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FIXED POINT THEOREMS FOR (ϕ, F)-CONTRACTION IN GENERALIZED ASYMMETRIC METRIC SPACES

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Abstract. In the last few decades, a lot of generalizations of the Banach contraction principle have been introduced. In this paper, we present the notion of (ϕ, F)-contraction in generalized asymmetric metric spaces and we investigate the existence of fixed points of such mappings. We also provide some illustrative examples to show that our results improve many existing results.

1. Introduction

Banach contraction principle is considered to be the initial result of the study of fixed point theory in metric spaces [2]. Various generalizations of it appeared in the literature, much mathematics steadied many interesting extensions and generalizations (see [6, 9, 15, 18]) and the recent works of Wardowski in [18, 19, 20].

In 2018, Wardowski [19] analysed a generalization of the Banach fixed point theorem on metric spaces in a new type of contraction mappings on metric space called F-ϕ-contraction. Very recently Kari *et al.* [9] extended Wardowskis ideas to the case of nonlinear F-contraction via w-distance and studied the solution of certain integral equations under a suitable set of hypotheses.

A well known, several generalizations of standard metric spaces have appeared. In particular, asymmetric metric spaces were introduced by Wilson [21] and then studied by many authors (see [1, 11, 13, 16]). In 2000, for the first time generalized metric spaces were introduced by Branciari [3], in such a way that triangle inequality is replaced by the quadrilateral inequality $d(x, y) \leq d(x, z) + d(z, u) + d(u, y)$ for all pairwise distinct points x, y, z and u. Any metric space is a generalized metric

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space but in general, generalized metric space might not be a metric space. Various fixed point results were established on such spaces (see [4, 5, 10, 17]) and references therein.

Combining conditions used for definitions of asymmetric metric and generalized metric spaces, Piri *et al.* [14] announced the notions of generalized asymmetric metric space, and formulated some first fixed point theorems for θ-contraction mapping in generalized asymmetric metric space.

In this paper, inspired by the interest aroused θ-ϕ-contraction introduced in [8], we introduce the notion of (ϕ, F)-contraction and establish some new fixed point theorems for mappings in the setting of complete generalized asymmetric metric spaces. Our results generalize, improve and extend the corresponding results due to Kannan and Reich. Moreover, an illustrative example is presented to support the obtained results.

2. Preliminaries

In the following, we recollect some definitions which will be useful in our main results.

Definition 2.1 ([3])**.** Let X be a non-empty set and $d : X \times X \to \mathbb{R}^+$ be a function such that for all $x, y \in X$ and for all distinct points $u, v \in X$, each of them different from x and y, one has

(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all distinct points $x, y \in X$;
(iii) $d(x, y) \le d(x, u) + d(u, v) + d(v, y)$ (quadrilateral inequality).

Then (X, d) is called a *generalized metric space.*

Definition 2.2 ([14])**.** Let X be a non-empty set and $d : X \times X \to \mathbb{R}^+$ be a function such that for all $x, y \in X$ and for all distinct points $u, v \in X$, each of them different from x and y, one has

(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) \le d(x, u) + d(u, v) + d(v, y)$ (quadrilateral inequality).

Then (X, d) is called a *generalized asymmetric metric space.*

Definition 2.3 ([14])**.** Let (X, d) is a generalized asymmetric metric space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X, and $x \in X$.

(i) We say that $\{x_n\}_{n\in\mathbb{N}}$ is *forward (backward) convergent* to x if

$$\lim_{n\to+\infty} d(x,x_n)=0 \qquad (\lim_{n\to+\infty} d(x_n,x)=0).$$

(ii) We say that $\{x_n\}_{n\in\mathbb{N}}$ is *forward (backward) Cauchy* if

$$\lim_{n,m\to+\infty, n<m} d(x_n,x_m)=0 \quad (\lim_{n,m\to+\infty, n<m} d(x_m,x_n)=0).$$

Example 2.4 ([7])**.** Let $X=A\cup B$, where $A=\{0,2\}$ and $B=\{\frac{1}{n},\ n\in\mathbb{N}^*\}$ and $d: X\times X\to[0,+\infty[$ be defined by

$$\begin{cases} d(0,2)=d(2,0)=1 \\ d\left(\frac{1}{n},0\right)=\frac{1}{n}, d\left(0,\frac{1}{n}\right)=1 \\ d\left(\frac{1}{n},2\right)=1, d\left(2,\frac{1}{n}\right)=\frac{1}{n} \\ d\left(\frac{1}{n},\frac{1}{m}\right)=d\left(\frac{1}{m},\frac{1}{n}\right)=1. \end{cases}$$

for all $n,m\in\mathbb{N}^*$, $n\neq m$. Then (X,d) is a generalized asymmetric metric space. However, we have the following:

1) (X,d) is not a metric space, since $d\left(\frac{1}{n},0\right)\neq d\left(0,\frac{1}{n}\right)$ for all $n>1$.
2) (X,d) is not a asymmetric metric space, since $d(2,0)=1>\frac{1}{2}=d\left(2,\frac{1}{4}\right)+d\left(\frac{1}{4},0\right)$.
3) (X,d) is not a rectangular metric space, since $d\left(\frac{1}{n},2\right)\neq d\left(2,\frac{1}{n}\right)$ for all $n>1$.

Remark 2.5 ([7])**.** Let (X,d) be as in Example 2.4 and $\{\frac{1}{n}\}_{n\in\mathbb{N}^*}$ be a sequence in X. Then we have the following:

i) $\lim_{n\to+\infty} d\left(\frac{1}{n},0\right)=0$, $\lim_{n\to+\infty} d\left(\frac{1}{n},2\right)=1$ and $\lim_{n\to+\infty} d\left(0,\frac{1}{n}\right)=1$, $\lim_{n\to+\infty} d\left(2,\frac{1}{n}\right)=0$. Thus the sequence $\{\frac{1}{n}\}$ is forward convergent to 2 and is backward convergent to 0. So the limit is not unique.
ii) $\lim_{n,m\to+\infty, m>n} d\left(\frac{1}{m},\frac{1}{n}\right)=\lim_{n,m\to+\infty, m<n} d\left(\frac{1}{m},\frac{1}{n}\right)=1$. So forward (backward) convergence does not imply forward (backward) Cauchy.

Lemma 2.6 ([14])**.** *Let (X,d) be a generalized asymmetric metric space and $\{x_n\}_n$ be a forward (or backward) Cauchy sequence with pairwise disjoint elements in X. If $\{x_n\}_n$ is forward convergent to $x\in X$ and backward convergent to $y\in X$, then $x=y$.*

Definition 2.7 ([14]). Let (X, d) be a generalized asymmetric metric space. Then X is said to be *forward (backward) complete* if every forward (backward) Cauchy sequence $\{x_n\}_n$ in X is forward (backward) convergent to $x \in X$.

Definition 2.8 ([14]). Let (X, d) be a generalized asymmetric metric space. Then X is said to be *complete* if X is forward and backward complete.

The following definition was introduced by Wardowski.

Definition 2.9 ([18]). Let $\mathcal{F}$ be the family of all functions $F\colon \mathbb{R}^+ \to \mathbb{R}$ such that

(i) F is strictly increasing;

(ii) for each sequence $\{x_n\}_{n\in\mathbb{N}}$ of positive numbers,

$$\lim_{n\to 0} x_n = 0 \quad \text{if and only if} \quad \lim_{n\to\infty} F(x_n) = -\infty;$$

(iii) there exists $k \in]0, 1[$ such that $\lim_{x\to 0} x^k F(x) = 0$.

Recently, Piri and Kuman [12] extended the result of Wardowski [18] by changing the condition (iii) in Definition 2.9 as follows:

Definition 2.10 ([12]). Let Γ be the family of all functions $F\colon \mathbb{R}^+ \to \mathbb{R}$ such that

(i) F is strictly increasing;

(ii) for each sequence $\{x_n\}_{n\in\mathbb{N}}$ of positive numbers,

$$\lim_{n\to\infty} x_n = 0 \quad \text{if and only if} \quad \lim_{n\to\infty} F(x_n) = -\infty;$$

(iii) F is continuous.

The following result introduced by Wardowski [19] will be used to prove our result.

Definition 2.11 ([19]). Let $\mathbb{F}$ be the family of all functions $F\colon \mathbb{R}^+ \to \mathbb{R}$ and Φ be the family of all functions $\phi :]0, +\infty[\to]0, +\infty[$ satisfying the following.

(i) F is strictly increasing;

(ii) for each sequence $\{x_n\}_{n\in\mathbb{N}}$ of positive numbers,

$$\lim_{n\to\infty} x_n = 0 \quad \text{if and only if} \quad \lim_{n\to\infty} F(x_n) = -\infty;$$

(iii) $\liminf_{s\to\alpha^+} \phi(s) > 0$ for all $\alpha > 0$;

(iv) there exists $k \in]0, 1[$ such that

$$\lim_{x\to 0^+} x^k F(x) = 0.$$

By replacing the condition (iv) in Definition 2.11, we introduce new class of F-ϕ-contraction.

Definition 2.12. Let $\Im$ be the family of all functions $F\colon\ \mathbb{R}^+ \to \mathbb{R}$ and Φ be the family of all functions $\phi :]0, +\infty[\to]0, +\infty[$ satisfying the following.

(i) F is strictly increasing;

(ii) for each sequence $\{x_n\}_{n\in\mathbb{N}}$ of positive numbers,

$$\lim_{n\to\infty} x_n = 0 \quad \text{if and only if} \quad \lim_{n\to\infty} F(x_n) = -\infty;$$

(iii) $\liminf_{s\to\alpha^+} \phi(s) > 0$ for all $\alpha > 0$;

(iv) F is continuous.

