phi^*, T\phi^*\big) = 0$. Thus

$$T\phi^* = \phi^*.$$

Hence, ϕ^* is a fixed point of T.

UNIQUENESS OF FIXED POINT: On the contrary, suppose ψ^* is another fixed point of T, then we have

$T\phi^* = \phi^*$ and $T\psi^* = \psi^*$. Now

$$\begin{aligned} d_r\big(\phi^*, \psi^*\big) &= d_r\big(T\phi^*, T\psi^*\big) \\ &\le \rho d_r\big(\phi^*, \psi^*\big) \\ (1-\rho) d_r\big(\phi^*, \psi^*\big) &\le 0 \\ \text{but} \quad 1-\rho \ne 0 \quad \therefore \quad & d_r\big(\phi^*, \psi^*\big) = 0. \end{aligned}$$

Therefore $\phi^* = \psi^*$, which completes the proof. □

Theorem 2.2. *Let (H, d_r, w) be complete δ-convex rectangular metric space. Let a self mapping $T : H \to H$ satisfy the condition*

$$d_r(T\phi, T\psi) \le \beta \max\big\{d_r(\phi, T\phi), d_r(\psi, T\psi)\big\} \quad \text{for } \phi, \psi \in H \quad \text{and } \beta \in [0, \frac{1}{2})$$

Take $\phi_0 \in H$ such that $d_r(\phi_0, T\phi_0) = M < \infty$ and $\phi_n = w\big(\phi_{n-1}, T\phi_{n-1}; \alpha_{n-1}, \delta\big)$ where $0 < \alpha_{n-1} < \frac{1}{2}$ and $n \in \mathbb{N}$. Then T has a unique fixed point in H.

Proof. For any $n \in N$, we have inequality (2.2)

$$d_r\big(\phi_n, \phi_{n+1}\big) \leq (1-\alpha_n)\delta d_r\big(\phi_n, T\phi_n\big)$$

and

$$\begin{aligned}
d_r\big(\phi_n, T\phi_n\big) &= d_r\big(w(\phi_{n-1}, T\phi_{n-1}; \alpha_{n-1}, \delta), T\phi_n\big) \\
&\leq (\alpha_{n-1}) d_r\big(\phi_{n-1}, T\phi_n\big) + (1-\alpha_{n-1})\delta d_r\big(T\phi_{n-1}, T\phi_n\big) \\
&\leq (\alpha_{n-1})\Big\{d_r\big(\phi_{n-1}, \phi_n\big) + d_r\big(\phi_n, T\phi_{n-1}\big) + d_r\big(T\phi_{n-1}, T\phi_n\big)\Big\} \\
&\quad + (1-\alpha_{n-1})\delta d_r\big(T\phi_{n-1}, T\phi_n\big) \\
&\leq \alpha_{n-1}\{(1-\alpha_{n-1})\delta d_r\big(\phi_{n-1}, T\phi_{n-1}\big) \\
&\quad + d_r\big(w(\phi_{n-1}, T\phi_{n-1}; \alpha_{n-1}, \delta), T\phi_{n-1}\big)\} \\
&\quad + \{(1-\alpha_{n-1})\delta + \alpha_{n-1}\}\beta[\max\{d_r\big(\phi_{n-1}, T\phi_{n-1}\big), d_r\big(\phi_n, T\phi_n\big\}] \\
&\leq \alpha_{n-1}\{(1-\alpha_{n-1})\delta d_r\big(\phi_{n-1}, T\phi_{n-1}\big) + \alpha_{n-1} d_r(\phi_{n-1}, T\phi_{n-1})\} \\
&\quad + \{(1-\alpha_{n-1})\delta + \alpha_{n-1}\}\beta[\max\{d_r\big(\phi_{n-1}, T\phi_{n-1}\big), d_r\big(\phi_n, T\phi_n\big\}] \\
&\leq \alpha_{n-1}\{\sigma d_r\big(\phi_{n-1}, T\phi_{n-1}\big) + \alpha_{n-1} d_r(\phi_{n-1}, T\phi_{n-1})\} \\
&\quad + \{(\sigma + \alpha_{n-1}\}\beta[\max\{d_r\big(\phi_{n-1}, T\phi_{n-1}\big), d_r\big(\phi_n, T\phi_n\big)\}] \\
&\leq \alpha_{n-1}(\sigma + \alpha_{n-1}) d_r(\phi_{n-1}, T\phi_{n-1}) \\
&\quad + (\sigma + \alpha_{n-1})\beta[\max\{d_r\big(\phi_{n-1}, T\phi_{n-1}\big), d_r\big(\phi_n, T\phi_n\big)\}]
\end{aligned} \tag{2.5}$$