Definition 2.13 ([19])**.** Let (X, d) be a metric space. A mapping $T : X \to X$ is called a (ϕ, F)-*contraction* on (X, d), if there exist $F \in \mathbb{F}$ and $\phi \in \Phi$ such that

$$F(d(Tx, Ty)) + \phi\,(d(x, y)) \leq F\,(d(x, y))$$

for all $x, y \in X$ with $Tx \neq Ty$.

Theorem 2.14 ([19])**.** *Let (X, d) be a complete metric space and $T : X \to X$ be an F-ϕ-contraction. Then T has a unique fixed point.*

3. Main Results

In this paper, using the idea introduced by Wardowski, we present the concept of F-ϕ-contraction in generalized asymmetric metric spaces and we prove some fixed point results in such spaces.

Definition 3.1. Let (X, d) be a generalized asymmetric metric space and $T : X \to X$ be a mapping.

(1) T is said to be a (ϕ, F)-*contraction of type* $(\mathbb{F})$ if there exist $F \in \mathbb{F}$ and $\phi \in \Phi$ such that for all $x, y \in X$ with $\max\{d\,(Tx, Ty)\,, d\,(Ty, Tx)\} > 0$, we have

$$F\left[d\left(\frac{d\,(Tx, Ty) + d\,(Ty, Tx)}{2}\right)\right] + \phi\left[d\left(\frac{d\,(x, y) + d\,(y, x)}{2}\right)\right]$$
$$\leq F\left[d\left(\frac{d\,(x, y) + d\,(y, x)}{2}\right)\right].$$

(2) T is said to be a (ϕ, F)-*contraction of type* $(\Im)$ if there exist $F \in \Im$ and $\phi \in \Phi$ such that for all $x, y \in X$ with $\max\{d(Tx,Ty), d(Ty,Tx)\} > 0$, we have

$$F\left[d\left(\frac{d(Tx,Ty)+d(Ty,Tx)}{2}\right)\right]+\phi\left[d\left(\frac{d(x,y)+d(y,x)}{2}\right)\right] \leq F[M(x,y)],$$

where

$$M(x,y) = \max\left\{d\left(\frac{d(x,y)+d(y,x)}{2}\right), d\left(\frac{d(x,Tx)+d(Tx,x)}{2}\right), d\left(\frac{d(y,Ty)+d(Ty,y)}{2}\right)\right\}.$$

(3) T is said to be a (ϕ, F)-Kannan-type $(\Im)$ contraction if there exist $F \in \Im$ and $\phi \in \Phi$ such that for all $x, y \in X$ with $\max\{d(Tx,Ty), d(Ty,Tx)\} > 0$, we have

$$F\left[d\left(\frac{d(Tx,Ty)+d(Ty,Tx)}{2}\right)\right]+\phi\left[d\left(\frac{d(x,y)+d(y,x)}{2}\right)\right] \leq F\left(\frac{d(x,Tx)+d(Tx,x)+d(y,Ty)+d(Ty,y)}{4}\right).$$

(4) T is said to be a (ϕ, F)-Reich-type $(\Im)$ contraction if there exist $F \in \mathbb{F}$ and $\phi \in \Phi$ such that for all $x, y \in X$ with $\max\{d(Tx,Ty), d(Ty,Tx)\} > 0$, we have

$$F\left[d\left(\frac{d(Tx,Ty)+d(Ty,Tx)}{2}\right)\right]+\phi\left[d\left(\frac{d(x,y)+d(y,x)}{2}\right)\right] \leq F\left(\frac{d(x,y)+d(y,x)+d(x,Tx)+d(Tx,x)+d(y,Ty)+d(Ty,y)}{6}\right).$$

Theorem 3.2. *Let* (X, d) *be a generalized asymmetric metric space and* $T : X \to X$ *be a mapping. Suppose that there exist* $F \in \mathbb{F}$ *and* $\phi \in \Phi$ *such that for all* $x, y \in X$ *with* $\max\{d(Tx,Ty), d(Ty,Tx)\} > 0$*, we have*

$$F\left[\frac{d(Tx,Ty)+d(Ty,Tx)}{2}\right]+\phi\left[d\left(\frac{d(x,y)+d(y,x)}{2}\right)\right] \leq F\left[\frac{d(x,y)+d(y,x)}{2}\right]. \tag{3.1}$$

Then T *has a unique fixed point.*

Proof. Let $x_0 \in X$ be fixed and define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n = T^{n+1}x_0$ for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that $d(x_{n_0}, x_{n_0+1}) = 0$ or $d(x_{n_0+1,}x_{n_0}) = 0$, then the proof is finished.

Now, suppose that $d(x_n, x_{n+1}) > 0$ and $d(x_{n+1,}x_n) > 0$ for all $n \in \mathbb{N}$. Then we have

$$max\{d(x_n, x_{n+1}), d(x_{n+1,}x_n)\} > 0.$$

Letting $x = x_{n-1}$ and $y = x_n$ in (3.1) for all $n \in \mathbb{N}$, we have

$$F\left[\frac{d(x_n, x_{n+1}) + d(x_{n+1,}x_n)}{2}\right]$$
$$\leq F\left(\frac{d(x_{n-1}, x_n) + d(x_n, x_{n-1})}{2}\right) - \phi\left(\frac{d(x_{n-1}, x_n) + d(x_n, x_{n-1})}{2}\right), \forall n \in \mathbb{N}.$$

Now, we set $D(x_n, x_m) = d(x_n, x_m) + d(x_m, x_n)$. Then

$$F\left(\frac{D(x_n, x_{n+1})}{2}\right) \leq F\left(\frac{D(x_{n-1}, x_n)}{2}\right) - \phi\left[\frac{D(x_{n-1}, x_n)}{2}\right].$$

Repeating this step, we conclude that

$$\begin{aligned} F\left(\frac{D(x_n, x_{n+1})}{2}\right) &\leq F\left(\frac{D(x_{n-1}, x_n)}{2}\right) - \phi\left[\frac{D(x_{n-1}, x_n)}{2}\right] \\ &\leq F\left(\frac{D(x_{n-2}, x_{n-1})}{2}\right) - \phi\left[\frac{D(x_{n-1}, x_n)}{2}\right] - \phi\left[\frac{D(x_{n-2}, x_{n-1})}{2}\right] \\ &\leq ... \leq F\left(\frac{D(x_0, x_1)}{2}\right) - \sum_{i=0}^{n} \phi\left[\frac{D(x_i, x_{i+1})}{2}\right]. \end{aligned}$$

Since F is increasing, we get

$$D(x_n, x_{n+1}) < D(x_{n-1}, x_n). \tag{3.2}$$

Since $\liminf_{s\to\alpha^+} \phi(s) > 0$, we have $\liminf_{n\to\infty} \phi(D(x_{n-1}, x_n)) > 0$. From the definition of the limit, there exist $n_0 \in \mathbb{N}$ and $A > 0$ such that for all $n \geq n_0$, $\phi(D(x_{n-1}, x_n)) > A$. Thus

$$\begin{aligned} F(D(x_n, x_{n+1})) &\leq F(D(x_0, x_1)) - \sum_{i=0}^{n_0-1} \phi(D(x_i, x_{i+1})) - \sum_{i=n_0-1}^{n} \phi(D(x_i, x_{i+1})) \\ &\leq F(D(x_0, x_1)) - \sum_{i=n_0-1}^{n} A \\ &= F(D(x_0, x_1)) - (n - n_0)A \end{aligned}$$

for all $n \geq n_0$. Taking the limit as $n \to \infty$ in the above inequality, we get

$$\lim_{n\to\infty} F(D(x_n, x_{n+1})) \leq \lim_{n\to\infty} [F(D(x_0, x_1)) - (n - n_0)A], \tag{3.3}$$

that is, $\lim_{n\to\infty} F(D(x_n, x_{n+1})) = -\infty$. From the condition (ii) of Definition 2.11, we conclude that

$$\lim_{n\to\infty} D(x_n, x_{n+1}) = 0. \tag{3.4}$$

Next, we shall prove that

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0 \text{ and } \lim_{n\to\infty} d(x_{n+2}, x_n) = 0.$$

Assume that $x_n \neq x_m$ for all $n, m \in$ with $\mathbb{N},\ n \neq m$. Indeed, suppose that $x_n = x_m$ for some $n = m + k$ with $k > 0$. Then we have $x_{n+1} = Tx_n = Tx_m = x_{m+1}$.

So, from the assumption of the theorem, we get

$$\begin{aligned} F\left(\frac{D(x_m, x_{m+1})}{2}\right) &= F\left(\frac{D(x_n, x_{n+1})}{2}\right) \\ &\leq F\left(\frac{D(x_{n-1}, x_n)}{2}\right) - \phi\left(\frac{D(x_{n-1}, x_n)}{2}\right) < F\left(\frac{D(x_{n-1}, x_n)}{2}\right). \end{aligned}$$

By (3.2), we have

$$D(x_m, x_{m+1}) = D(x_n, x_{n+1}) < D(x_{n-1}, x_n).$$

Continuing this process, we can obtain that

$$D(x_m, x_{m+1}) < D(x_m, x_{m+1}).$$

This is a contradiction. Therefore,

$$\max\{d(x_m, x_n), d(x_n, x_m)\} > 0$$

for all $n, m \in \mathbb{N}$ with $n \neq m$.