where $\sigma = (1-\alpha_{n-1})\delta \leq 1$.

Case i. Assume that $\max\Big\{d_r\big(\phi_{n-1}, T\phi_{n-1}\big), d_r\big(\phi_n, T\phi_n\big)\Big\} = d_r\big(\phi_{n-1}, T\phi_{n-1}\big)$. Then by inequality (2.5), we get

$$\begin{aligned}
d_r\big(\phi_n, T\phi_n\big) &\leq (\alpha_{n-1} + \beta)(\sigma + \alpha_{n-1}) d_r\big(\phi_{n-1}, T\phi_{n-1}\big) \\
&\leq \lambda_1 d_r\big(\phi_{n-1}, T\phi_{n-1}\big)
\end{aligned}$$

Since $0 < \alpha_{n-1} < \frac{1}{2}$, $\sigma \leq 1$ and $\beta \in [0, \frac{1}{2})$ then $\lambda_1 \leq 1$.

Case ii. If $\max\Big\{d_r\big(\phi_{n-1}, T\phi_{n-1}\big), d_r\big(\phi_n, T\phi_n\big)\Big\} = d_r\big(\phi_n, T\phi_n\big)$. Then by inequality (2.5), we get

$$\begin{aligned}
d_r\big(\phi_n, T\phi_n\big) &\leq \alpha_{n-1}(\sigma + \alpha_{n-1}) d_r(\phi_{n-1}, T\phi_{n-1}) \\
&\quad + (\sigma + \alpha_{n-1})\beta d_r\big(\phi_n, T\phi_n\big) \\
\{1 - (\sigma + \alpha_{n-1})\beta\} d_r\big(\phi_n, T\phi_n\big) &\leq \alpha_{n-1}(\sigma + \alpha_{n-1}) d_r(\phi_{n-1}, T\phi_{n-1})
\end{aligned}$$

$$d_r\big(\phi_n, T\phi_n\big) \leq \frac{\alpha_{n-1}(\sigma + \alpha_{n-1})}{1-(\sigma + \alpha_{n-1})\beta} d_r(\phi_{n-1}, T\phi_{n-1})$$
$$\leq \lambda_2 d_r\big(\phi_{n-1}, T\phi_{n-1}\big),$$

where

$$\lambda_2 = \frac{\alpha_{n-1}(\sigma + \alpha_{n-1})}{1-(\sigma + \alpha_{n-1})\beta} \leq 1.$$

Let $\lambda = \max\{\lambda_1, \lambda_2\}$. Then

$$d_r\big(\phi_n, T\phi_n\big) \leq \lambda d_r\big(\phi_{n-1}, T\phi_{n-1}\big)$$

which implies that $d_r\big(\phi_n, T\phi_n\big)$ is a decreasing sequence of positive reals.