Letting $x = x_{n-1}$ and $y = x_{n+1}$ in (3.1) for all $n \in \mathbb{N}$, we have

$$F\left(\frac{D(x_n, x_{n+2})}{2}\right) \leq \left(F\left(\frac{D(x_{n-1}, x_{n+1})}{2}\right)\right) - \phi\left[\frac{D(x_{n-1}, x_{n+1})}{2}\right].$$

Repeating this step, we conclude that

$$\begin{aligned} F\left(\frac{D(x_n, x_{n+2})}{2}\right) &\leq \left(F\left(\frac{D(x_{n-1}, x_{n+1})}{2}\right)\right) - \phi\left[\frac{D(x_{n-1}, x_{n+1})}{2}\right] \\ &\leq F\left(\frac{D(x_{n-2}, x_n)}{2}\right) - \phi\left[\frac{D(x_{n-1}, x_{n+1})}{2}\right] - \phi\left[\frac{D(x_{n-2}, x_n)}{2}\right] \\ &\leq \dots \leq F\left(\frac{D(x_0, x_2)}{2}\right) - \sum_{i=0}^{n} \phi\left[\frac{D(x_i, x_{i+2})}{2}\right]. \end{aligned}$$

Since $\liminf_{s\to\alpha^+}\phi(s) > 0$, we have $\liminf_{n\to\infty}\phi(D\left(x_{n-1}, x_{n+1}\right)) > 0$. From the definition of the limit, there exist $n_1 \in \mathbb{N}$ and $B > 0$ such that for all $n \geq n_0$, $\phi(D\left(x_{n-1}, x_n\right)) > B$. Thus

$$\begin{aligned} F\left(D\left(x_n, x_{n+2}\right)\right) &\leq F\left(D\left(x_0, x_2\right)\right) - \sum_{i=0}^{n_1-1}\phi(D\left(x_i, x_{i+2}\right)) - \sum_{i=n_1-1}^{n}\phi(D\left(x_i, x_{i+2}\right)) \\ &\leq F\left(D\left(x_0, x_2\right)\right) - \sum_{i=n_1-1}^{n} B \\ &= F\left(D\left(x_0, x_2\right)\right) - (n-n_1)B \end{aligned}$$

for all $n \geq n_1$. Taking the limit as $n \to \infty$ in the above inequality, we get

$$\lim_{n\to\infty} F\left(D\left(x_n, x_{n+2}\right)\right) \leq \lim_{n\to\infty}\left[F\left(D\left(x_0, x_2\right)\right) - (n-n_1)B\right],$$

that is, $\lim_{n\to\infty} F\left(D\left(x_n, x_{n+2}\right)\right) = -\infty$. From the condition (ii) of Definition 2.11, we conclude that

$$\lim_{n\to\infty} D\left(x_{n,}x_{n+2}\right) = 0. \tag{3.5}$$

Next, we shall prove that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, i.e., $\lim_{n,m\to\infty} d\left(x_{n,}x_m\right) = 0$ for all $n, m \in \mathbb{N}$. Now, from (iv) of Definition 2.11, there exists $k \in \left]0,1\right[$ such that

$$\lim_{n\to\infty}\left[D\left(x_n, x_{n+1}\right)\right]^k F\left(D\left(x_n, x_{n+1}\right)\right) = 0.$$

Since

$$F\left[D\left(x_n, x_{n+1}\right)\right] \leq F\left[D\left(x_0, x_1\right)\right] - (n-n_0)A,$$

we have

$$\begin{aligned} &\left[D\left(x_n, x_{n+1}\right)\right]^k F\left[D\left(x_n, x_{n+1}\right)\right] \\ &\quad\leq \left[D\left(x_n, x_{n+1}\right)\right]^k F\left[D\left(x_0, x_1\right)\right] - \left[(n-n_0)A\right]\left[D\left(x_n, x_{n+1}\right)\right]^k \end{aligned}$$

Therefore,

$$\begin{aligned} \left[D\left(x_n, x_{n+1}\right)\right]^k F\left[D\left(x_n, x_{n+1}\right)\right] - \left[D\left(x_n, x_{n+1}\right)\right]^k F\left[D\left(x_0, x_1\right)\right] &\\ \leq -\left[(n-n_0)A\right]\left[D\left(x_n, x_{n+1}\right)\right]^k &\\ \leq 0. & \end{aligned}$$

Taking the limit $n \to \infty$ in the above inequality, we conclude that

$$\lim_{n\to\infty} D\left(x_{n,}x_{n+1}\right)^k (n-n_0)A = 0.$$

Then there exists $h \in \mathbb{N}$ such that

$$D\left(x_{n,}x_{n+1}\right) \leq \frac{1}{\left[(n-n_0)A\right]^k} \; for\ all\ n \geq h. \tag{3.6}$$

Now, from (iv) of Definition 2.11, there exists $k \in]0,1[$ such that

$$\lim_{n\to\infty}\left[D\left(x_n,x_{n+2}\right)\right]^k F\left(D\left(x_n,x_{n+2}\right)\right)=0.$$

Since

$$F\left[D\left(x_n,x_{n+2}\right)\right] \leq F\left[D\left(x_0,x_2\right)\right]-(n-n_1)B,$$

we have

$$\begin{aligned}&\left[D\left(x_n,x_{n+2}\right)\right]^k F\left[D\left(x_n,x_{n+2}\right)\right]\\&\quad\leq \left[D\left(x_n,x_{n+2}\right)\right]^k F\left[D\left(x_0,x_2\right)\right]-\left[(n-n_1)B\right]\left[D\left(x_n,x_{n+2}\right)\right]^k.\end{aligned}$$

Therefore,

$$\begin{aligned}\left[D\left(x_n,x_{n+2}\right)\right]^k F\left[D\left(x_n,x_{n+2}\right)\right]-\left[D\left(x_n,x_{n+2}\right)\right]^k F\left[D\left(x_0,x_2\right)\right]&\\ \leq -\left[(n-n_1)B\right]\left[D\left(x_n,x_{n+2}\right)\right]^k&\\ \leq 0.&\end{aligned}$$

Taking the limit $n\to\infty$ in the above inequality, we conclude that

$$\lim_{n\to\infty} D\left(x_{n,}x_{n+2}\right)^k(n-n_1)B=0.$$

Then there exists $l \in \mathbb{N}$ such that

$$D\left(x_{n,}x_{n+2}\right) \leq \frac{1}{\left[(n-n_1)B\right]^k}, \quad \forall\, n \geq l. \tag{3.7}$$

Next, we show that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, i.e.,

$$\lim_{n\to\infty} D\left(x_n,x_{n+r}\right)=0, \quad \forall r \in \mathbb{N}^*.$$

The cases $r=1$ and $r=2$, are proved, respectively, by (3.4) and (3.5).

Now, we take $r \geq 3$. It is sufficient to examine two cases:

Case I: Suppose that $r=2m+1$, where $m \geq 1$.

By using the quadrilateral inequality and (3.6), we have

$$\begin{aligned}
D(x_n, x_{n+r}) &= D(x_n, x_{n+2m+1}) \\
&\leq D(x_n, x_{n+1}) + D(x_{n+1}, x_{n+2}) + D(x_{n+2}, x_{n+2m+1}) \\
&\leq D(x_n, x_{n+1}) + D(x_{n+1}, x_{n+2}) + ... + D(x_{n+2m}, x_{n+2m+1}) \\
&\leq \frac{1}{[(n-n_0)A]^k} + \frac{1}{[(n+1-n_0)A]^k} + ... + \frac{1}{[(n-n+2m-n_0)A]^k} \\
&= \sum_{i=n}^{i=2m+n} \frac{1}{[(i-n_0)A]^k} \\
&\leq \sum_{i=n}^{i=\infty} \frac{1}{[(i-n_0)A]^k}.
\end{aligned}$$

Case II: Suppose that $r = 2m$, where $m \geq 1$.
Let $n_2 = \max\{n_0, n_1\}$ and $C = \max\{A, B\}$.
By the quadrilateral inequality and (3.6) and (3.7), we have

$$\begin{aligned}
D(x_n, x_{n+r}) &= D(x_n, x_{n+2m}) \\
&\leq D(x_n, x_{n+2}) + D(x_{n+2}, x_{n+3}) + D(x_{n+3}, x_{n+2m}) \\
&\leq D(x_n, x_{n+2}) + D(x_{n+2}, x_{n+3}) + ... + D(x_{n+2m-1}, x_{n+2m}) \\
&\leq \frac{1}{[(n-n_1)B]^k} + \frac{1}{[(n+2-n_0)A]^k} + \frac{1}{[(n+3-n_0)A]^k} \\
&\quad + ... + \frac{1}{[(n-n+2m-1-n_0)A]^k} \\
&= \frac{1}{[(n-n_1)B]^k} + \sum_{i=n+2}^{i=n+2m-1} \frac{1}{[(i-n_0)A]^k} \\
&\leq \sum_{i=n+1}^{i=\infty} \frac{1}{[(i-n_2)C]^k}.
\end{aligned}$$

From the convergence of the series $\sum_i \frac{1}{[(n-n_0)A]^k}$ and $\sum_i \frac{1}{[(n-n_2)C]^k}$, since $0 < k < 1$,

$$\lim_{n\to\infty} D(x_n, x_{n+r}) = 0, \ i.e, \ \lim_{n\to\infty} d(x_n, x_{n+r}) = 0 \ and \ \lim_{n\to\infty} d(x_{n+r}, x_n) = 0.$$

Hence $\{x_n\}$ is a forward and backward Cauchy sequence in X. By completeness of (X, d), there exist $z, u \in X$ such that

$$\lim_{x\to\infty} d(x_n, z) = \lim_{x\to\infty} d(u, x_n) = 0.$$

So, from Lemma 2.6, we get $z = u$.

Now, we show that $d(Tz, z) = 0$ or $d(z, Tz) = 0$. Arguing by contradiction, we assume that

$$d(Tz, z) > 0 \quad and \quad d(z, Tz) > 0.$$

Therefore,

$$\max\{d(Tz, z), d(z, Tz)\} > 0.$$

Now, by the quadrilateral inequality, we get

$$d(Tx_n, Tz) \le d(Tx_n, x_n) + d(x_n, z) + d(z, Tz), \tag{3.8}$$

$$d(z, Tz) \le d(z, x_n) + d(x_n, Tx_n) + d(Tx_n, Tz). \tag{3.9}$$

By letting $n \to \infty$ in (3.8) and (3.9), we obtain

$$d(z, Tz) \le \lim_{n\to\infty} d(Tx_n, Tz) \le d(z, Tz).$$

Therefore,

$$\lim_{n\to\infty} d(Tx_n, Tz) = d(z, Tz) \tag{3.10}$$

On the other hand,

$$d(Tz, Tx_n) \le d(Tz, z) + d(z, x_n) + d(x_n, Tx_n) \tag{3.11}$$

and

$$d(Tx_n, Tz) \le d(Tx_n, x_n) + d(x_n, z) + d(z, Tz). \tag{3.12}$$

By letting $n \to \infty$ in (3.11) and (3.12), we obtain

$$d(Tz, z) \le \lim_{n\to\infty} d(Tz, Tx_n) \le d(Tz, z).$$

Therefore,

$$\lim_{n\to\infty} d(Tz, Tx_n) = d(Tz, z). \tag{3.13}$$

By (3.10), and from the definition of the limit, there exists $n_3 \in \mathbb{N}$ such that

$$d(Tx_n, Tz) > d(z, Tz) > 0, \ \forall n \ge n_3.$$

Similarly, by (3.13), there exists $n_4 \in \mathbb{N}$ such that

$$d(Tz, Tx_n) > d(Tz, z) > 0, \ \forall n \ge n_4.$$

Let $N = \max\{n_3, n_4\}$. Then we conclude

$$\max\{d(Tz, Tx_n), d(Tx_n, Tz)\} > 0, \ \forall n \ge N.$$

Applying (3.1) with $x = z$ and $y = x_n$, we have

$$F\left(\frac{D(Tz, Tx_n)}{2}\right) \leq F\left(\frac{D(z, x_n)}{2}\right) - \phi\left(\frac{D(z, x_n)}{2}\right) < F\left(\frac{D(z, x_n)}{2}\right), \ \forall n \geq N.$$

Since F is increasing, we get

$$D(Tz, Tx_n) < D(z, x_n). \tag{3.14}$$

By letting $n \to \infty$ in (3.14) and using (3.10) and (3.13), we obtain

$$\lim_{n\to\infty} D(Tz, Tx_n) = D(Tz, z) \leq \lim_{n\to\infty} D(z, x_n) = 0.$$

Thus $d(Tz, z) = 0$ and $d(z, Tz)$. Hence $Tz = z$.

Now, suppose that $z, u \in X$ are two fixed points of T such that $u \neq z$. Then we have

$$d(Tz, Tu) = d(z, u) > 0$$

and

$$d(Tu, Tz) = d(u, z) > 0.$$

Therefore

$$\max\{d(Tu, Tz), d(Tz, Tu)\} > 0.$$

Applying (3.1) with $x = z$ and $y = u$, we have

$$F\left(\frac{D(Tz, Tu)}{2}\right) = F\left(\frac{D(z, u)}{2}\right) \leq F\left(\frac{D(z, u)}{2}\right) - \phi\left(\frac{D(z, u)}{2}\right) < F\left(\frac{D(z, u)}{2}\right),$$

which is a contradiction. Therefore $u = z$. □

Corollary 3.3. *Let $d(X, d)$ be a complete generalized asymmetric metric space and T be a self mapping on X. If for all $x, y \in X$ we have*

$$\begin{aligned}\max\{d(Tx, Ty), d(Ty, Tx)\} > 0 \Rightarrow & d(Tx, Ty) + d(Ty, Tx) \\ & \leq e^{\frac{-1}{1+d(x,y)+d(y,x)}} [d(x, y) + d(y, x)],\end{aligned}$$

then T has a unique fixed point.

Proof. Since $\max\{d(Tx, Ty), d(Ty, Tx)\} > 0$, we can take natural logarithm in both sides to get

$$\begin{aligned}ln[d(Tx, Ty) + d(Ty, Tx)] &\leq ln\left[e^{\frac{-1}{1+d(x,y)+d(y,x)}} (d(x, y) + d(y, x))\right] \\ &= \frac{-1}{1 + d(x, y) + d(y, x)} + ln[d(x, y) + d(y, x)].\end{aligned}$$

Hence

$$F[d(Tx, Ty) + d(Ty, Tx)] + \phi(d(x, y) + d(y, x)) \leq F(d(x, y) + d(y, x))$$

with $F(t) = ln(t)$ and $\phi(t) = \frac{1}{1+t}$. Therefore, as in the proof of Theorem 3.2, T has a unique fixed point $z \in X$. □

Corollary 3.4. *Let (X, d) be a complete generalized asymmetric metric space and $T : X \to X$ be a given mapping. Suppose that there exist $F \in \mathbb{F}$ and $\tau \in]1, +\infty[$ such that for all $x, y \in X$ with* $\max\{d(Tx, Ty), d(Ty,Tx)\} > 0$,

$$F\left[\frac{d(Tx, Ty) + d(Ty, Tx)}{2}\right] + \tau \leq \left[F\left(\frac{d(x, y) + d(y, x)}{2}\right)\right].$$

Then T has a unique fixed point.

Example 3.5. Let $X = A \cup B$, where $A = \{0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}\}$ and $B = [1, 2]$.
Define $d : X \times X \to [0, +\infty[$ as follows:

$$\begin{cases} d(x, y) = d(y, x), \quad \forall x, y \in B; \\ d(x, y) = 0 \Leftrightarrow y = x, \quad \forall x, y \in X \end{cases}$$

and

$$\begin{cases} d\left(\frac{1}{3}, \frac{1}{4}\right) = d\left(0, \frac{1}{2}\right) = 0.3 \\ d\left(\frac{1}{3}, 0\right) = d\left(\frac{1}{4}, \frac{1}{2}\right) = 0.2 \\ d\left(0, \frac{1}{3}\right) = d\left(\frac{1}{2}, \frac{1}{4}\right) = 0.35 \\ d\left(\frac{1}{3}, \frac{1}{2}\right) = d\left(\frac{1}{3}, \frac{1}{2}\right) = 0.6 \\ d(x, y) = |x - y| \ otherwise. \end{cases}$$

Then (X, d) is a generalized asymmetric metric space. However, we have the following:

1) (X, d) is not a metric space, since $d\left(\frac{1}{3}, \frac{1}{2}\right) = 0.6 > 0.5 = d\left(\frac{1}{3}, \frac{1}{4}\right) + d\left(\frac{1}{4}, \frac{1}{2}\right)$.
2) (X, d) is not a generalized metric space, since $d\left(\frac{1}{2}, \frac{1}{4}\right) = 0.35 \neq d\left(\frac{1}{4}, \frac{1}{2}\right) = 0.2$.

Define a mapping $T : X \to X$ by

$$T(x) = \begin{cases} x^{\frac{1}{4}} \ if \ x \in [1, 2] \\ 1 \ if \ x \in A. \end{cases}$$

Evidently, $T(x) \in X$. Let $F(t) = ln(t) + t$, $\phi(t) = \frac{1}{1+t}$. It is obvious that $F \in \mathbb{F}$ and $\phi \in \Phi$.

Consider the following possibilities:
Case 1: $x, y \in [1, 2]$ with $x \neq y$. Then

$$T(x) = x^{\frac{1}{4}},\ T(y) = y^{\frac{1}{4}},\ D\left(Tx, Ty\right) = 2\left(x^{\frac{1}{4}} - y^{\frac{1}{4}}\right),\ D(x, y) = 2(x - y).$$

On the other hand,

$$F\left[\frac{D\left(Tx, Ty\right)}{2}\right] = ln(x^{\frac{1}{4}} - y^{\frac{1}{4}}) + (x^{\frac{1}{4}} - y^{\frac{1}{4}}),$$

$$F\left[\frac{D\left(x, y\right)}{2}\right] = ln(x - y) + (x - y)$$

and

$$\phi\left[d\left(x, y\right)\right] = \frac{1}{[1 + (x - y)]}.$$

We have

$$\begin{aligned}
&F\left[\frac{D\left(Tx, Ty\right)}{2}\right] + \phi\left[\frac{D\left(x, y\right)}{2}\right] - F\left[\frac{D\left(x, y\right)}{2}\right] \\
&= ln(x^{\frac{1}{4}} - y^{\frac{1}{4}}) - ln(x - y) + (x^{\frac{1}{4}} - y^{\frac{1}{4}}) - (x - y) + \frac{1}{[1 + (x - y)]} \\
&= ln(x^{\frac{1}{4}} - y^{\frac{1}{4}}) - ln(x - y) + (x^{\frac{1}{4}} - y^{\frac{1}{4}}) - (x - y) + \frac{1}{[1 + (x - y)]} \\
&= -ln\left(x^{\frac{1}{4}} + y^{\frac{1}{4}}\right) - ln\left(x^{\frac{1}{2}} + y^{\frac{1}{2}}\right) \\
&\quad + (x^{\frac{1}{4}} - y^{\frac{1}{4}})\left[1 - \left(x^{\frac{1}{4}} + y^{\frac{1}{4}}\right)\left(x^{\frac{1}{2}} + y^{\frac{1}{2}}\right)\right] + \frac{1}{[1 + (x - y)]}.
\end{aligned}$$