Letting $n \to \infty$, we have

$$\lim_{n\to\infty} d_r\big(\phi_n, T\phi_n\big) = 0.$$

Moreover, by inequality (2.2), we obtain

$$d_r\big(\phi_n, \phi_{n+1}\big) \leq (1-\alpha_n)\delta d_r\big(\phi_n, T\phi_n\big) < d_r\big(\phi_n, T\phi_n\big).$$

That is,

$$\lim_{n\to\infty} d_r\big(\phi_n, \phi_{n+1}\big) = 0.$$

Next, we show that $\{\phi_n\}$ is a Cauchy sequence.
Suppose $\{\phi_n\}$ is not a Cauchy sequence, then there exists $\epsilon_0 > 0$ and we can find two sub sequences $\{\phi_{m'i}\}$ and $\{\phi_{n'i}\}$ of $\{\phi_n\}$ such that n'_i is the littlest positive number for which

$$n'_i > m'_i > i; \;\; d_r\big(\phi_{m'i}, \phi_{n'i}\big) \geq \epsilon.$$

This means

$$d_r\big(\phi_{m'i}, \phi_{n'i-1}\big) < \epsilon.$$

Now using rectangular inequality we get

$$\epsilon \leq d_r\big(\phi_{m'i}, \phi_{n'i}\big) \leq d_r\big(\phi_{m'i}, \phi_{n'i-2}\big) + d_r\big(\phi_{n'i-2}, \phi_{n'i-1}\big) + d_r\big(\phi_{n'i-1}, \phi_{n'i}\big).$$

Letting $i \to \infty$ in the above inequality, we get

$$\lim_{i\to\infty} \sup d_r\big(\phi_{m'i}, \phi_{n'i}\big) \leq \epsilon.$$

Now,

$$\begin{aligned} d_r\big(\phi_{m'i+1},\phi_{n'i}\big) &= d_r\big(w\big(\phi_{n'i-1},T\phi_{n'i-1};\alpha_{n'i-1},\delta\big),\phi_{m'i+1}\big)\\ &= \alpha_{n'i-1}d_r\big(\phi_{n'i-1},\phi_{m'i+1}\big)+\big(1-\alpha_{n'i-1}\big)\delta d_r\big(T\phi_{n'i-1},\phi_{m'i+1}\big)\\ &\le \alpha_{n'i-1}d_r\big(\phi_{n'i-1},\phi_{m'i+1}\big)+\big(1-\alpha_{n'i-1}\big)\delta\Big[d_r\big(T\phi_{n'i-1},\phi_{n'i-1}\big)\\ &\quad + d_r\big(\phi_{n'i-1},\phi_{n'i}\big)+d_r\big(\phi_{n'i},\phi_{m'i+1}\big)\Big]\end{aligned}$$

therefore

$$\{1-\big(1-\alpha_{n'i-1}\big)\delta\}d_r\big(\phi_{m'i+1},\phi_{n'i}\big)\le \alpha_{n'i-1}d_r\big(\phi_{n'i-1},\phi_{m'i+1}\big)+\big(1-\alpha_{n'i-1}\big)\delta \Big[d_r\big(T\phi_{n'i-1},\phi_{n'i-1}\big)+d_r\big(\phi_{n'i-1},\phi_{n'i}\big)\Big].$$

Letting $i\to\infty$, we obtain $\lim_{i\to\infty}\sup d_r\big(\phi_{m'i+1},\phi_{n'i}\big)<\epsilon$, which is a contradiction. Thus $\{\phi_n\}$ is a Cauchy sequence in H. By completeness of H, there exists $\phi^*\in H$ such that

$$\lim_{n\to\infty} d_r\big(\phi_m,\phi^*\big)=0$$

Hence, $\{\phi_n\}_{n=1}^{\infty}$ is a Cauchy sequence in H. By the completeness of H, it follows that there exists $\phi^*\in H$ such that

$$\lim_{n\to\infty} d_r\big(\phi_n,\phi^*\big)=0.$$

Now we will show that ϕ^* is a fixed point of T. Since

$$\begin{aligned} d_r\big(\phi^*,T\phi^*\big) &\le d_r\big(\phi^*,\phi_n\big)+d_r\big(\phi_n,T\phi_n\big)+d_r\big(T\phi_n,T\phi^*\big)\\ &\le d_r\big(\phi^*,\phi_n\big)+d_r\big(\phi_n,T\phi_n\big)+\beta\big\{d_r(\phi_n,T\phi_n),d_r(\phi^*,T\phi^*)\big\}\end{aligned}$$