Since $x, y \in [1, 2]$,

$$x^{\frac{1}{4}} + y^{\frac{1}{4}} \geq 1 \Rightarrow -ln\left(x^{\frac{1}{4}} + x^{\frac{1}{4}}\right) \leq 0,$$

$$(x^{\frac{1}{4}} - y^{\frac{1}{4}})\left(1 - \left(x^{\frac{1}{4}} + y^{\frac{1}{4}}\right)\left(x^{\frac{1}{2}} + y^{\frac{1}{2}}\right)\right) \leq 0$$

and

$$-ln\left(x^{\frac{1}{2}} + y^{\frac{1}{2}}\right) - ln\left(x^{\frac{1}{4}} + y^{\frac{1}{4}}\right) + \frac{1}{1 + \left(x^{\frac{1}{4}} - y^{\frac{1}{4}}\right)} \leq 0.$$

Thus, for all $x, y \in [1, 2]$ with $x \neq y$, we have

$$F\left[\frac{D\left(Tx, Ty\right)}{2}\right] + \phi\left[\frac{D\left(x, y\right)}{2}\right] \leq F\left[\frac{D\left(x, y\right)}{2}\right].$$

Case 2: $x \in [1, 2]$ and $y \in A$. Then

$$T(x) = x^{\frac{1}{4}},\ T(y) = 1,\ D\left(Tx, Ty\right) = 2\left(x^{\frac{1}{4}} - 1\right),\ d(x, y) = 2(x - y).$$

On the other hand,

$$F\left[\frac{D\left(Tx,Ty\right)}{2}\right]=ln(x^{\frac{1}{4}}-1)+(x^{\frac{1}{4}}-1),$$

$$F\left[\frac{D\left(x,y\right)}{2}\right]=ln((x-y))+(x-y)$$

and

$$\phi\left[\frac{D\left(x,y\right)}{2}\right]=\frac{1}{1+(x-y)}.$$

We have

$$\begin{aligned}&F\left[\frac{D\left(x,y\right)}{2}\right]-F\left[\frac{D\left(Tx,Ty\right)}{2}\right]-\phi\left[\frac{D\left(x,y\right)}{2}\right]\\&=(x-y)-(x^{\frac{1}{4}}-1)+ln(x-y)-ln(x^{\frac{1}{4}}-1)-\frac{1}{[1+(x-y)]}\\&=ln\left[\frac{x-y}{(x^{\frac{1}{4}}-1)}\right]+(x-y)-(x^{\frac{1}{4}}-1)-\frac{1}{[1+(x-y)]}.\end{aligned}$$

Since $x\in[1,2]$ and $y\in A$,

$$(x-y)\geq\left(x-\frac{1}{2}\right)=\left(x-1+\frac{1}{2}\right)>(x-1)\,.$$

Hence

$$(x-y)>(x-1)=\left(x^{\frac{1}{4}}-1\right)\left(x^{\frac{1}{4}}+1\right)\left(x^{\frac{1}{2}}+1\right),$$

$$(x-y)-\left(x^{\frac{1}{4}}-1\right)>\left(x^{\frac{1}{4}}-1\right)\left[\left(x^{\frac{1}{4}}+1\right)\left(x^{\frac{1}{2}}+1\right)-1\right]$$

and

$$\frac{(x-y)}{\left(x^{\frac{1}{4}}-1\right)}>\left(x^{\frac{1}{4}}+1\right)\left(x^{\frac{1}{2}}+1\right).$$

Then we have

$$ln\left[\frac{x-y}{(x^{\frac{1}{4}}-1)}\right]>ln\left[\left(x^{\frac{1}{4}}+1\right)\left(x^{\frac{1}{2}}+1\right)\right]=ln\left(x^{\frac{1}{4}}+1\right)+ln\left(x^{\frac{1}{2}}+1\right).$$

Since $x\in[1.2]$,

$$ln\left[\left(x^{\frac{1}{4}}+1\right)\right]+ln\left[\left(x^{\frac{1}{2}}+1\right)\right]\geq\frac{1}{[1+(x-y)]}.$$

Hence, the condition (3.1) is satisfied. Therefore, T has a unique fixed point $z=1$.

Theorem 3.6. *Let (X, d) be a complete generalized asymmetric metric space and $T : X \to X$ be a mapping. Suppose that there exist $F \in \Im$ and $\phi \in \Phi$ such that for all $x, y \in X$ with $\max\{d(Tx, Ty), d(Ty, Tx)\} > 0$, we have*

$$F\left[\frac{d(Tx,Ty)+d(Ty,Tx)}{2}\right]+\phi\left(\frac{d(x,y)+d(y,x)}{2}\right)\leq F\left[M(x,y)\right], \tag{3.15}$$

where

$$M(x,y)=\max\left\{\frac{d(x,y)+d(y,x)}{2},\frac{d(x,Tx)+d(Tx,x)}{2},\frac{d(y,Ty)+d(Ty,y)}{2}\right\}.$$

Then T has a unique fixed point.

Proof. Let $x_0 \in X$ be fixed and define a sequence $\{x_n\}$ by

$$x_{n+1} = Tx_n = T^{n+1}x_0, \quad \forall n \in \mathbb{N}.$$

If there exists $n_0 \in \mathbb{N}$ such that $d(x_{n_0}, x_{n_0+1}) = 0$ or $d(x_{n_0+1}, x_{n_0}) = 0$, then the proof is finished.

We can suppose that $d(x_n, x_{n+1}) > 0$ and $d(x_{n+1}, x_n) > 0$ for all $n \in \mathbb{N}$. Then we have

$$\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_n)\} > 0.$$

Letting $x = x_{n-1}$ and $y = x_n$ in (3.15) for all $n \in \mathbb{N}$, we have

(3.16)

$$F\left[\frac{d(x_n,x_{n+1})+d(x_{n+1},x_n)}{2}\right]+\phi\left[\frac{d(x_{n-1},x_n)+d(x_{n+1},x_n)}{2}\right]\leq F\left[M(x_{n-1},x_n)\right],$$

where

$$\begin{aligned} M(x_{n-1},x_n) &= \left\{\frac{d(x_{n-1},x_n)+d(x_n,x_{n-1})}{2},\frac{d(x_{n-1},x_n)+d(x_n,x_{n-1})}{2},\right. \\ &\qquad \left.\frac{d(x_n,x_{n+1})+d(x_{n+1},x_n)}{2}\right\} \\ &= \left\{\frac{d(x_{n-1},x_n)+d(x_n,x_{n-1})}{2},\frac{d(x_n,x_{n+1})+d(x_{n+1},x_n)}{2}\right\}. \end{aligned}$$

Now, we set $D(x_n, x_m) = d(x_n, x_m) + d(x_m, x_n)$. Then

$$M(x_{n-1},x_n)=\left\{\frac{D(x_{n-1},x_n)}{2},\frac{D(x_n,x_{n+1})}{2}\right\}.$$

Suppose that for some n, $M(x_{n-1}, x_n) = \left\{\frac{D(x_n,x_{n+1})}{2}\right\}$. Using the continuity of F and the property of ϕ, it follows from (3.16) that

$$\begin{aligned} F\left[\frac{D(x_n, x_{n+1})}{2}\right] &\leq F\left[\frac{D(x_n, x_{n+1})}{2}\right] - \phi\left[\frac{D(x_{n-1}, x_n)}{2}\right] \\ &< F\left[\frac{D(x_n, x_{n+1})}{2}\right]. \end{aligned}$$

This implies that

$$D(x_n, x_{n+1}) < D(x_n, x_{n+1}),$$

which is a contradiction. Hence

$$M(x_{n-1}, x_n) = \frac{D(x_{n-1}, x_n)}{2}.$$

Therefore,

$$F\left(\frac{D(x_n, x_{n+1})}{2}\right) < F\left(\frac{D(x_{n-1}, x_n)}{2}\right) - \phi\left(\frac{D(x_{n-1}, x_n)}{2}\right).$$

Since F is increasing,

$$D(x_n, x_{n+1}) < D(x_{n-1}, x_n). \tag{3.17}$$

Repeating this step, we conclude that

$$\begin{aligned} F\left(\frac{D(x_n, x_{n+1})}{2}\right) &\leq \left(F\left(\frac{D(x_{n-1}, x_n)}{2}\right)\right) - \phi\left[\frac{D(x_{n-1}, x_n)}{2}\right] \\ &\leq F\left(\frac{D(x_{n-2}, x_{n-1})}{2}\right) - \phi\left[\frac{D(x_{n-1}, x_n)}{2}\right] - \phi\left[\frac{D(x_{n-2}, x_{n-1})}{2}\right] \\ &\leq \ldots \leq F\left(\frac{D(x_0, x_1)}{2}\right) - \sum_{i=0}^{n} \phi\left[\frac{D(x_i, x_{i+1})}{2}\right]. \end{aligned}$$

Since $\liminf_{\alpha \to s^+} \phi(s) > 0$, we have $\liminf_{n\to\infty} \phi(D(x_{n-1}, x_n)) > 0$. From the definition of the limit, there exist $n_0 \in \mathbb{N}$ and $A > 0$ such that for all $n \geq n_0$, $\phi(D(x_{n-1}, x_n)) > A$. Thus

$$\begin{aligned} F(D(x_n, x_{n+1})) &\leq F(D(x_0, x_1)) - \sum_{i=0}^{n_0-1} \phi(D(x_i, x_{i+1})) - \sum_{i=n_0-1}^{n} \phi(D(x_i, x_{i+1})) \\ &\leq F(D(x_0, x_1)) - \sum_{i=n_0-1}^{n} A \\ &= F(D(x_0, x_1)) - (n - n_0)A \end{aligned}$$

for all $n \geq n_0$. Taking the limit as $n \to \infty$ in the above inequality, we get

$$\lim_{n\to\infty} F(D(x_n, x_{n+1})) \leq \lim_{n\to\infty} \left[F(D(x_0, x_1)) - (n - n_0)A\right],$$

that is, $\lim_{n\to\infty} F\left(D\left(x_n, x_{n+1}\right)\right) = -\infty$. From the condition (ii) of Definition 2.12, we conclude that

$$\lim_{n\to\infty} D\left(x_n,x_{n+1}\right) = 0. \tag{3.18}$$

Next, we shall prove that

$$\lim_{n\to\infty} d\left(x_n, x_{n+2}\right) = 0 \text{ and } \lim_{n\to\infty} d\left(x_{n+2}, x_n\right) = 0.$$