Case i. If $\max\big\{d_r(\phi_n,T\phi_n),d_r(\phi^*,T\phi^*)\big\}=d_r\big(\phi_n,T\phi_n\big)$. Then

$$\begin{aligned} d_r\big(\phi^*,T\phi^*\big) &\le d_r\big(\phi^*,\phi_n\big)+d_r\big(\phi_n,T\phi_n\big)+\beta d_r(\phi_n,T\phi_n)\\ &\le d_r\big(\phi^*,\phi_n\big)+(1+\beta)d_r\big(\phi_n,T\phi_n\big)\end{aligned}$$

Letting $n\to\infty$, we get $\lim_{n\to\infty} d_r\big(\phi^*,T\phi^*\big)=0$.
Thus

$$T\phi^*=\phi^*.$$

Thus ϕ^* is a fixed point of T.

Case ii. If $\max\big\{d_r(\phi_n,T\phi_n),d_r(\phi^*,T\phi^*)\big\}=d_r\big(\phi^*,T\phi^*\big)$. Then

$$\begin{aligned} d_r\big(\phi^*,T\phi^*\big) &\le d_r\big(\phi^*,\phi_n\big)+d_r\big(\phi_n,T\phi_n\big)+\beta d_r\big(\phi^*,T\phi^*\big)\\ (1-\beta)d_r\big(\phi^*,T\phi^*\big) &\le d_r\big(\phi^*,\phi_n\big)+d_r\big(\phi_n,T\phi_n\big)\end{aligned}$$

Letting $n\to\infty$, we get $\lim_{n\to\infty} d_r\big(\phi^*,T\phi^*\big)=0$.

Thus

$$T\phi^* = \phi^*.$$

Hence, ϕ^* is a fixed point of T. Uniqueness is clear. □

Theorem 2.3. *Let (H, d_r, w) be complete δ-convex rectangular metric space and $T : H \to H$ be a contraction mapping, that is there exists $\delta \in [0, \frac{1}{2})$ such that*

$$d_r(T\phi, T\psi) \leq \delta \max\{d_r(\phi, T\phi), d_r(\psi, T\psi), d_r(\phi, \psi)\} \quad \text{for } \phi, \psi \in H.$$

Take $\phi_0 \in H$ such that $d_r(\phi_0, T\phi_0) = M < \infty$ and define $\phi_n = w(\phi_{n-1}, T\phi_{n-1}; \alpha_{n-1}, \delta)$ where $0 < \alpha_{n-1} < \frac{1}{2}$ and $n \in \mathbb{N}$ then, T has a unique fixed point in H.

Proof. Obviously proves. □

Now we prove the Kannan-type fixed point result for a δ-convex rectangular metric space.

Theorem 2.4. *Let (H, d_r, w) be a complete δ-convex rectangular metric space and let the mapping $T : H \to H$ be defined as*

$$d_r(T\phi, T\psi) \leq \mu\{d_r(\phi, T\phi) + d_r(\psi, T\psi)\}$$

for all $\phi, \psi \in H$. Take $\phi_0 \in H$ such that $d_r(\phi_0, T\phi_0) = M < \infty$ and define $\phi_n = w(\phi_{n-1}, T\phi_{n-1}; \alpha_{n-1}, \delta)$ for $n \in \mathbb{N}$ and $\alpha_{n-1} \in [0, 1)$. If $\mu \in [0, \frac{1}{2})$, then T has a unique fixed point of H.

Proof: For any $n \in N$, we have from (2.2)

$$d_r(\phi_n, \phi_{n+1}) \leq (1 - \alpha_n)\delta d_r(\phi_n, T\phi_n)$$

Now applying rectangular inequality, we get

$$\begin{aligned}
d_r(\phi_n, T\phi_n) &= d_r(w(\phi_{n-1}, T\phi_{n-1}; \alpha_{n-1}, \delta), T\phi_n) \\
&\leq (\alpha_{n-1})d_r(\phi_{n-1}, T\phi_n) + (1 - \alpha_{n-1})\delta d_r(T\phi_{n-1}, T\phi_n) \\
&\leq (\alpha_{n-1})\Big\{d_r(\phi_{n-1}, \phi_n) + d_r(\phi_n, T\phi_{n-1}) + d_r(T\phi_{n-1}, T\phi_n)\Big\} \\
&\quad + (1 - \alpha_{n-1})\delta d_r(T\phi_{n-1}, T\phi_n) \\
&\leq \alpha_{n-1}\{(1 - \alpha_{n-1})\delta d_r(\phi_{n-1}, T\phi_{n-1}) \\
&\quad + d_r(w(\phi_{n-1}, T\phi_{n-1}; \alpha_{n-1}, \delta), T\phi_{n-1})\} \\
&\quad + \{\alpha_{n-1} + (1 - \alpha_{n-1})\delta\}\mu\Big\{d_r(\phi_{n-1}, T\phi_{n-1}) + d_r(\phi_n, T\phi_n)\Big\}
\end{aligned}$$

$$
\begin{aligned}
&\leq \alpha_{n-1}\{(1-\alpha_{n-1})\delta d_r\big(\phi_{n-1}, T\phi_{n-1}\big) + \alpha_{n-1} d_r(\phi_{n-1}, T\phi_{n-1})\} \\
&\quad + \{\alpha_{n-1} + (1-\alpha_{n-1})\delta\}\mu\Big\{d_r\big(\phi_{n-1}, T\phi_{n-1}\big) + d_r\big(\phi_n, T\phi_n\big)\Big\} \\
&\leq \alpha_{n-1}(\sigma + \alpha_{n-1}) d_r(\phi_{n-1}, T\phi_{n-1}) \\
&\quad + \{\alpha_{n-1} + \sigma\}\mu\Big\{d_r\big(\phi_{n-1}, T\phi_{n-1}\big) + d_r\big(\phi_n, T\phi_n\big)\Big\}
\end{aligned}
$$

where $\sigma = (1-\alpha_{n-1})\delta \leq 1$.

$$
\begin{aligned}
(1-\sigma-\alpha_{n-1}) d_r\big(\phi_n, T\phi_n\big) &\leq (\alpha_{n-1}+\mu)(\sigma+\alpha_{n-1}) d_r(\phi_{n-1}, T\phi_{n-1}) \\
d_r\big(\phi_n, T\phi_n\big) &\leq \frac{(\alpha_{n-1}+\mu)(\sigma+\alpha_{n-1})}{1-\sigma-\alpha_{n-1}} d_r(\phi_{n-1}, T\phi_{n-1})
\end{aligned}
$$

Since, $\frac{(\alpha_{n-1}+\mu)(\sigma+\alpha_{n-1})}{1-\sigma-\alpha_{n-1}} < 1$, which implies that $d_r\big(\phi_n, T\phi_n\big)$ is a decreasing sequence of positive reals. Using process of Theorem 2.1 we have

$$\lim_{n\to\infty} d_r\big(\phi_n, \phi_{n+1}\big) = 0.$$

Also we can check that $\{\phi_n\}_{n=1}^{\infty}$ is a Cauchy sequence in H. By the completeness of H, there exists $\phi^* \in H$ such that

$$\lim_{n\to\infty} d_r\big(\phi_n, \phi^*\big) = 0.$$

Now we show that ϕ^* is a fixed point of T. Since

$$
\begin{aligned}
d_r\big(\phi^*, T\phi^*\big) &\leq d_r\big(\phi^*, \phi_n\big) + d_r\big(\phi_n, T\phi_n\big) + d_r\big(T\phi_n, T\phi^*\big) \\
&\leq d_r\big(\phi^*, \phi_n\big) + d_r\big(\phi_n, T\phi_n\big) + \mu\{d_r\big(\phi_n, T\phi_n\big) + d_r\big(\phi^*, T\phi^*\big)\} \\
(1-\mu) d_r\big(\phi^*, T\phi^*\big) &\leq d_r\big(\phi^*, \phi_n\big) + (1+\mu) d_r\big(\phi_n, T\phi_n\big)
\end{aligned}
$$

Letting $n \to \infty$ we get $\lim_{n\to\infty} d_r\big(\phi^*, T\phi^*\big) = 0$.