We assume that $x_n \neq x_m$ for all $n, m \in \mathbb{N}$ with $n \neq m$. Indeed, suppose that $x_n = x_m$ for some $n = m + k$ with $k > 0$. Then we have $x_{n+1} = Tx_n = Tx_m = x_{m+1}$.
By (3.17), we have

$$D\left(x_n, x_{m+1}\right) < D\left(x_{n-1}, x_n\right).$$

Continuing this process, we can have that

$$D\left(x_m, x_{n+1}\right) < D\left(x_m, x_{m+1}\right),$$

which is a contradiction. Therefore,

$$\max\left\{d\left(x_m, x_n\right), d\left(x_n, x_m\right)\right\} > 0, \quad \forall n,\ m \in \mathbb{N}, n \neq m.$$

Letting $x = x_n$ and $y = x_{n+2}$, we have

$$max\left\{d\left(x_n, x_{n+2}\right), d\left(x_{n+2,}x_n\right)\right\} > 0.$$

Applying (3.15) with $x = x_{n-1}$ and $y = x_{n+1}$, we have

$$F\left[\frac{D\left(x_n, x_{n+2}\right)}{2}\right] + \phi\left(\frac{D\left(x_{n-1}, x_{n+1}\right)}{2}\right) \leq F\left(M\left(x_{n-1}, x_{n+1}\right)\right),$$

where

$$\begin{aligned} M\left(x_{n-1}, x_{n+1}\right) &= \max\left\{\frac{D\left(x_{n-1}, x_n\right)}{2}, \frac{D\left(x_{n-1}, x_{n+1}\right)}{2}, \frac{D\left(x_{n+1}, x_{n+2}\right)}{2}\right\} \\ &= \max\left\{\frac{D\left(x_{n-1}, x_n\right)}{2}, \frac{D\left(x_{n-1}, x_{n+1}\right)}{2}\right\}. \end{aligned}$$

Therefore,

$$\begin{aligned} &F\left(\frac{D\left(x_n, x_{n+2}\right)}{2}\right) \\ &\quad \leq F\left(\max\left\{\frac{D\left(x_{n-1}, x_n\right)}{2}, \frac{D\left(x_{n-1}, x_{n+1}\right)}{2}\right\}\right) - \phi\left(\frac{D\left(x_{n-1}, x_{n+1}\right)}{2}\right). \end{aligned} \tag{3.19}$$

Take $a_n = D\left(x_n, x_{n+2}\right)$ and $b_n = D\left(x_n, x_{n+1}\right)$. Since F is increasing, we have

$$a_n < \max\left\{a_{n-1}, b_{n-1}\right\}.$$

Again by (3.17),

$$b_n \leq b_{n-1} \leq \max\left\{a_{n-1}, b_{n-1}\right\}.$$

Therefore,

$$\max\{a_n, b_n\} \leq \max\{a_{n-1}, b_{n-1}\}, \ \forall n \in \mathbb{N}.$$

Then the sequence $\max\{a_{n-1}, b_{n-1}\}_{n\in\mathbb{N}}$ is monotone non-increasing, and so it converges to some $\beta \geq 0$ such that

$$\lim_{n\to\infty} \max\{a_n, b_n\} = \beta.$$

By (3.18), for $\beta > 0$, we have

$$\lim_{n\to\infty} \sup a_n = \lim_{n\to\infty} \sup \max\{a_n, b_n\} = \lim_{n\to\infty} \max\{a_n, b_n\}.$$

Taking the $\limsup_n \to \infty$ in (3.19) and using the properties of F and ϕ, we obtain

$$\begin{aligned} F\left(\lim_{n\to\infty} \sup a_n\right) &\leq F\left(\lim_{n\to\infty} \sup \max\{a_{n-1}, b_{n-1}\}\right) - \lim_{n\to\infty} \sup \phi\left(\frac{D(x_{n-1}, x_{n+1})}{2}\right) \\ &\leq F\left(\lim_{n\to\infty} \sup \max\{a_{n-1}, b_{n-1}\}\right) - \lim_{n\to\infty} \inf \phi\left(\frac{D(x_{n-1}, x_{n+1})}{2}\right) \\ &< F\left(\lim_{n\to\infty} \sup \max\{a_{n-1}, b_{n-1}\}\right). \end{aligned}$$

Therefore,

$$F\left(\frac{\beta}{2}\right) < F\left(\frac{\beta}{2}\right).$$

This is a contradiction. Thus

$$\lim_{n\to\infty} D(x_{n+2}, x_n) = 0. \tag{3.20}$$

Next, we shall prove that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, i.e., $\lim_{n,m\to\infty} D(x_n, x_m) = 0$, for all $n, m \in \mathbb{N}$. Suppose to the contrary. Then there is an $\varepsilon > 0$ such that for an integer k there exist two sequences $\{n_{(k)}\}$ and $\{m_{(k)}\}$, $m_{(k)} > n_{(k)} > k$,
such that

$$D\left(x_{m_{(k)}}, x_{n_{(k)}}\right) \geq \varepsilon, \ D\left(x_{m_{(k)-1}}, x_{n_{(k)}}\right) < \varepsilon.$$

Now, using (3.18), (3.20) and the quadrilateral inequality, we find

$$\begin{aligned} \varepsilon \leq D\left(x_{m_{(k)}}, x_{n_{(k)}}\right) &\leq D\left(x_{m_{(k)}}, x_{m_{(k)+1}}\right) + D\left(x_{m_{(k)+1}}, x_{m_{(k)-1}}\right) + D\left(x_{m_{(k)-1}}, x_{n_{(k)}}\right) \\ &\leq D\left(x_{m_{(k)}}, x_{m_{(k)+1}}\right) + D\left(x_{m_{(k)+1}}, x_{m_{(k)-1}}\right) + \varepsilon. \end{aligned}$$

Then

$$\lim_{k\to\infty} D\left(x_{m_{(k)}}, x_{n_{(k)}}\right) = \varepsilon. \tag{3.21}$$

Now, by the quadrilateral inequality, we have

$$D\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) \leq D\left(x_{m_{(k)+1}}, x_{m_{(k)}}\right) + D\left(x_{m_{(k)}}, x_{n_{(k)}}\right) + D\left(x_{n_{(k)}}, x_{n_{(k)+1}}\right),$$
$$D\left(x_{m_{(k)}}, x_{n_{(k)}}\right) \leq D\left(x_{m_{(k)}}, x_{m_{(k)+1}}\right) + D\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) + D\left(x_{n_{(k)+1}}, x_{n_{(k)}}\right).$$

Letting $k \to \infty$ in the above inequalities, we obtain

$$\lim_{k\to\infty} D\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) = \varepsilon. \tag{3.22}$$

By (3.22) there exists $n_0 \in \mathbb{N}$ such that

$$D\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) = d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) + d\left(x_{n_{(k)+1}}, x_{m_{(k)+1}}\right) \geq \frac{\varepsilon}{2}, \quad \forall n \geq n_0.$$

Therefore,

$$\max\left\{d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right), d\left(x_{n_{(k)+1}}, x_{m_{(k)+1}}\right)\right\} \geq \frac{\varepsilon}{4}, \quad \forall n \geq n_0.$$

So

$$\max\left\{d\left(Tx_{m_{(k)}}, Tx_{n_{(k)}}\right), d\left(x_{n_{(k)}}, Tx_{m_{(k)}}\right)\right\} \geq \frac{\varepsilon}{4}, \quad \forall n \geq n_0.$$

Applying (3.15) with $x = x_{m_{(k)}}$ and $y = x_{n_{(k)}}$, we have

$$F\left(\frac{D\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right)}{2}\right) \leq F\left(M\left(x_{m_{(k)}}, x_{n_{(k)}}\right)\right) - \phi\left(\frac{D\left(x_{m_{(k)}}, x_{n_{(k)}}\right)}{2}\right), \tag{3.23}$$

where

$$M\left(x_{m_{(k)}}, x_{n_{(k)}}\right) = \max\left\{\frac{D\left(x_{m_{(k)}}, x_{n_{(k)}}\right)}{2}, \frac{D\left(x_{m_{(k)}}, x_{m_{(k)+1}}\right)}{2}, \frac{D\left(x_{n_{(k)}}, x_{n_{(k)+1}}\right)}{2}\right\}.$$

By (3.18) and (3.21), we have

$$\lim_{k\to\infty} M\left(x_{m_{(k)}}, x_{n_{(k)}}\right) = \frac{\varepsilon}{2}.$$

By letting $k \to \infty$ in (3.23) and using the continuity of F and using the property of ϕ, we obtain

$$\begin{aligned} F\left(\frac{\varepsilon}{2}\right) &\leq F\left(\frac{\varepsilon}{2}\right) - \lim_{k\to\infty} \sup \phi\left(\frac{D\left(x_{m_{(k)}}, x_{n_{(k)}}\right)}{2}\right) \\ &\leq F\left(\frac{\varepsilon}{2}\right) - \lim_{k\to\infty} \inf \phi\left(\frac{D\left(x_{m_{(k)}}, x_{n_{(k)}}\right)}{2}\right) \\ &< F\left(\frac{\varepsilon}{2}\right), \end{aligned}$$

which implies that

$$\varepsilon < \varepsilon.$$

This is s a contradiction. Thus

$$\lim_{n,m\to\infty} D\left(x_m, x_n\right) = 0.$$

Hence

$$\lim_{n,m\to\infty} d\left(x_m, x_n\right) = \lim_{n,m\to\infty} d\left(x_n, x_m\right) = 0.$$

So $\{x_n\}$ is a forward and backward Cauchy sequence in X. By completeness of (X, d), there exist $z, u \in X$ such that

$$\lim_{n\to\infty} d\left(x_n, z\right) \ = \ \lim_{n\to\infty} d\left(u, x_n\right) = 0.$$

So, from Lemma 2.6, we get $z = u$.

Now, we show that $d\left(Tz, z\right) = 0 = d\left(z, Tz\right)$. Arguing by contradiction, we assume that

$$d\left(Tz, z\right) > 0 \quad \text{and} \quad d\left(z, Tz\right) > 0.$$

Therefore,

$$\max\left\{d\left(Tz, z\right), d\left(z, Tz\right)\right\} > 0.$$

As in the proof of Theorem 3.2, we conclude that

$$\lim_{n\to\infty} d\left(Tz, Tx_n\right) = d\left(Tz, z\right) \tag{3.24}$$

and

$$\lim_{n\to\infty} d\left(Tx_n, Tz\right) = d\left(z, Tz\right). \tag{3.25}$$

By (3.24) and (3.25), there exists $q \in \mathbb{N}$ such that

$$\max\left\{d\left(Tz, Tx_n\right), d\left(Tx_n, Tz\right)\right\} > 0, \ \forall\, n \geq q.$$

Since T is an F-ϕ-contraction, we obtain

$$F\left(\frac{D(Tz,Tx_n)}{2}\right) \leq F\left[\theta\left(M(z,x_n)\right)\right] - \phi\left(\frac{D(z,x_n)}{2}\right),\ \forall n \geq q, \tag{3.26}$$

where

$$M(z,x_n) = \max\left\{\frac{D(z,x_n)}{2}\frac{D(z,Tz)}{2}, \frac{D(x_n,Tx_n)}{2}\right\}.$$

Thus

$$\lim_{n\to\infty} M(z,x_n) = \max\left\{\frac{D(z,x_n)}{2}\frac{D(z,Tz)}{2}, \frac{D(x_n,Tx_n)}{2}\right\} = \frac{D(z,Tz)}{2}. \tag{3.27}$$

By letting $n \to \infty$ in (3.26), using (3.24), (3.25), (3.27) and property of ϕ, we obtain

$$\begin{aligned} F\left(\frac{D(Tz,z)}{2}\right) &\leq F\left(\frac{D(Tz,z)}{2}\right) - \lim_{n\to\infty} \sup \phi\left(\frac{D(z,x_n)}{2}\right) \\ &\leq F\left(\frac{D(Tz,z)}{2}\right) - \lim_{n\to\infty} \inf \phi\left(\frac{D(z,x_n)}{2}\right) \\ &< F\left(\frac{D(Tz,z)}{2}\right). \end{aligned}$$

Therefore,

$$D(z,Tz) < D(z,Tz),$$

which is a contradiction. Thus $z = Tz$. So T has a fixed point.

Let $z, u \in \text{Fix}(T)$ with $z \neq u$. Then

$$d(Tz,Tu) = d(z,u) > 0$$

and

$$d(Tu,Tz) = d(u,z) > 0.$$

Therefore,

$$\max\left\{d(Tz,Tu), d(Tu,Tz)\right\} > 0.$$

From assumption of the theorem, we get

$$F\left(\frac{D(Tz,Tu)}{2}\right) = F\left(\frac{D(z,u)}{2}\right) \leq F(M(z,u)) - \phi(M(z,u)),$$

where

$$M(z,u) = \max\left\{\frac{D(z,u)}{2}, \frac{D(z,Tz)}{2}, \frac{D(u,Tu)}{2}\right\} = \frac{D(z,u)}{2}.$$

Therefore, we have

$$F\left(\frac{D(Tz,Tu)}{2}\right) = F\left(\frac{D(z,u)}{2}\right) \leq F(M(z,u)) - \phi(D(z,u)) < F\left(\frac{D(z,u)}{2}\right),$$

which implies that $D(z,u) < D(z,u)$. This is a contradiction. Therefore $u = z$. $\square$

It follows from Theorem 3.6 that we obtain fixed point theorems for F-ϕ-Reich-type contraction and F-ϕ-Kannan-type contraction.

Theorem 3.7. *Let (X, d) be a complete generalized asymmetric space and $T : X \to X$ be a θ-ϕ-Kannan-type contraction. Then T has a unique fixed point.*

Proof. Since T is a $(\phi, F--$Kannan-type contraction, there exist $F \in \Im$ and $\phi \in \Phi$ such that

$$\begin{aligned} F & \left[\frac{d(Tx,Ty)+d(Ty,Tx)}{2}\right]+\phi\left(\frac{d(x,y)+d(y,x)}{2}\right) \\ & = F\left[\frac{D(Tx,Ty)}{2}\right]+\phi\left(\frac{D(x,y)}{2}\right) \\ & \leq F\left(\frac{D(Tx,x)+D(Ty,y)}{4}\right) \\ & \leq F\left(\max\left\{\frac{D(x,Tx)}{2},\frac{D(y,Ty)}{2}\right\}\right) \\ & \leq F\left(\max\left\{\frac{D(x,y)}{2},\frac{D(Tx,x)}{2},\frac{D(y,Ty)}{2}\right\}\right). \end{aligned}$$

Therefore, T is a (ϕ, F)-contraction. As in the proof of Theorem 3.6, we conclude that T has a unique fixed point. □

Theorem 3.8. *Let (X, d) be a complete generalized asymmetric space and $T : X \to X$ be a (ϕ, F)-Reich-type contraction. Then T has a unique fixed point.*

Proof. Since T is a (ϕ, F)-Reich-type contraction, there exist $F \in \Im$ and $\phi \in \Phi$ such that

$$\begin{aligned} F & \left[\frac{d(Tx,Ty)+d(Ty,Tx)}{2}\right]+\phi\left(\frac{d(x,y)+d(y,x)}{2}\right) \\ & = F\left[\frac{D(Tx,Ty)}{2}\right]+\phi\left(\frac{D(x,y)}{2}\right) \\ & \leq F\left(\frac{D(x,y)+D(Tx,x)+D(Ty,y)}{6}\right) \\ & \leq F\left(\max\left\{\frac{D(x,y)}{2},\frac{D(Tx,x)}{2},\frac{D(y,Ty)}{2}\right\}\right). \end{aligned}$$

Therefore, T is a (ϕ, F)-contraction. As in the proof of Theorem 3.6, we conclude that T has a unique fixed point. □

Corollary 3.9. *Let (X, d) be a complete metric space and $T : X \to X$ be a Kannan type mapping, i.e., there exists $\alpha \in \left]0, \frac{1}{2}\right[$ such that for all $x, y \in X$ with*

$\max\{d(Tx, Ty), d(Ty, Tx)\} > 0$,

$$\frac{d(Tx, Ty) + d(Ty, Tx)}{2} \leq \alpha \left[\frac{d(Tx, x) + d(x, Tx) + d(Ty, y) + d(y, Ty)}{2}\right].$$

Then T has a unique fixed point.

Proof. Let $F(t) = ln(t)$ for all $t \in]0, +\infty[$, and $\phi(t) = ln(\frac{1}{\alpha})$. We prove that T is a (ϕ, F)-Kannan-type contraction. Indeed,

$$\begin{aligned} &F\left(\frac{d(Tx, Ty) + d(Ty, Tx)}{2}\right) \\ &= ln\left(\frac{d(Tx, Ty) + d(Ty, Tx)}{2}\right) \\ &\leq ln\left(\frac{d(Tx, x) + d(x, Tx) + d(Ty, y) + d(y, Ty)}{4}\right) + ln(\alpha). \end{aligned}$$

Thus

$$\begin{aligned} &ln\left(\frac{d(Tx, Ty) + d(Ty, Tx)}{2}\right) + ln\left(\frac{1}{\alpha}\right) \\ &\leq ln\left(\frac{d(Tx, x) + d(x, Tx) + d(Ty, y) + d(y, Ty)}{4}\right). \end{aligned}$$

Therefore, as in the proof of Theorem 3.7, T has a unique fixed point $x \in X$. □

Corollary 3.10. *Let (X, d) be a complete metric space and $T : X \to X$ be a Reich type mapping, i.e., there exists $\lambda \in \left]0, \frac{1}{3}\right[$ such that for all $x, y \in X$ with* $\max\{d(Tx, Ty), d(Ty, Tx)\} > 0$, *we have*

$$\begin{aligned} &\frac{d(Tx, Ty) + d(Ty, Tx)}{2} \\ &\leq \lambda \left[\frac{d(x, y) + (d(y, x) + d(Tx, x) + d(x, Tx) + d(Ty, y) + d(y, Ty)}{2}\right]. \end{aligned}$$

Then T has a unique fixed point.