Thus

$$T\phi^* = \phi^*.$$

Hence, ϕ^* is a fixed point of T. Uniqueness is clear. □

Finally, we prove the Reich-type fixed point result for a δ-convex rectangular metric space.

Theorem 2.5. *Let (H, d_r, w) be a complete δ-convex rectangular metric space and let the mapping $T : H \to H$ be defined as*

$$d_r(T\phi, T\psi) \leq \theta d_r(\phi, \psi) + \eta\{d_r(\phi, T\phi) + d_r(\psi, T\psi)\}$$

for all $\phi, \psi \in H$. *Take* $\phi_0 \in H$ *such that* $d_r(\phi_0, T\phi_0) = M < \infty$ *and define* $\phi_n = w(\phi_{n-1}, T\phi_{n-1}; \alpha_{n-1}, \delta)$ *for* $n \in \mathbb{N}$ *and* $\alpha_{n-1} \in [0,1)$. *If* $\theta, \eta \in [0,1)$, *then* T *has a unique fixed point of* H.

Proof. For any $n \in N$, we have from (2.2)

$$d_r\big(\phi_n, \phi_{n+1}\big) \leq (1-\alpha_n)\delta d_r\big(\phi_n, T\phi_n\big)$$

Now using rectangular inequality, we get

$$\begin{aligned}
d_r\big(\phi_n, T\phi_n\big) &= d_r\big(w(\phi_{n-1}, T\phi_{n-1}; \alpha_{n-1}, \delta), T\phi_n\big) \\
&\leq (\alpha_{n-1}) d_r\big(\phi_{n-1}, T\phi_n\big) + (1-\alpha_{n-1})\delta d_r\big(T\phi_{n-1}, T\phi_n\big) \\
&\leq (\alpha_{n-1})\Big\{d_r\big(\phi_{n-1}, \phi_n\big) + d_r\big(\phi_n, T\phi_{n-1}\big) + d_r\big(T\phi_{n-1}, T\phi_n\big)\Big\} \\
&\quad + (1-\alpha_{n-1})\delta d_r\big(T\phi_{n-1}, T\phi_n\big) \\
&\leq \alpha_{n-1}\{(1-\alpha_{n-1})\delta d_r\big(\phi_{n-1}, T\phi_{n-1}\big) \\
&\quad + d_r\big(w(\phi_{n-1}, T\phi_{n-1}; \alpha_{n-1}, \delta), T\phi_{n-1}\big)\} \\
&\quad + \{\alpha_{n-1} + (1-\alpha_{n-1})\delta\}[\theta d_r\big(\phi_{n-1}, \phi_n\big) + \eta\{d_r\big(\phi_{n-1}, T\phi_{n-1}\big) \\
&\quad + d_r\big(\phi_n, T\phi_n\big)\}] \\
&\leq \alpha_{n-1}\{\sigma d_r\big(\phi_{n-1}, T\phi_{n-1}\big) + \alpha_{n-1} d_r(\phi_{n-1}, T\phi_{n-1})\} + (\alpha_{n-1} + \sigma) \\
&\quad [\theta(1-\alpha_{n-1}) d_r\big(\phi_{n-1}, T\phi_{n-1}\big) + \eta\{d_r\big(\phi_{n-1}, T\phi_{n-1}\big) + d_r\big(\phi_n, T\phi_n\big)\}]
\end{aligned}$$