Proof. Let $F(t) = ln(t)$ for all $t \in]0, +\infty[$, and $\phi(t) = ln(\frac{1}{\lambda})$. We prove that T is a (ϕ, F)- Kannan-type contraction. Indeed,

$$\begin{aligned} &F\left(\frac{d(Tx, Ty) + d(Ty, Tx)}{2}\right) = ln\left(\frac{d(Tx, Ty) + d(Ty, Tx)}{2}\right) \\ &\leq ln\left(\frac{d(x, y) + d(y, x)\, d(Tx, x) + d(x, Tx) + d(Ty, y) + d(y, Ty)}{6}\right) + ln(\lambda). \end{aligned}$$

Thus

$$ln\left(\frac{d\left(Tx,Ty\right)+d\left(Ty,Tx\right)}{2}\right)+ln\left(\frac{1}{\lambda}\right)$$
$$\leq ln\left(\frac{d\left(x,y\right)+d\left(y,x\right)d\left(Tx,x\right)+d\left(x,Tx\right)+d\left(Ty,y\right)+d\left(y,Ty\right)}{6}\right)+ln\left(\lambda\right).$$

Therefore, as in the proof of Theorem 3.8, T has a unique fixed point $x \in X$. □

Example 3.11. Let $X = A \cup B$, where $A = \{0, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\}$ and $B = \left[\frac{3}{4}, 2\right]$.
Define $d : X \times X \to [0, +\infty[$ as follows:

$$\begin{cases} d(x,y) = d(y,x), & \forall x,y \in B; \\ d(x,y) = 0 \Leftrightarrow y = x, & \forall x,y \in X \end{cases}$$

and

$$\begin{cases} d\left(\frac{1}{3},\frac{1}{4}\right) = d\left(\frac{1}{5},0\right) = 0.3 \\ d\left(\frac{1}{3},\frac{1}{5}\right) = d\left(\frac{1}{4},0\right) = 0.21 \\ d\left(\frac{1}{5},\frac{1}{3}\right) = d\left(0,\frac{1}{4}\right) = 0.34 \\ d\left(\frac{1}{3},0\right) = d\left(\frac{1}{3},0\right) = 0.6 \\ d\left(x,y\right) = |x-y| \ otherwise. \end{cases}$$

Then (X, d) is a generalized asymmetric metric space. However, we have the following:

1) (X, d) is not a metric space, since $d\left(\frac{1}{3}, 0\right) = 0.6 > 0.51 = d\left(\frac{1}{3}, \frac{1}{4}\right) + d\left(\frac{1}{4}, 0\right)$.
2) (X, d) is not a generalized metric space, since $d\left(0, \frac{1}{4}\right) = 0.34 \neq d\left(\frac{1}{4}, 0\right) = 0.21$.

Define a mapping $T : X \to X$ by

$$T(x) = \begin{cases} \sqrt{x} \ if \ x \in \left[\frac{3}{4}, 2\right] \\ 1 \ if \ x \in A. \end{cases}$$

Then $T(x) \in \left[\frac{3}{4}, 2\right]$. Let $F(t) = lnt \ for \ all \ t \in]0, +\infty[$, $\phi(t) = \frac{1}{2+t}$. It is obvious that $F \in \Im$ and $\phi \in \Phi$.
Consider the following possibilities:

Case 1: $x, y \in \left[\frac{3}{4}, 2\right]$ with $x \neq y$. Assume that $x > y$. Then

$$\begin{aligned} D(Tx, Ty) &= d(Tx, Ty) + d(Ty, Tx) \\ &= |\sqrt{x} - \sqrt{y}| + |\sqrt{y} - \sqrt{x}| \\ &= 2\left(\sqrt{x} - \sqrt{y}\right) \end{aligned}$$

and

$$\begin{aligned} D(x, y) &= d(x, y) + d(y, x) \\ &= |x - y| + |y - x| \\ &= 2\left(x - y\right). \end{aligned}$$

Therefore,

$$F\left(\frac{D(Tx, Ty)}{2}\right) = ln\left(\sqrt{x} - \sqrt{y}\right)$$

and

$$\phi\left(\frac{D(x, y)}{2}\right) = \left[\frac{1}{2 + (x - y)}\right].$$

On the other hand,

$$\begin{aligned} &F\left(\frac{D(Tx, Ty)}{2}\right) + \phi\left(\frac{D(x, y)}{2}\right) - F\left(\frac{D(x, y)}{2}\right) \\ &= ln\left(\sqrt{x} - \sqrt{y}\right) + \left[\frac{1}{2 + (x - y)}\right] - ln\left(x - y\right). \\ &= ln\left(\frac{\sqrt{x} - \sqrt{y}}{x - y}\right) + \left[\frac{1}{2 + (x - y)}\right] \\ &= ln\left(\frac{1}{\sqrt{x} + \sqrt{y}}\right) + \left[\frac{1}{2 + (x - y)}\right] \\ &= -ln\left(\sqrt{x} + \sqrt{y}\right) + \left[\frac{1}{2 + (x - y)}\right]. \end{aligned}$$

Since $x, y \in \left[\frac{3}{4}, 2\right]$, we have

$$-ln\left(\sqrt{x} + \sqrt{y}\right) \leq -ln(\sqrt{3})$$

and

$$\left[\frac{1}{2 + (x - y)}\right] \leq ln(\sqrt{3}).$$

Thus

$$F\left(\frac{D(Tx, Ty)}{2}\right) + \phi\left(\frac{D(x, y)}{2}\right) \leq F\left(\frac{D(x, y)}{2}\right) \leq F\left(M(x, y)\right).$$

Case 2: $x \in \left[\frac{3}{4}, 2\right], y \in A$ or $y \in \left[\frac{3}{4}, 2\right], x \in A$.
Then $T(x) = \sqrt{x}$, $T(y) = 1$ and so $d(Tx, Ty) = (|\sqrt{x} - 1|)$.
In this case, consider two possibilities:
$i)$ $x > 1$: Then $\sqrt{x} > 1$. Thus

$$D(Tx, Ty) = 2\left(\sqrt{x} - 1\right).$$

So we have

$$F\left(\frac{D(Tx, Ty)}{2}\right) = ln\left(\sqrt{x} - 1\right)$$

and

$$\begin{aligned} M(x, y) &= \max\left\{\frac{D(x, y)}{2}, \frac{D(x, Tx)}{2}, \frac{D(y, Ty)}{2}\right\} \\ &\geq \frac{D(x, y)}{2} \\ &\geq \frac{D\left(x, \frac{1}{3}\right)}{2} \\ &= x - \frac{1}{3} \\ &\geq x - 1. \end{aligned}$$

On the other hand,

$$\begin{aligned} &F\left(\frac{D(Tx, Ty)}{2}\right) + \phi\left(\frac{D(x, y)}{2}\right) - F\left(\frac{D(x, y)}{2}\right) \\ &= ln\left(\sqrt{x} - 1\right) + \left[\frac{1}{2 + (x - y)}\right] - ln\,(x - y) \\ &\leq ln\left(\sqrt{x} - 1\right) + \left[\frac{1}{2 + (x - y)}\right] - ln\,(x - 1) \\ &= ln\left(\frac{\sqrt{x} - 1}{x - 1}\right) + \left[\frac{1}{2 + (x - y)}\right] \\ &= ln\left(\frac{1}{\sqrt{x} + 1}\right) + \left[\frac{1}{2 + (x - y)}\right] \\ &= -ln\left(\sqrt{x} + 1\right) + \left[\frac{1}{2 + (x - y)}\right]. \end{aligned}$$

Since $x \in]1, 2]$, we have

$$-ln\left(\sqrt{x} + 1\right) \leq -ln\,(2)$$

and

$$\left[\frac{1}{2 + (x - y)}\right] \leq \frac{1}{2} \leq ln\,(2).$$

Thus

$$F\left(\frac{D(Tx,Ty)}{2}\right)+\phi\left(\frac{D(x,y)}{2}\right)\leq F\left(\frac{D(x,y)}{2}\right)\leq F\left(M(x,y)\right).$$

ii) $x<1$: Then $\sqrt{x}<1$. Thus

$$D(Tx,Ty)=2\left(1-\sqrt{x}\right).$$

So we have

$$F\left(\frac{D(Tx,Ty)}{2}\right)=ln\left(1-\sqrt{x}\right)$$

and

$$\begin{aligned}M(x,y)&=\max\left\{\frac{D\left(x,y\right)}{2},\frac{D\left(x,Tx\right)}{2},\frac{D\left(y,Ty\right)}{2}\right\}\\&\geq\frac{D\left(y,Ty\right)}{2}=1-y\\&\geq 1-\frac{1}{3}=\frac{2}{3}\end{aligned}$$

and

$$F\left(\frac{2}{3}\right)=ln\left(\frac{2}{3}\right).$$

On the other hand,

$$\begin{aligned}&F\left(\frac{D(Tx,Ty)}{2}\right)+\phi\left(\frac{D(x,y)}{2}\right)-F\left(M(x,y)\right)\\&=ln\left(1-\sqrt{x}\right)+\left[\frac{1}{2+(x-y)}\right]-F\left(M(x,y)\right)\\&\leq ln\left(1-\sqrt{x}\right)+\left[\frac{1}{2+(x-y)}\right]-ln\left(\frac{2}{3}\right)\\&=ln\left(\frac{3}{2}\left(1-\sqrt{x}\right)\right)+\left[\frac{1}{2+(x-y)}\right].\end{aligned}$$

Since $x\in\left[\frac{3}{4},1\right[$,

$$ln\left(\frac{3}{2}\left(1-\sqrt{x}\right)\right)+\left[\frac{1}{2+(x-y)}\right]\leq 0.$$

This implies that

$$F\left(\frac{D(Tx,Ty)}{2}\right)+\phi\left(\frac{D(x,y)}{2}\right)\leq F\left(\frac{D(x,y)}{2}\right)\leq F\left(M(x,y)\right).$$

Hence T satisfies the assumption of the theorem and $z=1$ is the unique fixed point of T.

DECLARATIONS

Availablity of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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