where $\sigma = (1-\alpha_{n-1})\delta \leq 1$.
Therefore

$$\begin{aligned}
&\{1 - (\alpha_{n-1} - \sigma)\eta\} d_r\big(\phi_n, T\phi_n\big) \\
&\quad \leq \alpha_{n-1}(\sigma + \alpha_{n-1}) d_r(\phi_{n-1}, T\phi_{n-1})\} + (\alpha_{n-1} + \sigma) \\
&\{\theta(1-\alpha_{n-1}) d_r\big(\phi_{n-1}, T\phi_{n-1}\big) + \eta d_r\big(\phi_{n-1}, T\phi_{n-1}\big)\} \\
&\quad \leq (\sigma + \alpha_{n-1})(\alpha_{n-1} + \theta(1-\alpha_{n-1}) + \eta) d_r\big(\phi_{n-1}, T\phi_{n-1}\big)
\end{aligned}$$

$$d_r\big(\phi_n, T\phi_n\big) \leq \frac{(\sigma + \alpha_{n-1})(\alpha_{n-1} + \theta(1-\alpha_{n-1}) + \eta)}{1 - (\alpha_{n-1} - \sigma)\eta} d_r\big(\phi_{n-1}, T\phi_{n-1}\big).$$

Since, $\frac{(\sigma+\alpha_{n-1})(\alpha_{n-1}+\theta(1-\alpha_{n-1})+\eta)}{1-(\alpha_{n-1}-\sigma)\eta} < 1$, which implies that $d_r\big(\phi_n, T\phi_n\big)$ is a decreasing sequence of positive reals. Hence

$$\lim_{n\to\infty} d_r\big(\phi_n, T\phi_n\big) = 0$$

and $\lim_{n\to\infty} d_r\big(\phi_n, \phi_{n+1}\big) = 0$.

Using Theorem 2.1, we can easily check that $\{\phi_n\}_{n=1}^{\infty}$ is a Cauchy sequence in H. By the completeness of H, there exists $\phi^* \in H$ such that

$$\lim_{n\to\infty} d_r\big(\phi_n, \phi^*\big) = 0.$$

Now we will show that ϕ^* is a fixed point of T. Since

$$\begin{aligned}
d_r\big(\phi^*, T\phi^*\big) &\le d_r\big(\phi^*, \phi_n\big) + d_r\big(\phi_n, T\phi_n\big) + d_r\big(T\phi_n, T\phi^*\big) \\
&\le d_r\big(\phi^*, \phi_n\big) + d_r\big(\phi_n, T\phi_n\big) + \theta d_r\big(\phi_n, \phi^*\big) \\
&\quad + \eta\{d_r\big(\phi_n, T\phi_n\big) + d_r\big(\phi^*, T\phi^*\big)\} \\
(1-\eta) d_r\big(\phi^*, T\phi^*\big) &\le d_r\big(\phi^*, \phi_n\big) + d_r\big(\phi_n, T\phi_n\big) + \theta d_r\big(\phi_n, \phi^*\big) + \eta d_r\big(\phi_n, T\phi_n\big) \\
&\le (1+\theta) d_r\big(\phi^*, \phi_n\big) + (1+\eta) d_r\big(\phi_n, T\phi_n\big) \\
d_r\big(\phi^*, T\phi^*\big) &\le \frac{1+\theta}{1-\eta} d_r\big(\phi^*, \phi_n\big) + \frac{1+\eta}{1-\eta} d_r\big(\phi_n, T\phi_n\big).
\end{aligned}$$

Letting $n \to \infty$, we get $\lim_{n\to\infty} d_r\big(\phi^*, T\phi^*\big) = 0$. Thus ϕ^* is a fixed point of T. Uniqueness is clear. □

3. Conclusion

In this paper, we established a new notion of δ-convex rectangular metric space with the help of a convex structure. We proved several innovative fixed point results for δ-convex contraction mapping in the reference of rectangular metric spaces. Finally, we obtained Kannan-type and Reich-type fixed point results in such spaces. Our effort can be enlarged in many ways by extending the class of this metric space.

4. Conflict of Interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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